Hypersurfaces in hyperbolic space
— Summer school of Xiamen University

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A brief resivion of Riemannian Geometry

Riemannian manifold and Riemannian metric

Suppose M^n is an n-dimensional differentiable manifold (that is a A_2, T_2 space, locally homeomorphic to an open set of $\mathbb{R}^n$, with a C^∞ differentiable structure). We use g to denote its Riemannian metric, that is, to give any point T_pM (tangent space at p) a positive definite inner product, which is smooth when you write it in local coordinate.

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In local coordinate $\{x_i\}_{i=1}^n$, we write $g_{ij}(x) = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})|_{x=(x_1,\cdots,x_n)}$. We say g is smooth (or C^∞) if $g_{ij}(x)$ is a smooth function of x. This property is independent of the local coordinate chosen. If $X = X^i\frac{\partial}{\partial x_i}, Y = Y^i\frac{\partial}{\partial x_i}$ are two vector fields, then $g(X,Y) = X^iY^jg_{ij}$.

Levi-Civita connection

We use ∇ to denote the unique affine connection which is compatiable with the metric g and torsion free. We use $\Gamma_{ij}^k \partial_k = \nabla_{\partial_i} \partial_j$ to define the Christoffel symbol. ∇ induces covariant derivative $\frac{D}{dt}$ along a curve $\alpha(t)$, such that $\frac{D}{dt} X(\alpha(t)) = \nabla_Y X$, if $Y = \dot{\alpha}(t)$ along $\alpha(t)$.

The following formula is an exercise:

$$\Gamma_{ij}^k = \frac{1}{2} g^{mk} (\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}).$$

We denote

$$C^\infty(M) = \{f(x); f(x) \text{ is smooth upto infinitely many order.}\}$$
$$\chi(M) = \{X : X \text{ is a smooth vector field upto infinitely many order.}$$

Curvature

The curvature tensor is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, X, Y, Z \in \chi(M)$$

in which $[X,Y]$ is the Lie brackets of X, Y. We denote $R(X,Y,Z,W) = g(R(X,Y)W,Z)$.

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In local coordinate $\{x_i\}$, we write $R^l_{ijk}\partial_l = R(\partial_i,\partial_j)\partial_k$ and $R_{ijkl} = R(\partial_i,\partial_j,\partial_k,\partial_l)$. We have

$$R_{ijkl} = g^{ls}(\Gamma^m_{ik}\Gamma^s_{jm} - \Gamma^m_{jk}\Gamma^s_{im} + \partial_j\Gamma^s_{ik} - \partial_i\Gamma^s_{jk}).$$

Sectional curvature

Suppose $\sigma \subset T_pM$ is a two dimensional subspace of T_pM, we let $\sigma = span\{X, Y\}$. The sectional curvature of σ is $K(\sigma) = \frac{R(X,Y,X,Y)}{|X \wedge Y|^2}$, where $|X \wedge Y|$ is the signed area of the parallelogram spanned by X, Y.

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Ricci tensor

Ricci tensor $Ric(X, Y)$ is defined by $\sum_{i=1}^{n} R(X, e_i, Y, e_i)$, $\{e_i\}$ is orthonormal basis. $Ric(\frac{X}{|X|}, \frac{X}{|X|})$ is called the Ricci curvature in X direction.

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Scalar curvature

Scalar curvature R is defined by $R = Ric(e_i, e_i)$.

Submanifold and hypersurface

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Second fundamental form

Let N is an immersed submanifold of M, with induced metric h, we use $\bar{\nabla}$ to denote the Levi-Civita connection of N. Then $\forall p \in N$, $T_pM = T_pN \oplus T_p^{\perp}N$. The second fundamental form $\vec{A} : T_pN \times T_pN \to T_p^{\perp}N$ is defined by $\vec{A}(X,Y) = \bar{\nabla}_X Y - \nabla_X Y$. If N is codimension 1 and locally we choose a unit normal vector field η, then $\vec{A} = A\eta$ and A is a real valued symmetric two tensor.

Space form

Among Riemannian manifolds, the constant sectional curvature ones are important, which are called space forms. Simply connected space forms can be rescaled to the following 3 kinds, 1. $K \equiv 1$, sphere $\mathbb{S}^n$; 2. $K \equiv 0$, Euclidean space $\mathbb{R}^n$; 3. $K \equiv -1$, Hyperbolic space $\mathbb{H}^n$.

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We will study the hyperbolic space $\mathbb{H}^n, n \geq 2$. It is often given by the following half space model.

Half space model of $\mathbb{H}^n$

Let $\mathbb{R}^n_+ = \{(x_1, \cdots, x_{n-1}, x_n); x_n > 0, x_i \in \mathbb{R}, i = 1, \cdots, n-1\}$ be the half Euclidean space. We define the metric $g_{\mathbb{H}^n} = \frac{1}{x_n^2}(dx_1^2 + \cdots + dx_n^2)$.

The calculation of the curvature of $\mathbb{H}^n$.

We calculate the curvature of $g_{ij} = \frac{\delta_{ij}}{F^2}$, where $F > 0$ is a smooth function. We write $g^{ij} = F^2\delta_{ij}$ as the inverse matrix of g_{ij}. Let $f = \log F$. We denote $f_j = \frac{\partial f}{\partial x_j}$. We have

$$\frac{\partial g_{ik}}{\partial x_j} = -\delta_{ik}\frac{2}{F^3}F_j = -2\frac{\delta_{ik}}{F^2}f_j,$$

$$\Gamma_{ij}^k = \frac{1}{2}\sum_m g^{km}\{\frac{\partial}{\partial x_i}g_{jm} + \frac{\partial}{\partial x_j}g_{mi} - \frac{\partial}{\partial x_m}g_{ij}\}$$
$$= \frac{1}{2}F^2(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}) = -\delta_{jk}f_i - \delta_{ki}f_j + \delta_{ij}f_k.$$

So $\Gamma_{ij}^k = 0$, if all three indices are distinct, and if $i \neq j$,

$$\Gamma_{ij}^i = -f_j, \Gamma_{ii}^j = f_j, \Gamma_{ij}^j = -f_i, \Gamma_{ii}^i = -f_i.$$

So we have

$$R_{ijij} = \frac{1}{F^2}(\Gamma^l_{ii}\Gamma^j_{jl} - \Gamma^l_{ji}\Gamma^j_{il} + \frac{\partial}{\partial x_j}\Gamma^j_{ii} - \frac{\partial}{\partial x_i}\Gamma^j_{ji}).$$

Since $\frac{\partial}{\partial x_j}\Gamma^j_{ii} = f_{jj}$ and $\frac{\partial}{\partial x_i}\Gamma^j_{ji} = -f_{ii}$, we have

$$\begin{aligned} F^2 R_{ijij} &= -\sum_{l\neq i, l\neq j} f_l^2 + f_i^2 - f_j^2 - f_i^2 + f_j^2 + f_{jj} + f_{ii} \\ &= -\sum_l f_l^2 + f_i^2 + f_j^2 + f_{ii} + f_{jj}. \end{aligned}$$

And $R_{ijkl} = 0$ if all four indices are distinct and if i, j, k are distinct, we have

$$R_{ijki} = -F^{-2}(f_k f_j + f_{kj}),\ R_{ijkj} = F^{-2}(f_i f_k + f_{ki}),\ R_{ijkk} = 0. \tag{1}$$

$$K(\partial_i \wedge \partial_j) = \frac{R_{ijij}}{g_{ii}g_{jj}} = R_{ijij}F^4$$
$$= (-\sum_l f_l^2 + f_i^2 + f_j^2 + f_{ii} + f_{jj})F^2.$$

When $F^2 = x_n^2$, and $f = \log x_n$, if $i, j \neq n$, then

$$K_{ij} = -\frac{1}{x_n^2}x_n^2 = -1;$$

If $i = n, j \neq n$, then

$$K_{nj} = (-f_n^2 + f_n^2 + f_{nn})F^2 = -\frac{1}{x_n^2}x_n^2 = -1.$$

From (1), we know $R_{ijki} = R_{ijkj} = R_{ijkl} = 0$ and

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Connections of conformal metrics

Suppose (M, g, ∇) is a Riemannian manifold with Levi Civita connection ∇. If $\bar{g} = \mu g$ is a conformal metric and $\bar{\nabla}$ is the Levi Civita connection for $\bar{g}$, then

$$\bar{\nabla}_X Y = \nabla_X Y + S(X, Y),$$

where $S(X,Y) = \frac{1}{2\mu}\{(X\mu)Y + (Y\mu)X - g(X,Y)\nabla\mu\}$, where $\nabla\mu$ is the gradient of μ.
It is easy to check $\bar{\nabla}$ is an affine connection and is torsion free.

$$\begin{aligned}
&X(\bar{g}(Y,Z)) \\
=&X(\mu g(Y,Z)) = X(\mu)g(Y,Z) + \mu g(\nabla_X Y, Z) + \mu g(Y, \nabla_X Z) \\
=&X(\mu)g(Y,Z) + \mu g(\bar{\nabla}_X Y - S(X,Y), Z) + \mu g(Y, \bar{\nabla}_X Z - S(X,Z)) \\
=&\bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}_X Z).
\end{aligned}$$

Umbilic hypersurfaces of the hyperbolic space

Let (M^{n+1}, g, ∇) be a manifold and $f : N^n \to M^{n+1}$ be an immersion. We say it is (totally) umbilic if for all $p \in N$, the second fundamental form A at $p \in N$ satisfies $A(X, Y) = \lambda(p)g(X, Y)$ with respect chosen unit normal vector η at p.

If $\bar{g} = \mu g$, then we show an umbilic immersion in (M^{n+1}, g) continues to be umbilic in the metric $\bar{g}$. This is because if $< \nabla_X \eta, Y >_g = -\lambda < X, Y >_g$, then

$$\begin{aligned} < \bar{\nabla}_X(\frac{\eta}{\sqrt{\mu}}), Y >_{\bar{g}} &= \mu < \nabla_X(\frac{\eta}{\sqrt{\mu}}), Y >_g + \mu < S(X, \frac{\eta}{\sqrt{\mu}}), Y >_g \\ &= \frac{-2\lambda\mu + \eta(\mu)}{2\mu\sqrt{\mu}} < X, Y >_{\bar{g}} . \end{aligned}$$

This is because

$$\begin{aligned}\mu < \nabla_X(\frac{\eta}{\sqrt{\mu}}), Y >_g &= \sqrt{\mu} < \nabla_X \eta, Y >_g \\ &= -\lambda\sqrt{\mu} < X, Y >_g \\ &= -\frac{\lambda}{\sqrt{\mu}} < X, Y >_{\bar{g}};\end{aligned}$$

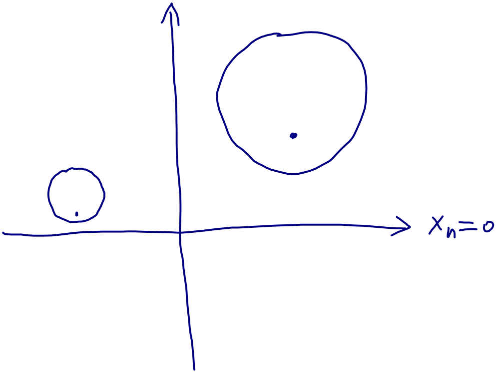
and

$$\begin{aligned}\mu < S(X, \frac{\eta}{\sqrt{\mu}}), Y >_g &= \frac{1}{2}(< X(\mu)\frac{\eta}{\sqrt{\mu}} + \frac{\eta(\mu)}{\sqrt{\mu}}X - g(X, \frac{\eta}{\sqrt{\mu}}), Y >_g) \\ &= \frac{1}{2}\frac{\eta(\mu)}{\sqrt{\mu}} < X, Y >_g = \frac{1}{2}\frac{\eta(\mu)}{\mu^{\frac{3}{2}}} < X, Y >_{\bar{g}} .\end{aligned}$$

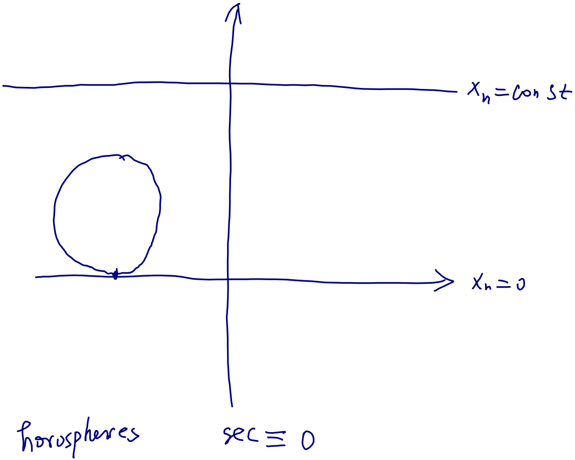
Classifications of umbilic hypersurfaces of hyperbolic space

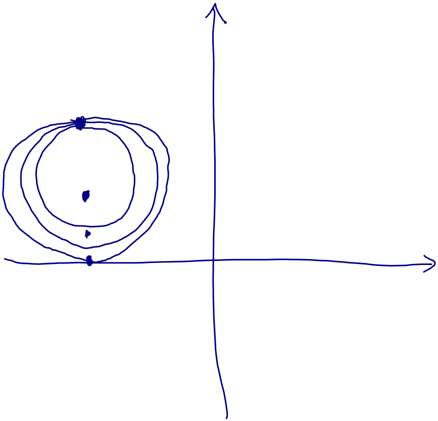
So a hypersuface in $\mathbb{H}^n$ is totally umbilic if and only if it is totally umbilic in $\mathbb{R}^n$, those are planes or spheres, which can be classified as follows.

- Geodesic spheres;
- Horospheres: A horosphere is an isometric embedding of flat $\mathbb{R}^{n-1}$ in $\mathbb{H}^n$, which is complete noncompact, the proof is left to the next section;
- Hyperspheres.

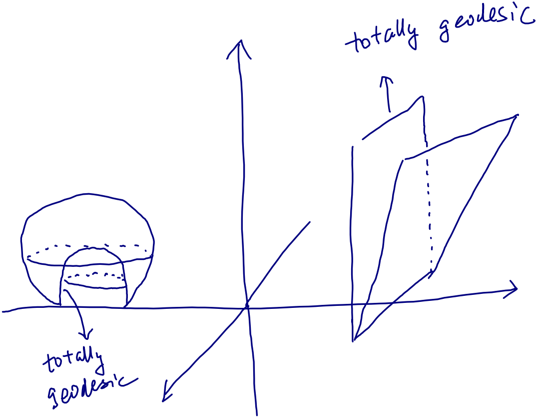


geodesic spheres, sec = const > 0


$x_n = const$
$x_n = 0$
horospheres
$sec \equiv 0$



geodesic sphere converges to a horosphere.



hyperspheres. See < 0.

Conformal map

Suppose $f : (M, g) \to (N, h)$ is an immersion such that $h(df(X), df(Y)) = \mu g(X, Y)$, then we call f a conformal map and μ is called the conformal factor. An injective conformal map from a domain of M to M is called a conformal transformation.

The isometries of $\mathbb{H}^n$ are closely related to the conformal transformations of $\mathbb{R}^n$.
If $n = 2$, a conformal transformation defined on $D \subset \mathbb{R}^2$ is given by a holomorphic or anti-holomorphic functions with non-zero derivative. The Riemann mapping theorem guarantees that given two simply connected proper open sets in $\mathbb{R}^2$, there is a conformal map taking one into the other.

For $\mathbb{R}^n, n > 2$, there are "much less" conformal transformations defined on a domain of $\mathbb{R}^n$. Actually if $f : D \to \mathbb{R}^n$ is a conformal map defined on $D \subset \mathbb{R}^n$, f have to be the restriction to D of three global defined conformal transformations.

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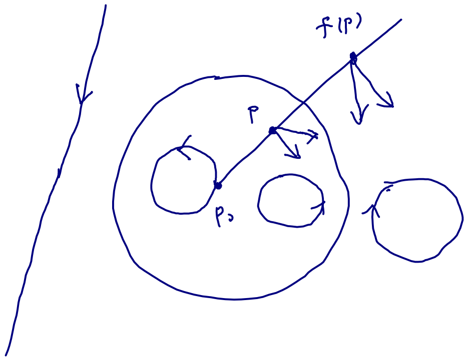
1. Isometry of $\mathbb{R}^n$ ($y = Ax + b, A \in O(n)$) is a conformal transformation of $\mathbb{R}^n$ with conformal factor $\mu \equiv 1$;
2. $f(p) = \lambda Id(p)$, where Id is the identity and $\lambda > 0$, is call a dilatation;
3. $f(p) = \frac{p-p_0}{|p-p_0|^2} + p_0, p \in \mathbb{R}^n \backslash \{p_0\}$ is the inversion with respect to the unit sphere centered at $p_0 \in \mathbb{R}^n$. The image of a sphere is either a sphere or a hyperplane. The image of a circle is either a circle or a line.



Inversion

To see f is conformal, observe that, if v is a vector at p,

$$df_p(v) = \frac{v|p-p_0|^2 - 2 < v, p-p_0 > (p-p_0)}{|p-p_0|^4},$$

where we use $<,>$ to denote the Euclidean metric and

$$\begin{aligned} |df_p(v)|^2 &= \frac{< v, v >}{|p-p_0|^4} + \frac{(4 < v, p-p_0 >^2 - 4 < v, p-p_0 >^2)|p-p_0|^2}{|p-p_0|^8} \\ &= \frac{< v, v >}{|p-p_0|^4}. \end{aligned}$$

Theorem (Liouville)

Let $f : U \to \mathbb{R}^n, n \geq 3$, be a conformal map defined on a domian $U \subset \mathbb{R}^n$. Then f is the restriction to U of a composition of isometries, dilatations and inversions, at most one of each.

Theorem (Liouville)

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Proof. Let $(x_1, \cdots, x_n)$ be the cartesian coordinate of $\mathbb{R}^n$ and $a_i = \frac{\partial}{\partial x_i}$. Let $\{e_i\}$ be parallel differentiable vector fields on U and $< e_i, e_j >= \delta_{ij}$. We assume

$$< df(e_i), df(e_k) >= \lambda^2 \delta_{ik}, i, k = 1, \cdots, n. \tag{2}$$

Let $d^2 f$ be the second differential of f, such that $d^2 f(a_i, a_j) = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Taking the indices i, j, k distinct and from (2) we have

$$< d^2 f(e_i, e_j), df(e_k) > + < df(e_i), d^2 f(e_k, e_j) >= 0,$$

Then by permutation, we have

$$< d^2f(e_j, e_k), df(e_i) > + < df(e_j), d^2f(e_i, e_k) > = 0,$$
$$< d^2f(e_k, e_i), df(e_j) > + < df(e_k), d^2f(e_j, e_i) > = 0.$$

Summing the first two and subtracting the third, we have

$$< d^2f(e_k, e_j), df(e_i) >= 0, i, j, k \text{ distinct.}$$

Fixing k, j and let i vary in the remaining $(n-2)$ indices, we know

$$d^2f(e_k, e_j) = \mu df(e_k) + \nu df(e_j).$$

Since $< df(e_k), df(e_k) >=< df(e_j), df(e_j) >= \lambda^2$,

$$\mu = \frac{< d^2f(e_k, e_j), df(e_k) >}{\lambda^2} = \frac{e_j(\lambda^2)}{\lambda^2} = \frac{d\lambda(e_j)}{\lambda}, \ \nu = \frac{d\lambda(e_k)}{\lambda}.$$

Then $d^2f(e_k, e_j) = \frac{1}{\lambda}(df(e_k)d\lambda(e_j) + df(e_j)d\lambda(e_k))$.

Let $\rho = \frac{1}{\lambda}$.

$$\begin{aligned} d(\rho f) &= f d\rho + \rho df; \\ d^2(\rho f)(e_k, e_j) &= d^2\rho(e_k, e_j) f + \rho d^2 f(e_k, e_j) + d\rho(e_k) df(e_j) \\ &\quad + d\rho(e_j) df(e_k) = d^2\rho(e_k, e_j) f + \frac{1}{\lambda} d^2 f(e_k, e_j) \\ &\quad - \frac{1}{\lambda^2}\{d\lambda(e_k) df(e_j) + d\lambda(e_j) df(e_k)\} = d^2\rho(e_k, e_j) f. \end{aligned}$$

We claim that $d^2\rho(e_k, e_j) = 0$ for $k \neq j$. We calculate $d^3(\rho f)$.

$$d^3(\rho f)(e_k, e_j, e_i) = d^3\rho(e_k, e_j, e_i) f + d^2\rho(e_k, e_j) df(e_i).$$

So we know $d^2\rho(e_k, e_j) df(e_i)$ are symmetric in i, j, k. So we know

$$d^2\rho(e_k, e_j) df(e_i) = d^2\rho(e_k, e_i) df(e_j).$$

Since $df(e_i)$ and $df(e_j)$ are linearly independent and i, j, k can be chosen distinct and arbitrarily, then $d^2\rho(e_k, e_j) = 0$ for all $j \neq k$.

Actually $d^2\rho(e_k, e_j) = 0, k \neq j$ hold for any orthomormal basis . So we know

$$0 = d^2\rho(\frac{e_j + e_k}{\sqrt{2}}, \frac{e_j - e_k}{\sqrt{2}}) = \frac{1}{2}(d^2\rho(e_j, e_j) - d^2\rho(e_k, e_k)),$$

which implies $d^2\rho(e_j, e_j) = d^2\rho(e_k, e_k)$ for all $j \neq k$. So

$$d^2\rho(e_j, e_k) = \sigma\delta_{jk}.$$

So we have

$$d\sigma(e_j) = e_j e_k e_k \rho = e_k e_j e_k \rho = 0,$$

so $\sigma \equiv const.$

If $\sigma = const \neq 0$ then

$$\rho = \frac{\sigma}{2}\sum x_i^2 + \sigma \sum b_i x_i + c, b_i, c \text{ constants.}$$

To prove this we let $\rho_1 = \frac{\sigma}{2}\sum x_i^2$, then $\frac{\partial^2(\rho-\rho_1)}{\partial x_i \partial x_j} = 0, \forall i, j = 1, \cdots n$. So

$$\frac{\partial(\rho-\rho_1)}{\partial x_i} = \sigma b_i.$$

Now

$$\frac{\partial(\rho-\rho_1-\sigma\sum_j b_j x_j)}{\partial x_i} = 0.$$

So

$$\rho = \frac{\sigma}{2}\sum x_i^2 + \sigma\sum_i b_i x_i + c.$$

So

$$\frac{1}{\lambda} = \rho = a_1|p - p_0|^2 + k_1, a_1 = \frac{\sigma}{2}, k_1 \equiv const, p_0 \in \mathbb{R}^n.$$

Then proof will be complete, for the case $\sigma \neq 0$, if we show $k_1 = 0$. This is because, considering the inversion $g : U \to \mathbb{R}^n$:

$$g(p) = \frac{p - p_0}{|p - p_0|^2} + p_0,$$

and let $h = g \circ f^{-1}$, we know h is a conformal transformation whose conformal factor is $a_1|p - p_0|^2 \frac{1}{|p-p_0|^2} = a_1$. Then the conformal factor of $a_1^{-1}h$ is 1. So we can choose a global isometry $h_1 : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$h_1(p) = a_1^{-1}h(p), dh_1(p) = d(a_1^{-1}h)(p).$$

Then $a_1^{-1}h \equiv h_1$ due to the following lemma.

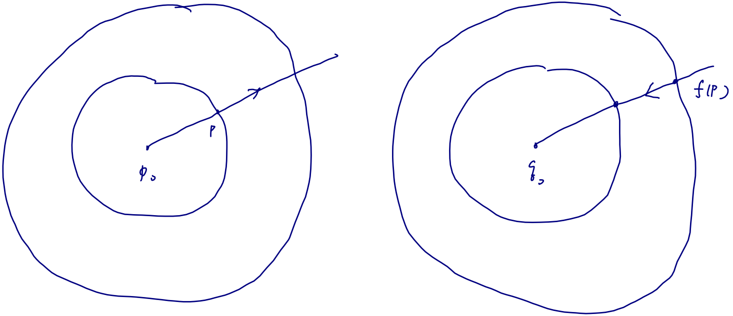
Lemma

Let $f_i : M \to N, i = 1, 2$ be two local isometries of the connected Riemannian manifold M to the Riemannian manifold N. Suppose that there exists $p \in M$ such that $f_1(p) = f_2(p)$ and $df_1(p) = df_2(p)$. Then $f_1 = f_2$.

Now we will prove $k_1 = 0$. Observe that applying to f^{-1} the above argument, we obtain $\lambda = a_2|f(p) - q_0|^2 + k_2$ or $\lambda = A_2(f(p)) + k_2$. We only prove the first case

$$(a_1|p - p_0|^2 + k_1)(a_2|f(p) - q_0|^2 + k_2) = 1. \qquad (3)$$

So a sphere of center p_0 is mapped by f into a sphere of center q_0. Since f preserves the angles, the radii of the first sphere are mapped into radii of the second. Let $p(s), 0 \leq s \leq s_0$ be a segment of a radius of the first sphere contained in U, where s is the arc length, and let $f \circ p(s)$ be its image.



Then length of the image segment is given by

$$\int_0^{s_0} |df(\frac{dp}{ds})|ds = \int_0^{s_0} \frac{ds}{a_1|p(s) - p_0|^2 + k_1} = |f(p(s_0)) - f(p(0))|.$$

If $k_1 \neq 0$, $|f(p(s_0)) - f(p(0))|$ is a transcendental function of $|p(s_0) - p|$ which is a contradiction with (3). So $k_1 = 0$.
It remains to consider the case $\sigma = 0$. Then

$$\rho = \frac{1}{\lambda} = \sum a_i x_i + c_1 = A_1(x) + c_1.$$

Similarly applying the initial argument to f^{-1}, we have

$$(A_1(x) + c_1)(a_2|f(x) - q_0|^2 + k_2) = 1,$$

$$\text{or } (A_1(x) + c_1)(A_2(f(x)) + c_2) = 1.$$

Since

$$|f(p(s_0)) - f(p(0))| = \int_0^{s_0} \frac{ds}{A_1(p(s)) + c_1},$$

is always a transcedental function, no matter $c_1 = 0$ or not, we know $A_1(x) \equiv 0$.

So if $\sigma = 0, \lambda = const$. In this case, the lengths of tangent vectors are multiplied by a constant λ and it's easy to verify f is an isometry followed by a dilatation. Then we proved the Liouville's theorem.

In the end of this section, we study the isometry group of $\mathbb{H}^n$.

Theorem

For $n \geq 2$, the isometries of $\mathbb{H}^n$ are the restrictions to $\mathbb{H}^n \subset \mathbb{R}^n$ of the conformal transformations of $\mathbb{R}^n$ that take $\mathbb{H}^n$ onto itself. (onto means surjective.)

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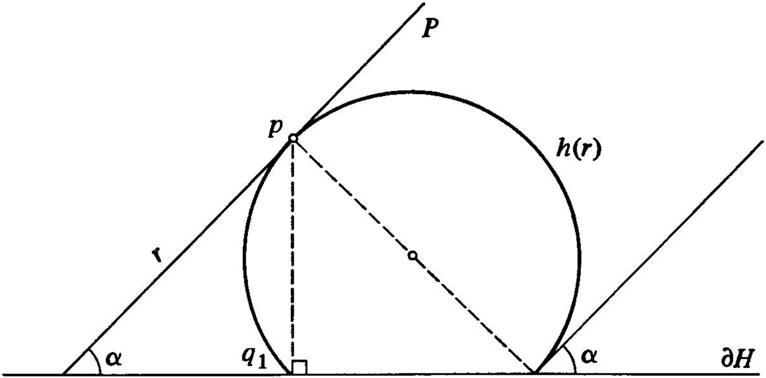
Proof. First we assume $n \geq 3$. Let $f : \mathbb{H}^n \to \mathbb{H}^n$ be an isometry in g_{ij}. Then by Liouville's theorem, f extends to a conformal map of $\mathbb{R}^n$ with metric δ_{ij}. It is obvious that f is surjective.

Conversely, let $f : \mathbb{H}^n \subset \mathbb{R}^n \to \mathbb{H}^n$ be a conformal transformation of $\mathbb{H}^n$ onto $\mathbb{H}^n$ and let $e_1, \cdots, e_n$ be an orhtonormal basis, in the metric g_{ij}, at a point $p \in \mathbb{H}^n$. Since $g_{ij} = x_n^{-2}\delta_{ij}$ and f is conformal, there is $\lambda^2 > 0$ such that $g(df_p(e_i), df_p(e_j)) = \lambda^2\delta_{ij}$. Therefore $\{\frac{df_p(e_i)}{\lambda}\}$ is orthonormal at $f(p)$, and by the following theorem which is a consequence Cartan's theorem, there is an isometry g of $\mathbb{H}^n$ taking p to $f(p)$, with $dg(e_i) = \frac{df(e_i)}{\lambda}$.

Theorem

Let M be a space of constant curvature let $p, q \in M$. Let $\{e_i\}$ and $\{f_j\}$ be arbitrary orthonormal basis of $T_p(M)$ and $T_q(M)$, respectively. There there are neighborhoods U of p and V of q, and an isometry $g : U \to V$ such that $dg_p(e_j) = f_j$. In particular, if both exp_p and exp_q are defined in whole T_pM and T_qM and both are diffeomorphism, then g is a global isometry.

So $h = g^{-1} \circ f$ is the restriction to $\mathbb{H}^n$ of a conformal mapping of $\mathbb{R}^n$ which takes $\mathbb{H}^n$ onto $\mathbb{H}^n$, leaving p fixed and satisfying $dh_p = \lambda I$. We will end the proof by showing $h = I$.
To see this, let P be a hyperplane passing through p. From Liouville's theorem, $h(P)$ is a hypersplane or a sphere passing through p. Because dh_p is a multiple of the identity, P and $h(P)$ are tangent at P. Because h takes the boundary $\partial\mathbb{H}^n$ of $\mathbb{H}^n$ into itself, and h is conformal, the angle of P with $\partial\mathbb{H}^n$ is the same as angle of $h(P)$ with $\partial\mathbb{H}^n$.
We claim that $h(P) = P$. To see this, consider a straight line r_1 passing through p and orthogonal to $\partial\mathbb{H}^n$, and let $q_1 = r_1 \cap \partial\mathbb{H}^n$. Since the image $h(r_1)$ of r_1 is a circle or a line, and makes the same angle with $\partial\mathbb{H}^n$ that r_1 does, it is clear that $h(r_1) = r_1$ and $h(q_1) = q_1$. This shows that if P is perpendicular to $\partial\mathbb{H}^n$, then $h(P) = P$.


P
p
h(r)
r
α
q1
α
∂H

If P is not orthogonal to $\partial\mathbb{H}^n$, let r be a line contained in P. Again, $h(r)$ is a circle or a line. If $h(r)$ is a circle, making the same angle α with $\partial\mathbb{H}^n$ that r does, then $q_1 \in h(r)$, which contradicts with the fact that $h(q_1) = q_1$. So $h(r)$ is a line, $h(r) = r$. So $h(P) = P$.
So h cannot include an inversion (since the image of most planes would be a spheres) or an isometry distinct from identity. From Liouville's theorem, $h = aA(x-p)+p, A \in O(n)$. So $A = I$ and $a = 1$, since it takes $\mathbb{H}^n$ onto itself.

For $n = 2$ case, the conformal transformation which maps $\mathbb{R}^2_+$ onto $\mathbb{R}^2_+$ is of the following form

$$f(z) = \frac{az+b}{cz+d}, z \in \mathbb{H}^2 \subset \mathbb{C}, a, b, c, d \in \mathbb{R}, ad - bc = 1.$$

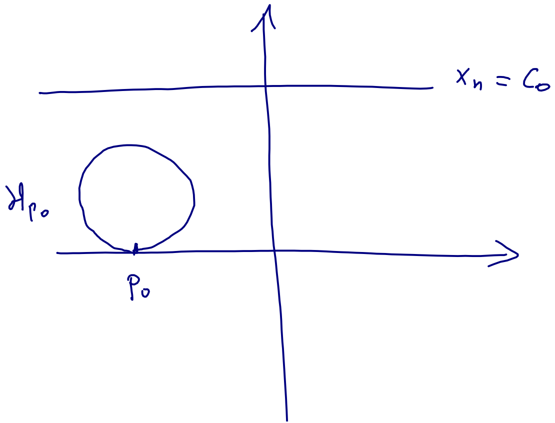
Due to easy calculation, we know it is isometry of $\mathbb{H}^2$. Moreover, it is not difficult to show that for a fixed point $p_0 \in \mathbb{H}^2$ and unit vector $v_0 \in T_{p_0}\mathbb{H}^2$, there exists a transformation f of the above form which takes any arbitrary p and unit vector $v \in T_p\mathbb{H}^2$ to (p_0, v_0). (Note that (p, v) are determined by 3 parameters and (a, b, c, d) are essentially 3 parameters since $ad - bc = 1$). Since there is a unique isometry of $\mathbb{H}^2$ which takes (p, v) to (p_0, v_0), we know all isometries of $\mathbb{H}^2$ are of the above form.

Now we prove that

Theorem

A horosphere in $\mathbb{H}^n$ is an isometric embedding of flat $\mathbb{R}^{n-1}$ in $\mathbb{H}^n$, which is complete and noncompact.

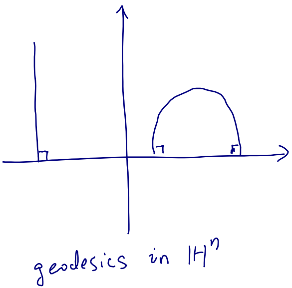
Proof. If a horosphere has equation $x_n = c_0$. The metric on it is $\frac{1}{c_0^2}(dx_1^2 + \cdots + dx_{n-1}^2)$. Clearly it is Euclidean metric on flat $\mathbb{R}^{n-1}$, upto scaling. It is obviously isometric embedding of $\mathbb{R}^{n-1}$ in $\mathbb{H}^n$. If a horosphere $\mathcal{H}_{p_0}$ is tangential to $x_n = 0$ at point $p_0 \in \partial\mathbb{R}^n_+$, then we can use a inversion $\frac{p-p_0}{|p-p_0|^2}$ to map it to the form $x_n = c_0$. Since such an inversion is an isometry of $\mathbb{H}^{n-1}$, we see $\mathcal{H}_{p_0}$ is also an isometric embedding of flat $\mathbb{R}^{n-1}$.

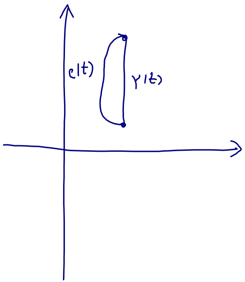

$x_n = c_0$
$\mathcal{H}_{p_0}$
p_0

We have studied some hypersurfaces in $\mathbb{H}^n$. Especially we know a horosphere is an embedding of flat $\mathbb{R}^{n-1}$ in $\mathbb{H}^n$. So it is complete noncompact, with sectional curvature 0 everywhere. We want to study some other examples of: complete noncompact hypersurface in $\mathbb{H}^n$, which has $Sec \geq 0$.
First, we study the geodesics of $\mathbb{H}^n$.

Theorem

The geodesics of $\mathbb{H}^n$ (half space model) are either straight rays which are orthogonal to $\partial\mathbb{H}^n$ or half circles which are orthogonal to $\partial\mathbb{H}^n$ at two ends.





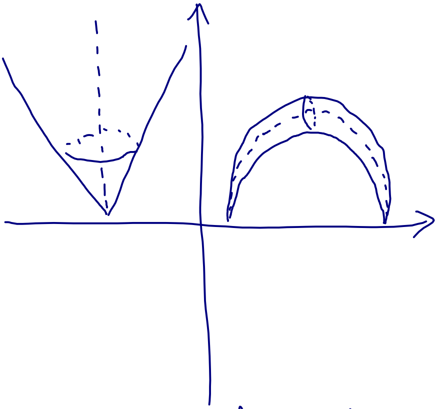
Proof.

A straight ray which is orthogonal to $\partial\mathbb{H}^n$ is given by $\gamma(t) = (x_1^0, \cdots, x_{n-1}^0, e^t), t \in (-\infty, +\infty)$. Note that $g_{\mathbb{H}^n}(\dot\gamma(t), \dot\gamma(t)) = 1$. We will prove that $\gamma(t), t_1 \le t \le t_2$ has least length among $\{c(t); c(t_i) = \gamma(t_i), i = 1, 2, c(t) \subset \mathbb{H}^n\}$.

We write $c(t) = (x_1(t), \cdots, x_n(t))$, then

$$l(c(t)) = \int_{t_1}^{t_2} \sqrt{g(\dot{c}(t), \dot{c}(t))}dt \geq \int_{t_1}^{t_2} \frac{1}{x_n^2(t)} \sqrt{\dot{x}_1(t)^2 + \cdots + \dot{x}_n(t)^2}dt$$
$$\geq \int_{t_1}^{t_2} \frac{\dot{x}_n(t)}{x_n}dt = \int_{e^{t_1}}^{e^{t_2}} \frac{1}{x_n}dx_n = t_2 - t_1 = l(\gamma(t)).$$

Any half circle which is orthogonal to $\partial\mathbb{H}^n$ at its two ends can be mapped to a straight ray orthogonal to $\partial\mathbb{H}^n$ by an isometry of $\mathbb{H}^n$. In the end, for any $p \in \mathbb{H}^n$ and $v \in T_p\mathbb{H}^n$, from basic geometry, we know there is a geodesic of the above type passing p with velocity v. Then any geodesic of $\mathbb{H}^n$ is of the above type from the uniqueness of the geodesic.



Equidistant hypersurface

Equidistant hypersurface (with respect to a given geodesic)

An equidistant hypersurface is a hypersurface which has everywhere the same distance to a given geodesic.

Equidistant hypersurface (with respect to a given geodesic)

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Theorem

For a given geodesic and a given distant $r > 0$, there is always such a hypersurface, which is complete, noncompact, embedded and nonnegatively curved. It is the isometric embedding of $\mathbb{R} \times \frac{1}{\alpha}\mathbb{S}^{n-2}$ in $\mathbb{H}^n$.

To prove the theorem, we assume the geodesic is $\gamma(t) = (x_1^0, \cdots, x_{n-1}^0, e^t), t \in (-\infty, +\infty)$. Otherwise, we can use an isometry of $\mathbb{H}^n$ to map the geodesic to such a geodesic. Then given r, the equidistant hypersurface is given by

$$x_n = \alpha\sqrt{(x_1 - x_1^0)^2 + \cdots + (x_{n-1} - x_{n-1}^0)^2}$$

and $\alpha = (\sinh r)^{-1}$.

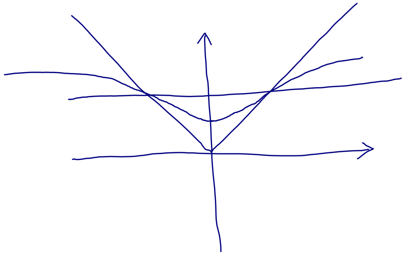
Now we prove that it is an embedded infinite cylinder. We let $e^t = s = \sqrt{(x_1 - x_1^0)^2 + \cdots + (x_{n-1} - x_{n-1}^0)^2}$.

Then on equidistant hypersurface

$$\begin{aligned}\frac{1}{x_n^2}(dx_1^2+\cdots+dx_{n-1}^2+dx_n^2) &= \frac{1}{\alpha^2 s^2}(ds^2+s^2 g_{\mathbb{S}^{n-2}}+\alpha^2 ds^2)\\ &= \frac{1}{\alpha^2}((1+\alpha^2)dt^2+g_{\mathbb{S}^{n-2}}).\end{aligned}$$

Obviously, it is an embedding of $\mathbb{R}\times\frac{1}{\alpha}\mathbb{S}^{n-2}$, which has nonnegative sectional curvature. Now we end the proof.
We notice that when the equidistant surface in $\mathbb{H}^3$, has topology $\mathbb{R}\times\mathbb{S}^1$, which has nontrivial covering space $\mathbb{R}^2$. So it can be regarded as the isometric immersion of $\mathbb{R}^2$ in $\mathbb{H}^3$. However, if the equidistant hypersurface in $\mathbb{H}^n, n\geq 4$, has topology $\mathbb{R}\times\mathbb{S}^{n-2}$, which is simply connected.

We know $x_n = const$ is a horosphere, and $x_n = \alpha\sqrt{x_1^2 + \cdots + x_{n-1}^2}$ is an equidistant hypersurface. There are also global graphs with $Sec \geq 0$ which lie in between at infinity. Refer to Alexander and Currier's paper for the expression of such surfaces.

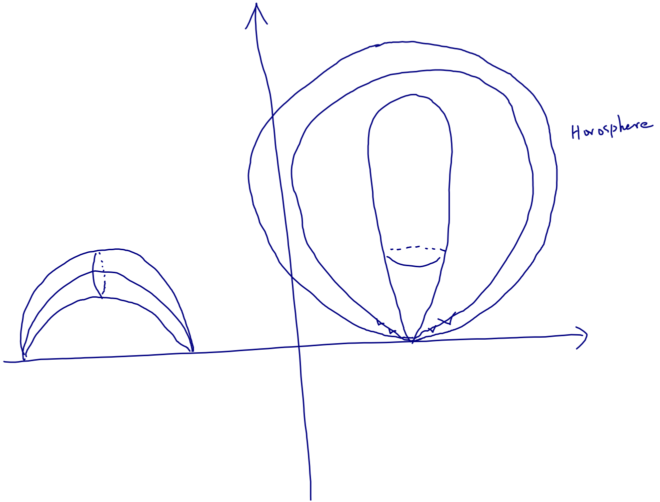


Based on the above examples, we can state the following theorem of Alexander and Currier.

Theorem(Alexander and Currier)

Let M be a nonnegatively curved, complete noncompact hypersurface of dimension $n-1, n \geq 3$, properly embedded in $\mathbb{H}^n$. Then it is of the 3 kinds below. 1. Horosphere; 2. Equidistant hypersurface; 3. A hypersurface which is asymptotical to an equidistant hypersurface, or "weakly asymptotical" to a rotation hypersurface which support every horosphere and supported by any equidistant hypersurface at infinity.

Here "properly embedded" means the preimage of a compact set is compact.


Horosphere

Conjecture(Alexander and Currier)

Except for covering maps of equidistant surfaces in $\mathbb{H}^3$, a complete, nonnegatively curved immersed hypersurface in hyperbolic space is necessarily properly embedded.

Conjecture(Alexander and Currier)

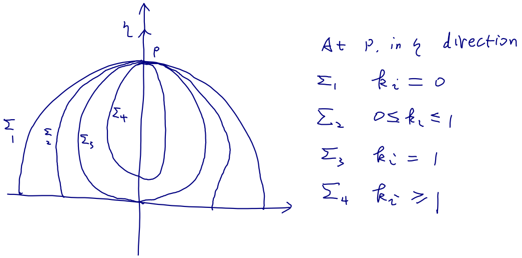
Except for covering maps of equidistant surfaces in $\mathbb{H}^3$, a complete, nonnegatively curved immersed hypersurface in hyperbolic space is necessarily properly embedded.

This conjecture is proved by V.Bonini, S.Ma and J.Qing. Now we briefly revise the "convex hypersurface problem" in $\mathbb{R}^n$ and $\mathbb{H}^n$. For a hypersurface of $\mathbb{R}^n$, at a point p, the following convexity conditions are equivalent. 1. The second fundamental form is semi-definite; ($\Longleftrightarrow$ all principal curvature k_i has same sign) 2. The hypersurface has nonnegative sectional curvature at p($\Longleftrightarrow$ $k_i k_j \geq 0, \forall i \neq j.$); 3. The hypersurface has nonnegative Ricci curvature at p (($\Longleftrightarrow$ $k_i(\sum_{j\neq i} k_j) \geq 0, \forall i \neq j.$)).

Immersion implies embedding type theorem in $\mathbb{R}^n$

- Hadamard: A compact, strictly positively curved $(K > 0)$ surface immersed in Euclidean space $\mathbb{R}^3$ is embedded as the boundary of a convex body.
- Stoker, Chern and Lashof, van Heijenoort extended Hadamard's theorem.
- Sacksteder made a most general extension to Hadamard's theorem: a complete, nonflat, nonnegatively curved $(Sec \geq 0)$ immersed hypersurface (with $dim \geq 2$) in Euclidean space is the boundary of a convex body.

For a hypersurface Σ in $\mathbb{H}^n$, we can impose several "convexity" conditions. At a point p, the following conditions become stronger and stronger. 1. $k_i \geq 0$ with respect to a given unit normal; 2. $Ric \geq 0$ at p; 3. $Sec \geq 0$ at p; 4. $k_i \geq 1$ (infinitesemal supported by horosphere) .



Immersion implies embedding type theorem in $\mathbb{H}^n$

- do Carmo and Warner: An isometric immersion in $\mathbb{H}^n$ of compact, connected orientable manifold with $sec \geq -1$ (consequence of $k_i \geq 0$) is a boundary of a convex body of $\mathbb{H}^n$, hence is embedded;
- For noncompact case, even with $k_i > 0$, a complete, immersed hypersurafce in $\mathbb{H}^n$ need not be embedded;
- Currier: a complete, immersed hypersurface in $\mathbb{H}^n$ with $k_i \geq 1$ is embedded. (Notice that equidistant hypersurface does not satisfies this assumption);
- Epstein: a complete, noncompact, immered surafce in $\mathbb{H}^3$, if $k_i > 0$ and the "boundary at infinity" consists of single point, is then embedded.

Question

Suppose

$$-\Delta u(x) = f(x) \geq 0, u(x) \geq 0, f(x) \in L^1(B_1(0)), x \in B_1(0)\backslash\{0\} \subset \mathbb{R}^2.$$

Let $\bar{u}(r) = \frac{1}{2\pi r}\int_{\partial B_r} u(x)dS$. We have $\frac{\bar{u}(r)}{\log\frac{1}{r}} \to m \geq 0, r \to 0^+$. Is it true that $\frac{u(x)}{\log\frac{1}{|x|}} \to m$?

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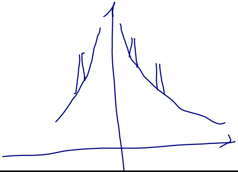
The answer is no, since we have the example below. However, if we remove a set which is "thin" at 0, it is true. (Arsove and Huber)



To prove the conjecture of Alexander and Currier, one of the preparations we need is the following theorem of Taliaferro.

Theorem(Taliaferro)

Let $u(x)$ be a C^2 positive solution to

$$0 \leq -\Delta u(x) \leq e^{2u(x)}, x \in B_1(0)\backslash\{0\} \subset \mathbb{R}^2$$

Then either u has C^1 extension to the origin or

$$\lim_{r \to 0^+} \frac{u(x)}{\log \frac{1}{|x|}} = m$$

for some finite positive number m.

To prove the above theorem, we need several lemmas.

Lemma

Suppose v is harmonic in $B_1(0)\backslash\{0\}$ and

$$\int_{|x|<\varepsilon} |v(x)|dx = o(\varepsilon), \text{ as } \varepsilon \to 0^+. \tag{4}$$

Then there is some β such that $v(x) - \beta \log \frac{1}{|x|}$ has harmonic extension to $B_1(0)$.

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Then there is some β such that $v(x) - \beta \log \frac{1}{|x|}$ has harmonic extension to $B_1(0)$.

Proof. We let

$$\bar{v}(r) = \frac{1}{2\pi} \int_0^{2\pi} v(r\cos\theta, r\sin\theta)d\theta$$

Since v is harmonic in $B_1(0)$, $\bar{v}(r)$ is also harmonic. So $\bar{v}'' + \frac{1}{r}\bar{v}' = 0$. The solution is $\bar{v}(r) = \beta \log \frac{1}{r} + b$.

Let $\gamma : \mathbb{R} \to [0,1]$ be a C^∞ decreasing function such that $\gamma(s) = 1$ for $s \le \frac{1}{2}$ and $\gamma(s) = 0$ for $s \ge 1$. For $\varepsilon > 0$, define $\psi_\varepsilon(x) = \gamma(|x|/\varepsilon)$. Let $\phi \in C_0^\infty(B_1(0))$ and $\hat{\phi} = \phi(x) - \phi(0)$. Then for small $\varepsilon > 0$,

$$\begin{aligned}\int (v - \beta \log \frac{1}{|x|} - b)\Delta\phi &= \int (v - \beta \log \frac{1}{|x|} - b)[\Delta(\phi\psi_\varepsilon) + \Delta(\phi(1 - \psi_\varepsilon))\\ &= \int (v - \beta \log \frac{1}{|x|} - b)\Delta(\phi\psi_\varepsilon)\\ &= I_1(\varepsilon) + \phi(0) I_2(\varepsilon),\end{aligned}$$

where

$$I_1(\varepsilon) = \int (v - \beta \log \frac{1}{|x|} - b)\Delta(\hat{\phi}\psi_\varepsilon),$$

$$I_2(\varepsilon) = \int (v - \beta \log \frac{1}{|x|} - b)\Delta\psi_\varepsilon.$$

From (4) and $\sup_{|x|<\varepsilon} |\Delta(\hat{\phi}\psi_\varepsilon)| = O(1/\varepsilon)$, so $I_1(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Also, since ψ_ε is radial, so $I_2(\varepsilon) \equiv 0$. So letting $\varepsilon \to 0^+$, we complete the proof.

Theorem 1 (Taliaferro)

Let $u(x) \geq 0$ be a C^2 solution to $\Delta u + f(x) = 0$ in $B_1(0)\backslash\{0\}$ and $f : B_1(0)\backslash\{0\} \to [0, \infty)$ is a continuous function. Then $f, u \in L^1(B_{\frac{1}{2}}(0))$ and for some $m \geq 0$ and some harmonic function $u_h : B_{\frac{1}{2}}(0) \to \mathbb{R}$ we have

$$u(x) = u_s(x) + u_\alpha(x) + u_h(x), x \in B_{\frac{1}{2}}(0)\backslash\{0\},$$

where $u_s(x) = m \log \frac{1}{|x|}$ and $u_\alpha(x) = \frac{1}{2\pi} \int_{|y|<\frac{1}{2}} \log \frac{1}{|x-y|} f(y)dy$.

Proof. Averaging u on ∂B_r, we get $\bar{u}(r)$, which satisfies

$$r\bar{u}'(r) = \frac{1}{2}\bar{u}'(\frac{1}{2}) + \frac{1}{2\pi}\int_{r<|x|<\frac{1}{2}} f(x)dx \text{ for } 0 < r \leq \frac{1}{2}.$$

Thus $f \in L^1(B_{\frac{1}{2}}(0))$, otherwise there is some r_0 such that when $0 < r \leq r_0$, $r\bar{u}'(r) \geq 1$. Then $\bar{u}'(r) \geq \frac{1}{r}$, which implies $\bar{u}(r) \to -\infty$ as $r \to 0^+$. Then we assume

$$\lim_{r\to 0^+}(\frac{1}{2}\bar{u}'(\frac{1}{2}) + \frac{1}{2\pi}\int_{r<|x|<\frac{1}{2}} f(x)dx) = -m \leq 0.$$

Then $r\bar{u}'(r) = -m + o(1)$ and $\frac{\bar{u}(r)}{\log\frac{1}{r}} = m + o(1)$, which implies

$$\int_{|x|<\varepsilon} udx = O(\varepsilon^2 \log\frac{1}{\varepsilon}), \text{ as } \varepsilon \to 0^+$$

which implies $u \in L^1(B_{r_1}(0))$.

Now we define $u_\alpha : \mathbb{R}^2 \to \mathbb{R}$ by

$$u_\alpha(x) = \frac{1}{2\pi} \int_{|y|<\frac{1}{2}} \log \frac{1}{|x-y|} f(y)dy.$$

Then $u_\alpha \in C^1(\mathbb{R}^2 \backslash \{0\}) \cap L^1(B_{r_1}(0))$ and

$$-\Delta u_\alpha = f \text{ in } \mathscr{D}'(B_{r_1}(0)).$$

Also, using the

$$\frac{1}{2\pi r} \int_{|x|=r} \log|x-x_0|dS = g(x_0) = \begin{cases} \log|x_0|, & |x_0| > r \\ \log r, & |x_0| \le r \end{cases}$$

it is straightforward to show the average of u_α on $|x| = r$, $\bar{u}_\alpha(r) = o(\log \frac{1}{r})$ as $r \to 0^+$.

This is because for fixed M large, we can choose r small such that

$$\begin{aligned}
\bar{u}_\alpha(r) &= \frac{1}{2\pi r}\int_{|x|=r} u_\alpha(x)dS \\
&= \frac{1}{2\pi}\int_{|y|<\frac{1}{2}} f(y)dy \cdot \frac{1}{2\pi r}\int_{|x|=r} \log\frac{1}{|x-y|}dx \\
&= -\frac{1}{2\pi}\int_{|y|<\frac{1}{2}} f(y)g(y)dy \\
&= -\frac{1}{2\pi}\int_{|y|<r} f(y)(\log r)dy - \frac{1}{2\pi}\int_{B_{r^{\frac{1}{M}}}\setminus B_r} f(y)\log|y|dy \\
&\quad -\frac{1}{2\pi}\int_{B_{\frac{1}{2}}\setminus B_{r^{\frac{1}{M}}}} f(y)\log|y|dy \\
&= o(\log r) + \frac{1}{M}O(\log r).
\end{aligned}$$

So

$$\int_{|x|<\varepsilon}|u-u_\alpha|dx \leq \int_{|x|<\varepsilon} udx+\int_{|x|<\varepsilon} u_\alpha dx = O(\varepsilon^2\log\frac{1}{\varepsilon}), \text{ as } \varepsilon\to 0^+.$$

So, by the lemma before, we have $u-u_\alpha-\beta\log\frac{1}{|x|}=u_h$ in $B_{r_1}(0)\backslash\{0\}$ some some constant β and some harmonic function u_h. It is easy to check $\beta=m$. Then we finish the proof of Theorem 1.

So

$$\int_{|x|<\varepsilon} |u-u_\alpha| dx \leq \int_{|x|<\varepsilon} u dx + \int_{|x|<\varepsilon} u_\alpha dx = O(\varepsilon^2 \log \frac{1}{\varepsilon}), \text{ as } \varepsilon \to 0^+.$$

So, by the lemma before, we have $u - u_\alpha - \beta \log \frac{1}{|x|} = u_h$ in $B_{r_1}(0)\backslash\{0\}$ some some constant β and some harmonic function u_h. It is easy to check $\beta = m$. Then we finish the proof of Theorem 1.

Theorem 2 (Taliaferro)

Let $u(x)$ be a C^2 positive solution of $0 \leq -\Delta u \leq e^{2u}$ in $B_1(0)\backslash\{0\} \subset \mathbb{R}^2$. Then $u(x) = O(\log \frac{1}{|x|})$ as $|x| \to 0^+$.

Proof. From Theorem 1 of Taliferro, we know

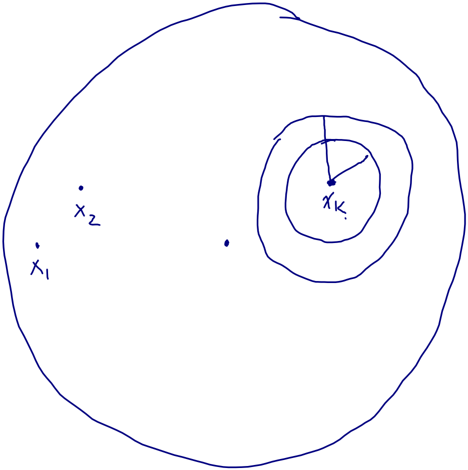
$$u(x) = m \log \frac{1}{|x|} + u_\alpha(x) + u_h(x), x \in B_{\frac{1}{2}}(0)$$

where u_h is harmonic and

$$u_\alpha(x) = \frac{1}{2\pi} \int_{|y|<\frac{1}{2}} \log \frac{1}{|x-y|}(-\Delta u(y))dy.$$

Now, for contradiction, there exists a sequence $\{x_k\}_{k=1}^{\infty} \subset B_{\frac{1}{2}}(0) \backslash \{0\}$ such that $|x_k| \to 0$ as $k \to \infty$, and

$$\lim_{k\to\infty} \frac{u(x_k)}{\log \frac{1}{|x_k|}} = \infty.$$


x_2
x_1
x_K

For $|x-x_k|<\frac{|x_k|}{4}$,

$$\int_{|y-x_k|>|x_k|/2}\log\frac{1}{|x-y|}(-\Delta u(y))dy\le(\log\frac{4}{|x_k|})\int_{\Omega}-\Delta u(y)dy$$

then

$$u(x)\le C\log\frac{1}{|x_k|}+\frac{1}{2\pi}\int_{|y-x_k|<\frac{|x_k|}{2}}\log\frac{1}{|x-y|}(-\Delta u(y))dy \quad (5)$$

for $|x-x_k|<\frac{|x_k|}{4}$, where $C>0$ does not depend on k or x. Then we must have

$$\frac{1}{\log\frac{1}{|x_k|}}\int_{|y-x_k|<\frac{|x_k|}{2}}\log\frac{1}{|x_k-y|}(-\Delta u(y))dy\to\infty,\ \text{as } k\to\infty. \quad (6)$$

Since $-\Delta u\in L^1$, we have

$$\int_{|y-x_k|<\frac{|x_k|}{\ldots}}-\Delta u(y)dy\to 0 \text{ as } k\to\infty. \quad (7)$$

For each integer $k > 0$, define $f_k : B_2 \to [0, \infty)$ by $f_k = -r_k^2 \Delta u(x_k + r_k \xi)$, where $r_k = |x_k|/4$.
Letting $y = x_k + r_k \zeta$, from (5)(6)(7), and $-\Delta u \leq e^{2u}$, we have

$$\int_{B_2} f_k(\zeta) d\zeta \to 0, \text{ as } k \to \infty, \tag{8}$$

$$\frac{1}{M_k} \int_{B_2} (\log \frac{4}{|\zeta|}) f_k(\zeta) d\zeta \to \infty, \text{ as } k \to \infty, \tag{9}$$

$$f_k(\xi) \leq exp(2M_k + \frac{1}{\pi} \int_{B_2} (\log \frac{4}{|\xi - \zeta|})) f_k(\zeta) d\zeta) \text{ for } \xi \in B_1, \tag{10}$$

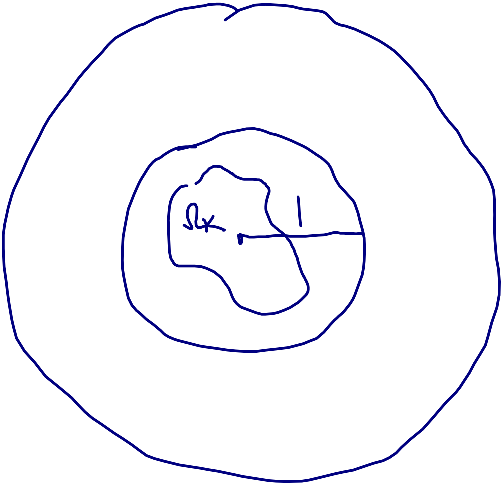
where $M_k = C \log \frac{1}{|x_k|}$ and C is a positive constant that does not depend on k or ξ.

Let $\Omega_k = \{\xi \in B_1 : u_k(\xi) > M_k\}$, where

$$u_k(\xi) = \frac{1}{2\pi} \int_{B_2} \log \frac{4}{|\xi - \zeta|} f_k(\zeta) d\zeta.$$

Then letting $p_k = \pi/2 \int_{B_2} f_k(\zeta) d\zeta$, it follows from (10) that

$$\begin{aligned}
&\int_{\Omega_k} f_k(\xi)^{p_k} d\xi \\
\leq &\int_{B_2} e^{4p_k u_k(\xi)} d\xi \\
\leq &\int_{B_2} exp(4p_k \frac{\|f_k(\zeta)\|_{L^1(B_2)}}{2\pi} \int_{B_2} \log \frac{4}{|\xi - \zeta|} \frac{f_k(\zeta)}{\|f_k(\zeta)\|_{L^1(B_2)}} d\zeta) d\xi \\
\leq &\int_{B_2} \int_{B_2} \frac{4}{|\xi - \zeta|} \frac{f_k(\zeta)}{\|f_k(\zeta)\|_{L^1(B_2)}} d\zeta d\xi \\
\leq &16\pi.
\end{aligned}$$


Ωx
1

Thus by (8) and Holder's inequality we have

$$\limsup_{k\to\infty}\int_{\Omega_k}\log\frac{4}{|\zeta|}f_k(\zeta)d\zeta<\infty.$$

We define $g_k : B_1 \to [0,\infty)$ by

$$g_k(\xi)=\begin{cases} f_k(\xi), & \xi\in B_1\backslash\Omega_k \\ 0, & \xi\in\Omega_k \end{cases}$$

then from (9)(8) we see

$$\frac{1}{M_k}\int_{B_1}(\log\frac{4}{|\zeta|})g_k(\zeta)d\zeta\to\infty, \text{as } k\to\infty. \tag{11}$$

By (9)(10), we have

$$\int_{B_1}g_k(\zeta)d\zeta\to 0, \text{as } k\to\infty. \tag{12}$$

And

$$g_k(\xi) \le e^{4M_k} \tag{13}$$

in B_1.

For fixed k, think of $g_k(\zeta)$ as the density of a distribution of mass in B_1 satisfying (11)(12)(13). By moving small pieces of the mass closer to 0 in a way that the new density does not violate (13), the total mass $\int_{B_1} g_k(\zeta)d\zeta$ will not change but $\frac{1}{M_k}\int_{B_1}(\log\frac{4}{|\zeta|})g_k(\zeta)d\zeta$ will increase. Thus for some $\rho_k \in (0,1)$ the functions

$$g_k(\zeta) = \begin{cases} e^{4M_k}, & |\zeta| < \rho_k \\ 0, & \rho_k < |\zeta| < 1 \end{cases}$$

satisfying (11)(12)(13), which is impossible because $M_k \to \infty$ as $k \to \infty$. This contradiction proves the theorem.

At last, we will prove the theorem of Taliaferro mention at the beginning of this section. From Theorem 2 of Taliaferro, we know $u(x) = O(\log \frac{1}{2|x|})$, then

$$0 \leq -\Delta u(x) \leq \frac{1}{(2|x|)^C}, x \in B_{\frac{1}{2}}(0) \backslash \{0\}. \tag{14}$$

Lemma (Taliaferro)

$$u_\alpha = \frac{1}{2\pi} \int_{B_{\frac{1}{2}}(0)} \log \frac{1}{|x-y|} (-\Delta u(y)) dy = o(\log \frac{1}{|x|}), \text{as } |x| \to 0^+.$$

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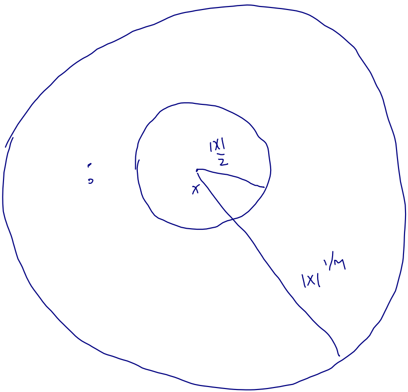
Proof. Let $\varepsilon > 0$ and $M = \frac{2}{\varepsilon}\int_{B_{\frac{1}{2}}} -\Delta u(y)dy + 1$. Fix $|x|$ small and positive we have $u_\alpha(x) = \frac{1}{2\pi}(I(x) + J(x))$.


|x|
2
x
|x| 1/M

Here

$$\begin{aligned}
I(x) : &= \int_{|y-x|>\frac{|x|}{2}} (\log \frac{1}{|x-y|})(-\Delta u(y))dy \\
&= \int_{|x|/2<|y-x|<|x|^{1/M}} (\log \frac{1}{|x-y|})(-\Delta u(y))dy \\
&+ \int_{|y-x|>|x|^{1/M}} \log \frac{1}{|x-y|}(-\Delta u(y))dy \\
&\leq (\log \frac{2}{|x|}) \int_{|y-x|<|x|^{1/M}} -\Delta u(y)dy + \frac{1}{M}(\log \frac{1}{|x|}) \int_{B_{\frac{1}{2}}} -\Delta u(y)dy \\
&\leq \varepsilon \log \frac{1}{|x|},
\end{aligned}$$

and where $J(x) := \int_{|y-x|<|x|/2} \log \frac{1}{|x-y|}(-\Delta u(y))dy.$

From (14), we know

$$0 \leq -\Delta u \leq \frac{1}{|x|^C}, x, y \in B_{\frac{1}{2}} \backslash \{0\} \text{ and } |y-x| < \frac{|x|}{2}. \qquad (15)$$

Let $r(x)^2 = (1/\pi)E(x)|x|^C$, where

$$E(x) := \int_{|y-x|<\frac{|x|}{2}} -\Delta u(y) dy \to 0 \text{ as } x \to 0.$$

Since

$$\int_{|y-x|<r(x)} \frac{dy}{|x|^C} = \frac{\pi r(x)^2}{|x|^C} = \int_{|y-x|<|x|/2} -\Delta u(y) dy,$$

it follows from (15) that

$$\begin{aligned}
J(x) &\le \frac{1}{|x|^C}\int_{|y-x|<r(x)}(\log\frac{1}{|x-y|})dy\\
&= \frac{1}{|x|^C}\int_{|\zeta|<r(x)}\log\frac{1}{|\zeta|}d\zeta = \frac{2\pi}{|x|^C}(\frac{r(x)^2}{2}\log\frac{1}{r(x)}+\frac{r(x)^2}{2})\\
&= O(E(x)\log\frac{1}{E(x)|x|^C}) = o(\log\frac{1}{|x|}),\ \text{as } |x|\to 0^+.
\end{aligned}$$

Then we proved the lemma.
By the above lemma, Taliaferro's theorem is true when $m>0$. When $m=0$, we see $-\Delta u(x) = O(|x|^{-\frac{1}{2}})$. Thus u_α and hence u is bounded in $B_{\frac{1}{2}}$. It follows that $-\Delta u$ is bounded in Ω. Therefore N and hence u has C^1 extension to 0. We completes the proof of Taliaferro's theorem.

Ball model of hyperbolic space

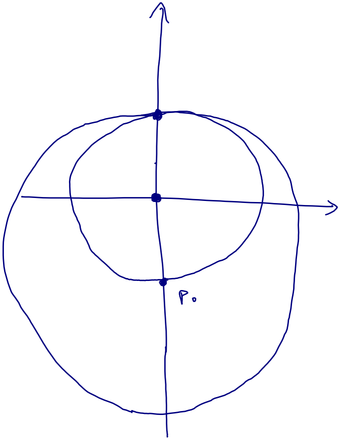
Let B^{n+1} be the unit ball of $\mathbb{R}^{n+1}$. Using the coordinate $(x_1, \cdots, x_{n+1})$, we introduce the following conformal metric $g_{B^{n+1}}$ on B^{n+1}.

$$g_{B^{n+1}} = \frac{4}{(1 - \sum_{i=1}^{n+1} x_i^2)^2}(dx_1^2 + \cdots + dx_{n+1}^2).$$

We prove that $(B^{n+1}, g_{B^{n+1}})$ is isometric to $(\mathbb{H}^{n+1}, g_{\mathbb{H}^{n+1}})$. For this, letting $p_0 = (0, \cdots, 0, -1)$ we define

$$f : B^{n+1} \to \mathbb{R}^{n+1}_+$$
$$p \mapsto \frac{2(p - p_0)}{|p - p_0|^2} + p_0,$$

it is straightforward to check the map is diffeomorphism.


P₀

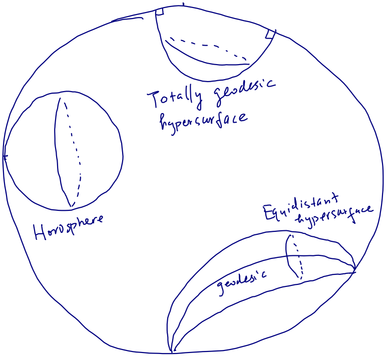
Due to easy calculations, if $v = (v_1, \cdots, v_{n+1}) \in T_pB^{n+1}$, we let

$$g_{\mathbb{R}^{n+1}}(df_p(v), df_p(v)) = \frac{4}{|p-p_0|^4} \sum_{i=1}^{n+1} v_i^2.$$

Notice that $f_{n+1}(p) = \frac{2(x_{n+1}(p)+1)}{|p-p_0|^2} - 1 = \frac{1-|p|^2}{|p-p_0|^2}$ (this is because $2x_{n+1} + 2 - (x_1^2 + \cdots + x_n^2 + (x_{n+1}+1)^2) = 1 - (x_1^2 + \cdots + x_{n+1}^2)$). Therefore

$$\frac{g_{\mathbb{R}^{n+1}}(df_p(v), df_p(v))}{f_{n+1}(p)^2} = \frac{4}{(1-|p|^2)^2} \sum_{i=1}^{n+1} v_i^2 = g_{B^{n+1}}(v, v).$$

So $(B^{n+1}, g_{B^{n+1}})$ gives another model for $\mathbb{H}^{n+1}$.



Definition

If $\phi : M \to B^{n+1}$ is an immersion, we define the asymptotical boundary at infinity $\partial_\infty M$ as $\{p \in \partial B^{n+1}; p \text{ is a limit point of } \phi(M)\}$.

Now we will work with B^{n+1}. Similarly we can prove

Theorem

The isometry group of B^{n+1} (regarded as the unit ball of $\mathbb{R}^{n+1}$) consists of the restrictions of conformal maps of $\mathbb{R}^{n+1}$ which map B^{n+1} onto itself.

Now we can prove

Theorem

For $n \geq 2$, the isometry group of $\mathbb{H}^{n+1} \simeq B^{n+1}$ is isomorphic to the conformal transformation group of $\mathbb{S}^n$.

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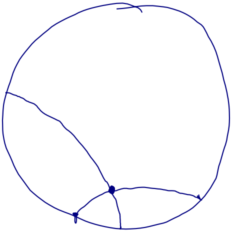
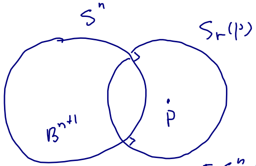
Proof. We identify $\mathbb{S}^n = \partial B^{n+1}$. Give any $\xi \in Iso(B^{n+1})$, ξ is a conformal transformation of $\mathbb{R}^{n+1}$, which maps B^{n+1} onto B^{n+1}. Then $\xi : \mathbb{S}^n \to \mathbb{S}^n$ is a conformal diffeomorphism.

If $\xi : \mathbb{S}^n \to \mathbb{S}^n$ is identity. Then ξ must maps any geodesic of B^{n+1} to itself, since the ends are fixed points of ξ. Then we know $\xi : B^{n+1} \to B^{n+1}$ is identity since any point of B^{n+1} is the intersection of two geodesics.

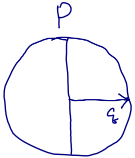


On the other hand, we prove that a conformal diffeomorphism $\xi : \mathbb{S}^n \to \mathbb{S}^n$ can be generated by the following two kinds of elements. Each one can be extended to a conformal diffeomorphism $\xi : B^{n+1} \to B^{n+1}$.



$p \in \mathbb{R}^{n+1} \setminus \overline{B^{n+1}}$，$S_r(p)$ 与 S^n 垂直相交

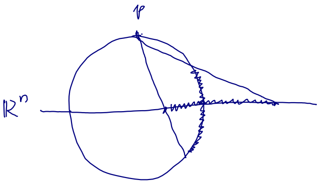
T_p 是关于 $S_r(p)$ 的反演



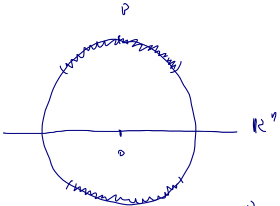
$\xi \in S^n$，P 是垂直于 ξ 的平面

R_ξ 是关于 P 的反射

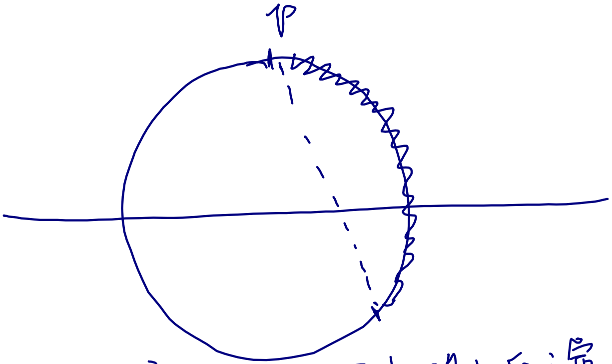
For this choose $p \in \mathbb{S}^n$ and consider stereographic projection $\pi_p : \mathbb{S}^n \to \mathbb{R}^n$. The conformal diffeomorphism on $\mathbb{S}^n$ has one-to-one correspondence with the global conformal map on $\mathbb{R}^n$, which is generated by isometries, dilatations and inversions. These three kinds can be generated by T_p and R_q.



$\mathbb{R}^n$上的反演可以由S^n上的反演或反射生成



$\mathbb{R}^n$上的伸缩可由S^n上反演生成



$\mathbb{R}^n$上的镜面反射可由S^n上反演或反射生成，因此等距也可以.

From Erlangen program, there is a correspondence between the Riemannian geometry of $\mathbb{H}^{n+1}$ and the conformal geometry of $\mathbb{S}^n$. This is often mentioned as "AdS-CFT correspondence" in theoretical physics.

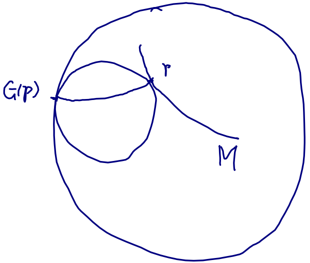
In our case, what we are interested in is the submanifold in $\mathbb{H}^{n+1}$, we will show how to reduce the problem to a conformal geometry problem on $\mathbb{S}^n$.

Key idea

A "nice" hypersurface M in $B^{n+1} = \mathbb{H}^{n+1}$ have a conformal metric $g_h = e^{2u} G^* g_{\mathbb{S}^n}$ on M, which is called "horospherical metric". And the principal curvatures of the hypersurfaces correspond to the eigenvalue of the Schouten tensor of g_h.

Hyperbolic Gauss map

Suppose $f : (M^n, g_M) \to (B^{n+1}, g_{B^{n+1}})$ is an isometric immersion. For $p \in M$, suppose U is a neighborhood of p in M. On U we choose a smooth normal vector field η. Suppose $\gamma_\eta(t), t \geq 0$ is a geodesic ray with $\gamma_\eta(0) = p$ and $\gamma'_\eta(0) = \eta$, suppose $\gamma_\eta(+\infty) = q \in \mathbb{S}^n = \partial B^{n+1}$. Then we define the hyperbolic Gauss map $G : M^n \to \mathbb{S}^n$ such that $G(p) = p$. Notice that the definition of Hyperbolic Gauss map depend on the choice of the normal vector field. An equivalent definition is that, for given p and η, there is a unit horosphere $\mathcal{H}_p$ which is tangential to M at p and have η as interior normal vector field, then $G(p)$ is defined as the infinity of $\mathcal{H}_p$.



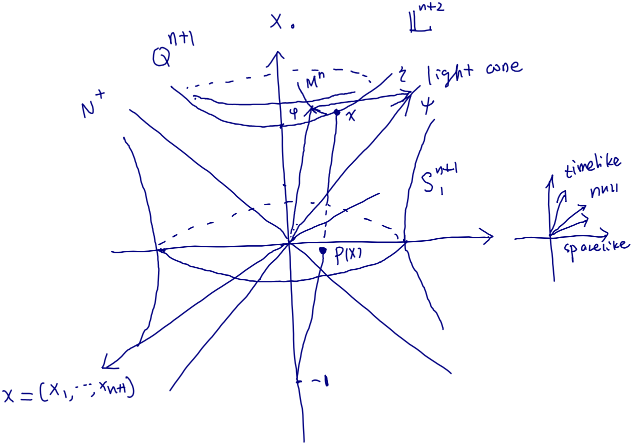
Hyperbolic Gauss map

Lorentzian space and hyperboloid

There is a third model for $\mathbb{H}^{n+1}$, for which we will introduce the Lorentzian space $\mathbb{L}^{n+2}$. The Lorentzian space $\mathbb{L}^{n+2} \simeq (\mathbb{R}^{n+2}, g_{\mathbb{L}^{n+2}})$. In the standard coordinate $(x_0, x_1, \cdots, x_n)$, $g_{\mathbb{L}^{n+2}} = -dx_0^2 + dx_1^2 + \cdots + dx_n^2$. We define hyperboloid

$$Q^{n+1} = \{(x_0, \cdots, x_{n+1}); x_0 > 0, -x_0^2 + x_1^2 \cdots + x_{n+1}^2 = -1\}.$$

Although $g_{\mathbb{L}^{n+2}}$ is not a Riemannian metric, its restriction on Q^{n+1}, $g_{Q^{n+1}}$ is Riemannian. We will prove that $(Q^{n+1}, g_{Q^{n+1}})$ is isometric to $(B^{n+1}, g_{B^{n+1}})$. So it is another model for $\mathbb{H}^{n+1}$. The idea of the proof is the following and the details are left to you.


X_0
$\mathbb{L}^{n+2}$
Q^{n+1}
M^n
light cone
N^+
φ
x
ψ
S_1^{n+1}
timelike
null
spacelike
P(x)
$x = (x_1, \cdots, x_{n+1})$
-1

Maps

Now we will consider $Q^{n+1} \subset \mathbb{L}^{n+2}$. Denote

$$N^+ = \{x; x_0 \geq 0, -x_0^2 + x_1^2 \cdots + x_{n+1}^2 = 0\};$$
$$\mathbb{S}_1^n = \{x; -x_0^2 + x_1^2 \cdots + x_{n+1}^2 = 1\}.$$

Suppose $\phi : M^n \to Q^{n+1} \subset \mathbb{L}^{n+2}$ is isometric immersion and suppose Σ^n is orientable and η is a global unit normal vector field of M^n in Q^{n+1}. We may also consider η as a map

$$\eta : M \to \mathbb{L}^{n+2}$$

since $\mathbb{L}^{n+2}$ is a linear space. We define

$$\psi = \phi + \eta : M^n \to \mathbb{L}^{n+2}.$$

Theorem

ψ maps M into light cone N^+. So it is called "light cone map"

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Proof. The Levi Civita connection of $\mathbb{L}^{n+1}$ is the same as the usual derivative of Euclidean space. Given $p \in M$, suppose $c(t)$ is a curve of Q^{n+1} which satisfies $c(0) = \phi(p)$. We know

$$0 = \frac{d}{dt} g_{\mathbb{L}^{n+2}}(c(t), c(t))|_{t=0} = 2g_{\mathbb{L}^{n+2}}(\dot{c}(t), \phi(p)).$$

So $\phi(p)$ is normal to Q^{n+1}. Since η is tangential to Q^{n+1}, we have $g_{\mathbb{L}^{n+2}}(\phi(p), \eta(p)) = 0$. So

$$g_{\mathbb{L}^{n+2}}(\psi(p), \psi(p)) = g_{\mathbb{L}^{n+2}}(\phi(p), \phi(p)) + g_{\mathbb{L}^{n+2}}(\eta(p), \eta(p)) = -1 + 1 = 0.$$

$\psi \in N^+$ since the tangential space of Q^{n+1} has no intersection with N^-.

For $p \in M$, we choose $\{e_i\}_{i=1}^n$ as an orthonormal frame at p. We also consider $\{e_i\}$ as an orthonormal frame at $\phi(p)$. Then $e_i \perp \phi$ and $e_i \perp \eta$. We can choose $\{e_i\}$ specially as the principal direction of M, which means ${}^{Q^{n+1}}\nabla_{e_i}\eta = k_i e_i$.

Notice

$$g_{\mathbb{L}^{n+2}}(d\eta(e_i), \phi) = -g_{\mathbb{L}^{n+2}}(\eta, d\phi(e_i)) = -g_{\mathbb{L}^{n+2}}(\eta, e_i) = 0,$$
$$g_{\mathbb{L}^{n+2}}(d\eta(e_i), \eta) = \frac{1}{2} e_i g_{\mathbb{L}^{n+2}}(\eta, \eta) = 0.$$

We see that

$$d\eta(e_i) = {}^{Q^{n+1}}\nabla_{e_i}\eta = k_i e_i.$$

Then

$$d\psi(e_i) = (1 + k_i)e_i.$$

Lemma

If $k_i > -1$, ψ is an immersion and $\psi(M)$ is codimension 2. The two normal vectors are ϕ and η. And the second fundamental form is given by

$$(\overrightarrow{A}_{\psi})_{ij} = (1+k_i)(-\phi + k_i\eta)g_{ij}.$$

Lemma

If $k_i > -1$, ψ is an immersion and $\psi(M)$ is codimension 2. The two normal vectors are ϕ and η. And the second fundamental form is given by

$$(\overrightarrow{A}_{\psi})_{ij} = (1 + k_i)(-\phi + k_i\eta)g_{ij}.$$

Proof. Since $k_i > -1$,

$$d\psi(e_i) = d\phi(e_i) + d\eta(e_i) = (1 + k_i)e_i \neq 0.$$

So ψ is immersion and it is obvious that $\psi(M)$ is codimension 2. We claim ϕ and η span the normal bundle of $\psi(M)$, since

$$g_{\mathbb{L}^{n+2}}(d\psi(e_i), \phi) = g_{\mathbb{L}^{n+2}}(e_i, \phi) + g_{\mathbb{L}^{n+2}}(d\eta(e_i), \phi) = 0,$$
$$g_{\mathbb{L}^{n+2}}(d\psi(e_i), \eta) = g_{\mathbb{L}^{n+2}}(e_i, \eta) + g_{\mathbb{L}^{n+2}}(d\eta(e_i), \eta) = 0.$$

Since

$$Hess_{d\psi^2}\psi(e_i, e_j) = e_j e_i \psi - ({}^{d\psi^2}\nabla_{e_j} e_i)\psi,$$

and $-{}^{d\psi^2}\nabla_{e_j} e_i = \sum_i \beta_i e_i$ we have

$$\begin{aligned} g_{\mathbb{L}^{n+2}}(\overrightarrow{A}_\psi(e_i, e_j), \phi) &= -g_{\mathbb{L}^{n+2}}(e_j e_i \psi - ({}^{d\psi^2}\nabla_{e_j} e_i)\psi, \phi) \\ &= -g_{\mathbb{L}^{n+2}}(e_j e_i \phi + e_j e_i \eta, \phi) \\ &= g_{\mathbb{L}^{n+2}}(e_i, e_j) + k_i g_{\mathbb{L}^{n+2}}(e_i, e_j) = (1 + k_i)\delta_{ij}. \end{aligned}$$

Similarly, we have

$$g_{\mathbb{L}^{n+2}}(\overrightarrow{A}_\psi(e_i, e_j), \eta) = (1 + k_i)k_j \delta_{ij}.$$

So

$$\begin{aligned} \overrightarrow{A}_\psi(e_i, e_j) &= (-(1 + k_i)\phi + (1 + k_i)k_j \eta)\delta_{ij} \\ &= (1 + k_i)(-\phi + k_i \eta) g_{ij}. \end{aligned}$$

Horospherical metric

We define the horospherical metric on M by $g_h = d\psi^2$, that is, the pulled back metric by the map ψ, which is also Riemannian if $k_i > -1$. If fact we have

$$g_h(e_i, e_j) = g_{\mathbb{L}^{n+2}}(d\psi(e_i), d\psi(e_j)) = (1+k_i)(1+k_j)\delta_{ij} = (1+k_i)^2\delta_{ij}.$$

Then we know

$$\overrightarrow{A}_\psi = (1 + k_i)^{-1}(-\phi + k_i\eta)g_h$$

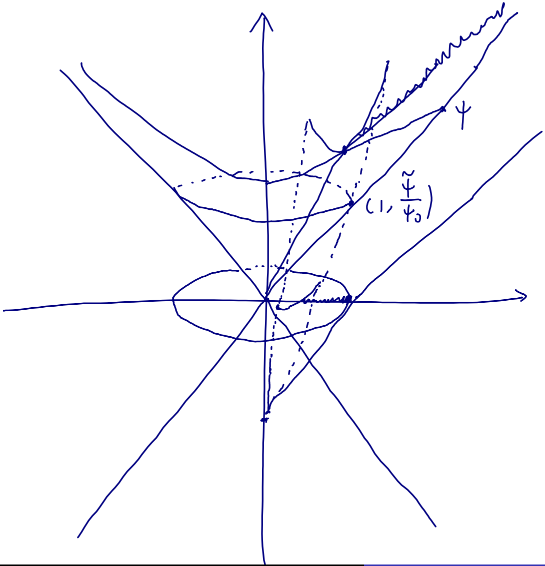
Light cone map and hyperbolic Gauss map

Since we have a way to identify Q^{n+1} with the unit disk $x_1^2 + \cdots + x_{n+1}^2 < 1, x_0 = 0$ by the stereographic projection map P. We have

Theorem

Let $\psi = (\psi_0, \psi_1, \cdots, \psi_{n+1}) = (\psi_0, \tilde{\psi})$, then $G \circ P \circ \phi = \frac{\tilde{\psi}}{\psi_0}$.

Proof. See the graph below.


ψ
$(1, \frac{\tilde{\psi}}{\psi_0})$

Definition

We define $\rho = \log \psi_0$ and call it horospherical support function.

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Theorem

$$g_h = e^{2\rho}(dG)^* g_{\mathbb{S}^n}.$$

Proof. Since $g_{\mathbb{L}^{n+2}}(\psi, \psi) = 0$ and $g_{\mathbb{L}^{n+2}}(\psi, e_i) = 0$

$$dG^2(e_i, e_j) = g_{\mathbb{L}^{n+2}}(d\frac{\psi}{\psi_0}(e_i), d\frac{\psi}{\psi_0}(e_j)) = \frac{1}{\psi_0^2} d\psi^2(e_i, e_j).$$

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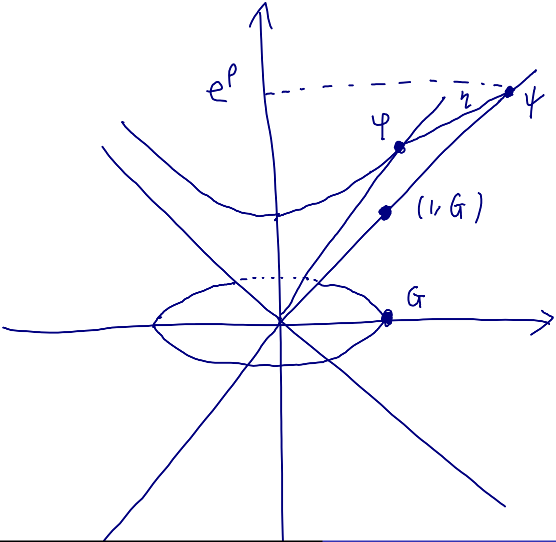
It is also easy to prove that : if $k_i > -1$, then G is local diffeomorphism.

Now if $\phi : M \to Q^{n+1}$ is an immersion, then we can define map η, ψ, G and function ρ. What is important for us is that ϕ can be expressed in terms of G and ρ.

Theorem (Representation formula)

Suppose $\phi : M \to Q^{n+1}$ is an immersion with $k_i > -1$ uniformly. Let ρ be the corresponding horospherical support function.Then

$$\phi = \frac{e^\rho}{2}(1 + e^{-2\rho}(1 + \|^{g_{\mathbb{S}^n}} \nabla\rho\|^2_{g_{\mathbb{S}^n}}))(1, x) + e^{-\rho}(0, -x +^{g_{\mathbb{S}^n}} \nabla\rho)$$


e^ρ
φ
η
ψ
$(1, G)$
G

Proof. First we prove that

$$\phi = \frac{1}{n}\Delta^{g_h}\psi + \frac{R(g_h) + n(n-1)}{2n(n-1)}\psi, \tag{16}$$

where R is the scalar curvature of g_h and $\Delta^{g_h}\psi$ is the Laplacian of ψ with respect to g_h. We do the calculation at p, using the orthonormal frame $\{e_i\}$, which are the principal directions with principal curvatures k_i . We let $v_i = \frac{e_i}{1+k_i}$. So $g_h(v_i, v_j) = \delta_{ij}$.
From

$$\overrightarrow{A}_\psi(v_i, v_j) = (-\frac{1}{1+k_i}\phi + \frac{k_i}{1+k_i}\eta)\delta_{ij}.$$

If $K(x, y)$ denotes the sectional curvature of ψ, from Gauss equation

$$K(v_i, v_j) =< \overrightarrow{A}_\psi(v_i, v_i), \overrightarrow{A}_\psi(v_j, v_j) > -\|\overrightarrow{A}_\psi(v_i, v_j)\|^2$$
$$= 1 - \frac{1}{1+k_i} - \frac{1}{1+k_j}.$$

So we know that

$$R(g_h) = n(n-1) - 2(n-1)\sum_{i=1}^{n}\frac{1}{1+k_i} \tag{17}$$

Also the mean curvature of ψ is

$$\begin{aligned}
\overrightarrow{H}_\psi &= \sum_{i=1}^{n}\overrightarrow{A}_\psi(v_i,v_i) = \sum_{i=1}^{n}(-\frac{1}{1+k_i}\phi + \frac{k_i}{1+k_i}\eta) \\
&= \sum_{i=1}^{n}(-\phi + \frac{k_i}{1+k_i}\phi + \frac{k_i}{1+k_i}\eta) \\
&= -n\phi + \sum_i \frac{k_i}{1+k_i}\psi. \\
&= -n\phi + (n - \sum_i \frac{1}{1+k_i})\psi.
\end{aligned}$$

Now we recall that

$$\Delta^{g_h}\psi = -\overrightarrow{H}_\psi = n\phi - (n - \sum_{i=1}^{n}\frac{1}{1+k_i})\psi.$$

So we have

$$\phi = \frac{1}{n}\Delta^{g_h}\psi + (1 - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+k_i})\psi.$$

From (17), we can prove (16).

Now we compute $\Delta^{g_h}\psi$. First because $g_h = e^{2\rho} g_{\mathbb{S}^n}$

$$\Delta^{g_h}\psi = e^{-2\rho}(\Delta^{g_{\mathbb{S}^n}}\psi + (n-2)g_{\mathbb{S}^n}({}^{g_{\mathbb{S}^n}}\nabla\rho, {}^{g_{\mathbb{S}^n}}\nabla\psi)). \qquad (18)$$

Let $\{u_0, \cdots, u_{n+1}\}$ be the canonical basis of $\mathbb{L}^{n+2}$ and write $\psi = (\psi_0, \cdots, \psi_{n+1})$. Let $(1, x) = (1, x_1, \cdots, x_n) \in \mathbb{S}^n$.

$$\begin{aligned}\Delta^{g_{\mathbb{S}^n}}\psi &= (\Delta^{g_{\mathbb{S}^n}} e^\rho, \Delta^{g_{\mathbb{S}^n}}(e^\rho)x + e^\rho\Delta^{g_{\mathbb{S}^n}}x + 2e^\rho\sum_{k=1}^{n+1} g_{\mathbb{S}^n}({}^{g_{\mathbb{S}^n}}\nabla x_k, {}^{g_{\mathbb{S}^n}}\nabla\rho)u_k \\ &= (e^{-\rho}\Delta^{g_{\mathbb{S}^n}}(e^\rho))\psi + (0, e^\rho\Delta^{g_{\mathbb{S}^n}}x + 2e^\rho\,\cdot{}^{g_{\mathbb{S}^n}}\nabla\rho).\end{aligned}$$

Now $\Delta^{g_{\mathbb{S}^n}}x = -nx$ and $\Delta^{g_{\mathbb{S}^n}}e^\rho = e^\rho(\Delta^{g_{\mathbb{S}^n}}\rho + \|{}^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}})$, we have

$$\Delta^{g_{\mathbb{S}^n}}\psi = (\Delta^{g_{\mathbb{S}^n}}\rho + \|{}^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}})\psi + e^\rho(0, -nx + 2{}^{g_{\mathbb{S}^n}}\nabla\rho). \quad (19)$$

Notice that, for any $Z \in S(TM(\mathbb{S}^n))$:

$$g_{\mathbb{S}^n}({}^{g_{\mathbb{S}^n}}\nabla\psi, Z) := \sum_{k=0}^{n+1} g_{\mathbb{S}^n}({}^{g_{\mathbb{S}^n}}\nabla\psi_k, Z)u_k.$$

Then if we choose f_i as the orthonormal basis of $\mathbb{S}^n$,

$$\begin{aligned} g_{\mathbb{S}^n}({}^{g_{\mathbb{S}^n}}\nabla\psi, {}^{g_{\mathbb{S}^n}}\nabla\rho) =& \sum_{i=1}^{n} f_i(\rho)f_i(\psi) \\ =& \sum_{i=1}^{n} f_i(\rho)(e^{\rho}(f_i(\rho))(1,x) + e^{\rho}(0,f_i)) \\ =& \|{}^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}}\psi + e^{\rho}(0, {}^{g_{\mathbb{S}^n}}\nabla\rho). \end{aligned} \tag{20}$$

From (18)(19)(20) we have that

$$\Delta^{g_h}\psi = e^{-2\rho}(\Delta^{g_{\mathbb{S}^n}}\rho + (n-1)\|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}})\psi + ne^{-\rho}(0, -x + {}^{g_{\mathbb{S}^n}}\nabla\rho).$$

On the other hand it is well known that if $g_h = e^{2\rho}g_{\mathbb{S}^n}$ we have

$$\Delta^{g_{\mathbb{S}^n}}\rho + \frac{n-2}{2}\|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}} - \frac{n}{2} + \frac{e^{2\rho}}{2(n-1)}R(g_h) = 0.$$

So

$$\Delta^{g_h}\psi = (-\frac{R(g_h)}{2(n-1)} + \frac{ne^{-2\rho}}{2}(1 + \|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}}))\psi + ne^{-\rho}(0, -x + {}^{g_{\mathbb{S}^n}}\nabla\rho)).$$

Plugging this into (16) we get the result.

Now we consider the tensor on M,

$$P = -^{g_{\mathbb{S}^n}}\nabla^2\rho + d\rho \otimes d\rho - \frac{1}{2}(\|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}} - 1)g_{\mathbb{S}^n}.$$

When $n \geq 3$, this is actually the Schouten tensor of g_h. That is

$$Sch = \frac{1}{n-2}(Ric - \frac{R}{2(n-1)}g).$$

The curvature has decomposition

$$R_{ijkl} = W_{ijkl} + g_{ik}Sch_{jl} - g_{jk}Sch_{il} - g_{il}Sch_{jk} + g_{jl}Sch_{ik}.$$

Theorem

Let e_i be the principal direction. Then e_i are also the eigendirection of P. Let λ_i be the corresponding eigenvalue with respect to g_h. Then we have $\lambda_i = \frac{1}{2} - \frac{1}{1+k_i}$.

Proof. Choose f_i as the geodesic frame at p in the metric $g_{\mathbb{S}^n}$,

$$P_{ij} = -^{g_{\mathbb{S}^n}}\nabla^2_{ij}\rho + \rho_i\rho_j - \frac{1}{2}(\|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}} - 1)(g_{\mathbb{S}^n})_{ij}$$

where $(g_{\mathbb{S}^n})_{ij} = \delta_{ij}$. Now we calculate

$$^{g_{\mathbb{S}^n}}\nabla^2_{ij}\rho = f_if_j\rho - (^{g_{\mathbb{S}^n}}\nabla_{f_i}f_j)\rho = f_if_j\rho,$$
$$^{g_h}\nabla^2_{ij}\rho = f_if_j\rho - (^{g_h}\nabla_{f_i}f_j)\rho = f_if_j\rho - (\Gamma_h)^k_{ij}\rho_k.$$

Since $g_h = e^{2\rho}g_{\mathbb{S}^n}$

$$(\Gamma_h)^k_{ij} = \frac{1}{2}e^{-2\rho}\delta_{kl}(\partial_i(e^{2\rho}(g_{\mathbb{S}^2})_{jl}) + \partial_j(e^{2\rho}(g_{\mathbb{S}^2})_{il}) - \partial_l(e^{2\rho}(g_{\mathbb{S}^2})_{ij}))$$
$$= \rho_i\delta_{kj} + \rho_j\delta_{ki} - \rho_k\delta_{ij}.$$

So we have

$$(^{g_h}\nabla^2\rho)_{ij} = (^{g_{\mathbb{S}^n}}\nabla^2\rho)_{ij} - 2\rho_i\rho_j + \rho_k^2\delta_{ij}.$$

$$P_{ij} = -({}^{g_h}\nabla^2\rho)_{ij} - \rho_i\rho_j + \frac{1}{2}(\|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}} + 1)(g_{\mathbb{S}^n})_{ij}.$$

Now we choose normal coordinate undar g_h. As $\rho = \log\psi_0$, we see that

$$\begin{aligned}({}^{g_h}\nabla^2\rho)_{ij} &= \partial^2_{ij}\rho = \partial_i(\frac{\partial_j\psi_0}{\psi_0}) = \frac{\psi_0\partial^2_{ij}\psi_0 - \partial_i\psi_0\partial_j\psi_0}{\psi_0^2} \\ &= \psi_0^{-1}\partial^2_{ij}\psi_0 - \rho_i\rho_j.\end{aligned}$$

So

$$P_{ij} = -\psi_0^{-1}({}^{g_h}\nabla^2\psi_0)_{ij} + \frac{1}{2}e^{-2\rho}(\|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}} + 1)(g_h)_{ij}.$$

From the expression of second fundamental form, we know that $-(\nabla^2\psi_0)_{ij} = (-\frac{1}{1+k_i}\phi_0 + \frac{k_i}{1+k_i}\eta_0)(g_h)_{ij}$. So

$$P_{ij} = \psi_0^{-1}(-\frac{1}{1+k_i}\phi_0 + \frac{k_i}{1+k_i}\eta_0 + \frac{1}{2}e^{-\rho}(\|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}} + 1))(g_h)_{ij}.$$

Now we know that, from Representation formula,

$$\phi_0 = \frac{e^\rho}{2}(1 + e^{-2\rho}(1 + \|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}})).$$

So

$$\frac{1}{2}e^{-\rho}(1 + \|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}}) = \phi_0 - \frac{e^\rho}{2}.$$

$$P_{ij} = \psi_0^{-1}(\frac{k_i}{1+k_i}\psi_0 - \frac{e^\rho}{2})(g_h)_{ij} = (\frac{1}{2} - \frac{1}{1+k_i})(g_h)_{ij}.$$

Now we let $n \geq 2$. From the decomposition R_{ijkl}, for locally conformal flat manifold, we know

$$R_{ijij} = Sch_{ii}g_{jj} + Sch_{jj}g_{ii} = \lambda_i + \lambda_j, i \neq j.$$

Lemma

Suppose $\phi : M^n \to Q^{n+1}$ is nonnegatively curved immersed hypersurface. Then ϕ is strictly convex $k_i > 0$ and the horospherical metric is also nonnegatively curved.

Proof. From $k_i k_j \geq 1$ for all $i, j = 1, \cdots, n$, we know $k_i > 0$ for all $i = 1, \cdots, n$. Since $\lambda_i = \frac{1}{2} - \frac{1}{1+k_i}$, we have

$$Sec_{g_h}(\frac{e_i}{1+k_i}, \frac{e_j}{1+k_j}) = \lambda_i + \lambda_j = \frac{k_i k_j - 1}{(1+k_i)(1+k_j)} \geq 0.$$

For a local conformal flat manifold, a conformal immersion

$$F : M^n \to \mathbb{S}^n$$

is called a developing map.

Theorem

If $n \geq 3$, and M^n is simply connected, locally conformal flat manifold, then there is a developing map from M^n to $\mathbb{S}^n$. And the map is unique up to the conformal transformation of $\mathbb{S}^n$.

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Theorem(Schoen-Yau)

Let (M^n, g) be a complete Riemannian manifold with $R_g \geq 0$. If $F : M^n \to \mathbb{S}^n$ is conformal, F is injective.

Theorem(Shunhui Zhu)

Let $\Omega \subset \mathbb{S}^n$ be a domain of standard $\mathbb{S}^n$. If there is a complete conformal metric g on Ω such that $Ric_g \geq 0$, then the Hausdorff of $\partial\Omega = \mathbb{S}^n \backslash \Omega$ is 0. If fact $\partial\Omega$ consists of at most two points.

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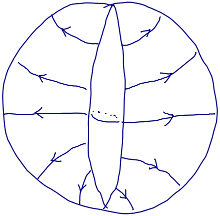
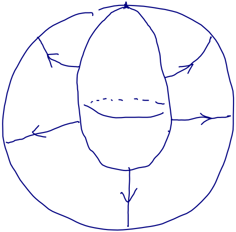
Theorem(Injective hyperbolic Gauss map)

For $n \geq 3$, let $\phi : M^n \to \mathbb{H}^{n+1}$ be a complete, nonnegatively curved, immersed hypersurface. Then the hyperbolic Gauss map is a developing map from (M^n, g_h) to $(\mathbb{S}^n, g_{\mathbb{S}^n})$ and is injective. Moreover, the Hausdorff dimension of $\partial G(M) = \mathbb{S}^n \backslash G(M)$ is zero. In fact $\partial G(M)$ consists of at most two points.

Proof.
Since (M^n, g_M) is nonnegatively curved, we know g_h is also nonnegatively curved. Then from the theorem of Schoen and Yau, we know, the hyperbolic Gauss map G as the developing map, is injective. Then from Shunhui Zhu's theorem, we know $\partial G(M)$ consists of at most two points. Then we end the proof.



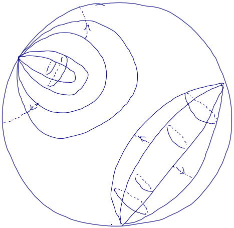


Let $n \geq 2$. Let Ω be a domain of $\mathbb{S}^n$. Let $\phi : \Omega \to Q^{n+1}$ be given by

$$\phi = \frac{e^{\rho}}{2}(1 + e^{-2\rho}(1 + \|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}}))(1, x) + e^{-\rho}(0, -x +^{g_{\mathbb{S}^n}} \nabla\rho)$$

whose hyperbolic Gauss map is $G(x) = x$. We can define a normal flow of $\phi(M)$, by

$$\phi_t = \frac{e^{\rho+t}}{2}(1+e^{-2(\rho+t)}(1+\|^{g_{\mathbb{S}^n}}\nabla\rho\|^2_{g_{\mathbb{S}^n}}))(1, x)+e^{-(\rho+t)}(0, -x+^{\mathbb{S}^n}\nabla\rho).$$



The properties of the normal flow.

- The normal flow preserves the hyperbolic Gauss map and boundary at infinity.
- The normal flow preserves the nonnegative sectional curvature condition.
- When ϕ is a horospere, ϕ_t is also a horosphere. When ϕ is an equidistant surface, ϕ_t is also an equidistant surface. If $\phi(M)$ is contained in a horosphere $\mathcal{H}$, then ϕ_t is also contained in $\mathcal{H}_t$.

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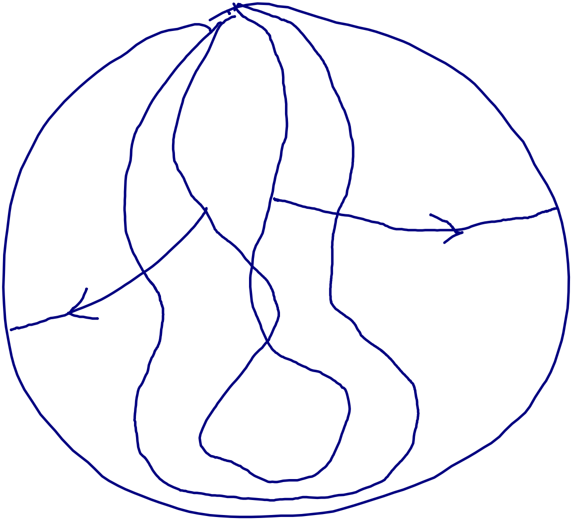
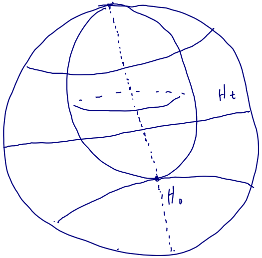
Theorem(Bonini,Qing,Espinar)

For $n \geq 2$, let $\phi : M^n \to \mathbb{H}^{n+1}$ be a complete hypersurface with injective Gauss map and $k_i > -1$ uniformly. Suppose that the asymptotic boundary $\partial_\infty \phi(M)$ at infinity of the hypersurface is a disjoint union of smooth closed submanifolds in $\mathbb{S}^n$. Then, along the normal flow, $\phi_t(M)$ eventually becomes embedded.



If $\partial_\infty \phi$ consists of two points, then we can use the normal flow ϕ_t, such that ϕ_t is embedded for t large. So from the classification theorem of Alexander and Currier, we know ϕ_t is an equidistant hypersurface. So ϕ is also an equidistant hypersurface.
If $\partial_\infty \phi$ contains only one point, then we can generalize Epstein's result to conclude that it is embedded.



HYPOTHESIS F. *Suppose there exists a smooth foliation of* $\mathbf{H}^3$ *by planes* $\{H_t : t \in \mathbf{R}\}$ *such that*

(a) *For all* t, $M \cap H_t$ *is a compact set.*

(b) H_0 *is a local support plane at a point where* M *is locally strictly convex.*

We have the following extension of Van Heijenoort's theorem.

THEOREM 4.1 [8]. *Let* M *be a connected topological surface and* ψ *an immersion of* M *into* $\mathbf{H}^3$ *such that:*

(1) ψ *is locally one-to-one;*

(2) *every point* p *in* M *has a neighborhood* N *such that* $\psi(N)$ *is part of the boundary of a compact convex set;*

(3) $\psi(M)$ *is locally strictly convex at some point (as in Hypothesis* F*);*

(4) *the metric on* M *defined by pulling back the hyperbolic metric via* ψ *is complete; and*

(5) *Hypothesis* F *holds.*

Then $\psi(M)$ *is the boundary of a convex set in* $\mathbf{H}^3$.

$n = 2$ case

$$K_{g_h} = \frac{K_{g_M}}{(1+k_1)(1+k_2)} \geq 0.$$

Since $\lambda_i = \frac{1}{2} - \frac{1}{1+k_i}$, we know $\lambda_i \in (-\frac{1}{2}, \frac{1}{2})$. Then $-\frac{1}{2}g_h < P < \frac{1}{2}g_h$. Let $\pi : \tilde{M}^2 \to M^2$ be the universal covering map. Let $\tilde{\phi} = \phi \circ \pi : \tilde{M} \to Q^3$, $\tilde{G} = G \circ \pi : \tilde{M} \to \mathbb{S}^2, \tilde{g}_h = \pi^* g_h$ and $K_{\tilde{g}_h} = K_{g_h} \circ \pi \geq 0$.

$$\tilde{P} = P \circ \pi = -^{\tilde{G}^* g_{\mathbb{S}^2}} \nabla^2 \tilde{\rho} + d\tilde{\rho} \otimes d\tilde{\rho} - \frac{1}{2}(\|d\tilde{\rho}\|^2_{\tilde{G}^* g_{\mathbb{S}^2}} - 1)\tilde{G}^* g_{\mathbb{S}^2}$$

where $\tilde{\rho} = \rho \circ \pi$ and $-\frac{1}{2}\tilde{g}_h < \tilde{P} < \frac{1}{2}\tilde{g}_h$. From Huber's theorem, we know $\tilde{M}$ is parabolic. And since it is simply connected, we know it is biholomorphic to $\mathbb{R}^2$.

Theorem ($n = 2$, flat cases)

Let ϕ be an isometric immersion flat Euclidean plane to Q^3. Then ϕ is either a covering map of an equidistant surface about a geodesic in Q^3 or it is an embedded horosphere.

Proof. First from

$$K_{g_h} = \frac{K_g}{(1+k_1)(1+k_2)} = 0,$$

$(\mathbb{R}^2, g_h)$ is isometric to the Euclidean plane. Let $z = (x, y)$ be the Euclidean coordinate for $(\mathbb{R}^2, g_h)$ so that

$$|dz|^2 = g_h = e^{2\rho} G^* g_{\mathbb{S}^2}.$$

As

$$P_{ij} = -({}^{g_h}\nabla^2\rho)_{ij} - \rho_i\rho_j + \frac{1}{2}(|d\rho|^2_{g_h} + e^{-2\rho})g_h,$$

in local coordinate (x, y) we have

$$P_{xx} = -\partial_x^2\rho + \frac{1}{2}(\rho_y^2 - \rho_x^2) + \frac{1}{2}e^{-2\rho},$$
$$P_{yy} = -\partial_y^2\rho + \frac{1}{2}(\rho_x^2 - \rho_y^2) + \frac{1}{2}e^{-2\rho},$$
$$P_{xy} = P_{yx} = -\partial^2_{x,y}\rho - \rho_x\rho_y.$$

We have

$$P_{xx} + P_{yy} = -\Delta_h\rho + e^{-2\rho} = 0.$$

And

$$P_{xxy} = -\partial^3_{xxy}\rho + \rho_y\rho_{yy} - \rho_x\rho_{xy} - e^{-\rho}\rho_y$$
$$P_{xyx} = -\partial^3_{xyx}\rho - \rho_{xx}\rho_y - \rho_x\rho_{yx}.$$

So

$$P_{xxy} - P_{xyx} = \rho_y \Delta_h \rho - e^{-\rho} \rho_y = 0.$$

Also

$$\begin{aligned} P_{xxx} &= -\partial^3_{xxx}\rho + \rho_y \rho_{yx} - \rho_x \rho_{xx} - e^{-2\rho}\rho_x, \\ P_{xyy} &= -\partial^3_{xyy}\rho - \rho_{xy}\rho_y - \rho_x \rho_{yy}. \end{aligned}$$

So

$$\begin{aligned} P_{xxx} + P_{xyy} &= -\partial_x(\Delta_h \rho) - \rho_x(\Delta_h \rho) - e^{-2\rho}\rho_x \\ &= 2e^{-2\rho}\rho_x - e^{-2\rho}\rho_x - e^{-2\rho}\rho_x \\ &= 0. \end{aligned}$$

So $P_{xx} - \sqrt{-1}P_{xy}$ or $P_{yy} + \sqrt{-1}P_{xy}$ is a holomorphic function on $(\mathbb{R}^2, z)$.

As

$$-\frac{1}{2}|dz|^2 < P < \frac{1}{2}|dz|^2$$

we know that this holomorphic function is bounded on the whole $\mathbb{C}$. So it is a constant. So the principal curvatures of the intrinsic surface is also constant. From the classification of isoparametric surfaces in hyperbolic 3-space, we have proved the theorem.

Now we consider a complete, noncompact, nonnegatively curved, nonflat, immersed $\phi : M \to Q^3$. First $(\tilde{M}, \tilde{g}_h)$ is also nonnegatively curved and complete.
Let's assume $z = x + \sqrt{-1}y$ is the complex coordinate on $\tilde{M}$. We have

$$\tilde{g}_h = e^{2\tilde{\rho}_0}|dz|^2 = e^{2\tilde{\rho}}\tilde{G}^* g_{\mathbb{S}^2}.$$

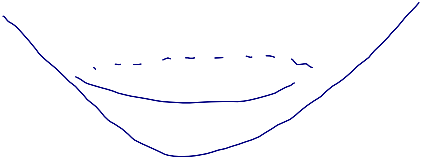
Rewrite the relation as

$$|dz|^2 = e^{2(\tilde{\rho}-\tilde{\rho}_0)}\tilde{G}^* g_{\mathbb{S}^2} = e^{2\rho_0}\tilde{G}^* g_{\mathbb{S}^2}$$

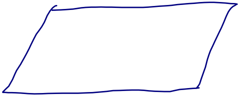
for $\rho_0 = \tilde{\rho} - \tilde{\rho}_0$ and consider the symmetric 2-tensor

$$P_0 = -\nabla^2_{\tilde{G}^* g_{\mathbb{S}^2}}\rho_0 + d\rho_0 \otimes d\rho_0 - \frac{1}{2}(|d\rho_0|^2_{\tilde{G}^* g_{\mathbb{S}^2}} - 1)\tilde{G}^* g_{\mathbb{S}^2}.$$

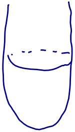
It is easy to know that P_0 also gives rise to a holomorphic function as $-\Delta_{\mathbb{R}^2}\rho_0 + e^{-2\rho_0} = 0$.



若 $\tilde{\rho}_0 = -m\log r + o(\log r)$ $\quad 0 \le m \le 1$



$m = 0$



$m = 1$

Lemma

Let

$$\tilde{P} = -^{\tilde{G}^* g_{\mathbb{S}^2}} \nabla^2 \tilde{\rho} + d\tilde{\rho} \otimes d\tilde{\rho} - \frac{1}{2}(|d\tilde{\rho}|^2_{\tilde{G}^* g_{\mathbb{S}^2}} - 1)\tilde{G}^* g_{\mathbb{S}^2}.$$

In (x, y) coordinates we have

$$(P_0)_{11} = \partial_x^2 \tilde{\rho}_0 - \frac{1}{2}((\partial_x \tilde{\rho}_0)^2 - (\partial_y \tilde{\rho}_0)^2) + \tilde{P}_{11},$$
$$(P_0)_{22} = \partial_y^2 \tilde{\rho}_0 - \frac{1}{2}((\partial_y \tilde{\rho}_0)^2 - (\partial_x \tilde{\rho}_0)^2) + \tilde{P}_{22},$$
$$(P_0)_{12} = (P_0)_{21} = \partial_x \partial_y \tilde{\rho}_0 - (\partial_x \tilde{\rho}_0)(\partial_y \tilde{\rho}_0) + \tilde{P}_{12}.$$

Now we use Taliaferro's result to prove that

$$\tilde{\rho}_0 = -m \log |z| + o(\log |z|) \text{ as } |z| \to \infty$$

for some $m \in (0, 1]$.
We first take an inversion. Let $\tilde{z} = \frac{z}{|z|^2}$ be the inversion map. Then

$$|dz|^2 = \frac{1}{|\tilde{z}|^4} |d\tilde{z}|^2 \text{ and } \tilde{g}_h = e^{2\tilde{\rho}_0} |dz|^2 = e^{2(\tilde{\rho}_0 - 2\log|\tilde{z}|)} |d\tilde{z}|^2 = e^{2v} |d\tilde{z}|^2$$

where

$$v(\tilde{z}) = \tilde{\rho}_0 (\frac{z}{|z|^2}) - 2 \log |\tilde{z}|. \quad (21)$$

First we assume v has a lower bound on $B_1(0)$. Since $0 \le -\tilde{\Delta} v = K_{g_h} e^{2v} \le e^{2v}$ according to Taliaferro's Theorem, we get

$$v(\tilde{z}) = m_1 \log \frac{1}{|\tilde{z}|} + o(\log \frac{1}{|\tilde{z}|}) \text{ as } \tilde{z} \to 0$$

for some $m_1 > 0$. Next we claim that $m_1 \ge 1$ since the metric $g = e^{2v}|d\tilde{z}|^2$ is complete and noncompact at the origin. Assume otherwise $m_1 < 1$. Then let $m_2 \in (m_1, 1)$ and r_s be sufficiently small so that $v < m_2 \log \frac{1}{|\tilde{z}|}$ for all $0 < |\tilde{z}| < r_s$, which implies

$$\exp(v) < |\tilde{z}|^{-m_2} \text{ for all } 0 < |\tilde{z}| < r_s,$$

and

$$\int_0^{r_s} \exp(v(t,0))dt < \int_0^{r_s} t^{-m_2} dt < \infty.$$

This is a contradiction.

Therefore, from (21) we know

$$u(z) = (2 - m_1) \log \frac{1}{|z|} + o(\log \frac{1}{|z|}) \text{ as } |z| \to \infty$$

where $m = 2 - m_1 \leq 1$.
To see $m > 0$ when g is nonnegatively curved and nonflat, we recall

$$-\Delta u = K_g e^{2u} \geq 0 \text{ in } \mathbb{R}^2.$$

For $0 < r_2 < r_1$, we have that

$$r_2 \bar{u}'(r_2) = r_1 \bar{u}'(r_1) + \frac{1}{2\pi} \int_{r_2 < |z| < r_1} K_g e^{2u},$$

where

$$\bar{u}(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r \cos\theta, r \sin\theta) d\theta.$$

Then

$$|\bar{u}'(r)| \leq \frac{1}{2\pi} \int_0^{2\pi} |\nabla u(r\cos\theta, r\sin\theta)| d\theta$$

and therefore

$$\lim_{r_2 \to 0^+} r_2 \bar{u}'(r_2) = 0.$$

So we have

$$r_1 \bar{u}'(r_1) = -\frac{1}{2\pi} \int_{|z|<r_1} K_g e^{2u}.$$

Now, from $u = m \log \frac{1}{|z|} + o(\log |z|)$ as $|z| \to \infty$, it follows that

$$\lim_{r_1 \to \infty} r_1 \bar{u}'(r_1) = -m = -\int_{\mathbb{R}^2} K_g e^{2u} < 0,$$

and $K_g \geq 0$ and is not identially 0.

Now we prove that v has lower bound in $B_1(0)$. We first observe that e^{-v} is a subharmonic function on $(\mathbb{R}^2, e^{2v}|d\tilde{z}|^2)$, that is

$$\Delta_g e^{-v} = e^{-v}|\nabla_g v|^2 - e^{-v}\Delta_g v = e^{-v}(|\nabla_g v|^2 + \tilde{K}_g) \geq 0.$$

Theorem (Li and Schoen)

Suppose that M^n is a Riemannian manifold with $Ric \geq -(n-1)k$. Let $x_0 \in M$ and r is a number that if M has no boundary, r is less than half of the diameter of M; if $\partial M \neq \phi$, $r < \frac{1}{5}d(x_0, \partial M)$. Then for a nonnegative subharmonic v_1, $\tau \in (0, \frac{1}{2})$

$$\sup_{B_{(1-\tau)r}(x_0)} v_1^2 \leq \tau^{-C(1+\sqrt{k}r)} \frac{1}{vol(B_r(x_0))} \int_{B_r(x_0)} v_1^2 dvol.$$

Theorem (Croke and Karcher)

If (M^2, g) is complete and nonnegatively curved, then there exists a constant $C(M)$ such that, for $r \leq 1$,

$$vol_g(B_r(x)) \geq C(M)r^2.$$

Thus, the fact that the conformal factor v is bounded from below follows from Li Schoen's inequality and and the fact that $vol_{|d\tilde{z}|^2}(B_r(x_0))$ is bounded.
Then we proved that

$$\tilde{\rho}_0 = -m \log|z| + o(\log|z|) \text{ as } |z| \to \infty$$

for some $m \in (0, 1]$.

Then from

$$-\frac{1}{2}\tilde{g}_h < \tilde{P} < \frac{1}{2}\tilde{g}_h,$$

we know

$$|\tilde{P}| \leq Ce^{2\tilde{\rho}_0} \leq \frac{C}{(1+|z|^2)^{\frac{m}{2}}}, \tag{22}$$

and hence $|\tilde{P}|$ belongs to $L^p(\mathbb{R}^2)$ for some large $p > 1$. From interior estimates to the Gaussian curvature equation

$$-\Delta\tilde{\rho}_0 = K_{\tilde{g}_h} e^{2\tilde{\rho}_0}$$

and the Schauder and L^p estimates, we have

$$\begin{cases} R^{2-\frac{2}{p}}\|\partial^2\tilde{\rho}_0\|_{L^p(B_R(0))} & \leq C(\|\tilde{\rho}_0\|_{C^0(B_{2R}(0))} + R^{2-\frac{2}{p}}\|K_{\tilde{g}_h}e^{2\tilde{\rho}_0}\|_{L^p(B_{2R}(0} \\ r\|\partial\tilde{\rho}_0\|_{C^0(B_r(z))} & \leq C(\|\tilde{\rho}_0\|_{C^0(B_{2r}(z))} + r^2\|K_{\tilde{g}_h}e^{2\tilde{\rho}_0}\|_{C^0(B_{2r}(z))}). \end{cases} \tag{23}$$

From (22) and the first inequality of (23) as $R \to \infty$, we have $\partial^2 \tilde{\rho}_0 \in L^p(\mathbb{R}^2)$ for p large since $K_{\tilde{g}_h}$ is bounded. Now in the second one of (23), we choose $r = (1+|z|^2)^{\frac{m}{4}} < \frac{1}{2}|z|$ at least when $|z| > 2\sqrt{2}$, we get

$$|\partial \tilde{\rho}_0(z)| \leq \frac{C}{(1+|z|^2)^{\frac{m}{4}}}(\log|z| + C),$$

which implies that $|\partial \tilde{\rho}_0(z)^2| \in L^p(\mathbb{R}^2)$ for p large. So $|P_0| \in L^p(\mathbb{R}^2)$.

With $P_0 = 0$, we know

$$\rho_0 = \log(C[(x - x_0)^2 + (y - y_0)^2] + \frac{1}{4C}).$$

Now we use Representation formula to generate a immersed surface in $\mathbb{H}^3$ with intrinsic Gaussian curvature 0. We know it is a horosphere.
So we know $\tilde{G} : \tilde{M} \to \mathbb{S}^2$ is injective and $\tilde{G}(\tilde{M}) = \mathbb{S}^2 \backslash \{q\}$. And the boundary at infinity of $\phi(\tilde{M})$ is a single point. So the covering map is a diffeomorphism.

$$\tilde{\rho} = \rho_0 + \tilde{\rho}_0 = (2 - m) \log |z| + o(\log |z|), \ m \in (0, 1]$$

as $z \to +\infty$. So $\tilde{\rho}(\tilde{G}^{-1}(\xi)) \to +\infty$ as $\xi \to q$. From an argument of Bonini, Espinar and Qing, we know $\partial_\infty \phi = \{q\}$. From the result of Epstein, we know that ϕ is embedding.

Thank you for your attention!