超曲面上的几何曲率流及应用 - Lecture 1

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Overall theme

Understanding the behavior of hypersurface curvature flows and their applications in the proof of geometric inequalities.

Topics covered:

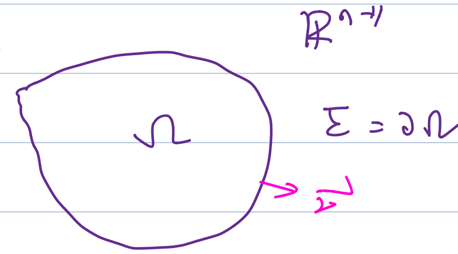
1. Curvature flows, Isoperimetric and Alexandrov-Fenchel inequalities in Euclidean space.
2. Quermassintegrals, Curvature flows, and Alexandrov-Fenchel inequalities in Hyperbolic space
3. Convex geometry in Euclidean space, Geometry of horospherical convex hypersurfaces in Hyperbolic space.
4. Tensor maximum principle, Maximum principle for functions on the frame bundle.

1. Introduction of curvature flow, Huisken's theorem on volume preserving mean curvature flow and Isoperimetric inequality.
2. Quermassintegrals in Euclidean space, McCoy's theorem on Quermassintegral preserving curvature flow and Alexandrov-Fenchel inequalities.
3. Sketch proof of McCoy's theorem.

Hypersurface in $\mathbb{R}^{n+1}$

$X : M^n \longrightarrow \mathbb{R}^{n+1}$ embedding

$\Sigma = X(M^n)$ hypersurface



- Induced metric g
- Second fundamental form

$\forall\, u, v \in T\Sigma,\quad D_u v = \nabla_u v - h(u,v)\vec{\nu}$

$\vec{\nu} : \Sigma \to S^n$

Weingarten operator : $\mathcal{W} = d\vec{\nu} : T_p\Sigma \longrightarrow T_{\nu(p)}S^n = T_p\Sigma$. self-adjoint

$g(\mathcal{W}(u), v) = h(u, v)$

In local coordinates: $g_{ij} = \langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \rangle$

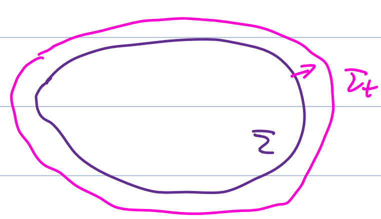
$h_{ij} = -\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \vec{\nu} \rangle$

$(h_i^j) = \sum_k g^{jk} h_{ik}$

principal curvatures $\kappa = (\kappa_1, \kappa_2, \cdots \kappa_n)$

- mean curvature $H = \sum_{i=1}^n \kappa_i$

$\frac{\partial X}{\partial t} = F\vec{\nu}$



$\Rightarrow \frac{d}{dt}|\Sigma_t| = -\int_{\Sigma_t} \langle \vec{H}, \frac{\partial X}{\partial t} \rangle d\mu_t$

$\vec{H} = -H\vec{\nu}$

- Gauss curvature $K = \kappa_1 \cdot \kappa_2 \cdots \kappa_n$
- k-th mean curvature: $\sigma_k = \sum_{i_1 < \cdots < i_k} \kappa_{i_1} \cdots \kappa_{i_k}$

(also, $E_k = \sigma_k / \binom{n}{k}$)

Convex: $\kappa_i > 0$, $i = 1 \ldots n$

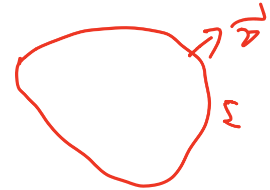
k-convex: $E_1 > 0 \cdots E_k > 0$.

Star-shaped: $\langle X, \vec{\nu} \rangle > 0$

$\Sigma = \{(\theta, r(\theta)),\ \theta \in S^n\}$

$r \in C^\infty(S^n)$

Let $\Sigma_0 \subset \mathbb{R}^{n+1}$ be a smooth hypersurface given by the embedding



$$X_0 : M^n \to \mathbb{R}^{n+1}.$$

we consider a smooth family of embeddings $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ satisfying

unit outer normal

$$\frac{\partial}{\partial t} X = F(x,t)\nu(x,t). \tag{1}$$

Curvature flow: $F = F(\kappa)$ is a smooth function of the principal curvatures of Σ_t.

Examples:

$$\frac{\partial X}{\partial t} = \vec{H}$$

contracting

1. Mean curvature flow $F = -H$ (Huisken 1984, application in topology)
2. Gauss curvature flow $F = -K$ (Firey 1974, fate of rolling stones, application in affine geometry, image analysis)

Expanding

3. Inverse mean curvature flow $F = 1/H$ (Geroch 1973, application in general relativity, geometric inequalities)

constrained

4. Volume preserving mean curvature flow (Huisken, 1987, application in geometric inequalities)

$$\frac{\partial}{\partial t} X = (\phi(t) - H)\vec{\nu}$$

Short time existence (cf. Huisken and Polden 1999).

If $F = F(\kappa)$ satisfies

$$-\frac{\partial F}{\partial \kappa_i} > 0, \quad i = 1, \cdots, n$$

holds everywhere on Σ_0, then the curvature flow has a smooth solution at least on a short time interval $[0, \varepsilon)$.

$$\frac{\partial X}{\partial t} = F \vec{\nu}$$

Evolution equations. Along the curvature flow, we have

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} =& 2Fh_{ij} \\ \Downarrow \quad & \\ \frac{\partial}{\partial t} d\mu_t =& FH d\mu_t \\ \frac{\partial}{\partial t} \nu =& -\nabla F \\ \frac{\partial}{\partial t} h_{ij} =& -\nabla_i \nabla_j F + F(h^2)_{ij}. \end{aligned}$$

$$d\mu_t = \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n$$

Evolution equations

$$\frac{\partial X}{\partial t} = F\vec{\nu}$$

① $\frac{\partial}{\partial t} g_{ij} = 2F h_{ij}$

$$g_{ij} = \langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \rangle$$

$$\Rightarrow \frac{\partial}{\partial t} g_{ij} = \langle \underbrace{\frac{\partial}{\partial x^i}\left(\frac{\partial X}{\partial t}\right)}, \frac{\partial X}{\partial x^j} \rangle + i \leftrightarrow j$$

$$= \langle \cancel{\frac{\partial F}{\partial x^i}\vec{\nu}} + F\frac{\partial \vec{\nu}}{\partial x^i}, \frac{\partial X}{\partial x^j} \rangle + i \leftrightarrow j$$

$$= 2F h_{ij}$$

② $|\nu| = 1 \Longrightarrow \frac{\partial}{\partial t}\vec{\nu} \perp \vec{\nu}$

$$\langle \frac{\partial}{\partial t}\vec{\nu}, \frac{\partial X}{\partial x^i} \rangle = -\langle \vec{\nu}, \frac{\partial}{\partial x^i}\left(\frac{\partial X}{\partial t}\right) \rangle$$

$$= -\frac{\partial F}{\partial x^i}$$

$$\Longrightarrow \frac{\partial}{\partial t}\vec{\nu} = -\nabla F = -\frac{\partial F}{\partial x^i} g^{ij} \frac{\partial X}{\partial x^j}$$

③ $\frac{\partial}{\partial t} h_{ij} = -\frac{\partial}{\partial t} \langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \vec{\nu} \rangle$

$$= -\langle \frac{\partial^2}{\partial x^i \partial x^j}(F\vec{\nu}), \vec{\nu} \rangle - \langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \frac{\partial}{\partial t}\vec{\nu} \rangle$$

$$= -\langle \frac{\partial}{\partial x^i}\left(\partial_j F \vec{\nu} + F \partial_j \vec{\nu}\right), \vec{\nu} \rangle + \langle \partial_i \partial_j X, \partial_k F g^{kl} \partial_l X \rangle$$

$$= -\frac{\partial^2 X}{\partial x^i \partial x^j} F + \Gamma_{ij}^k \frac{\partial F}{\partial x^k} - F \langle \partial_i \partial_j \vec{\nu}, \vec{\nu} \rangle$$

$$= -\nabla_i \nabla_j F + F \langle \partial_i \vec{\nu}, \partial_j \vec{\nu} \rangle$$

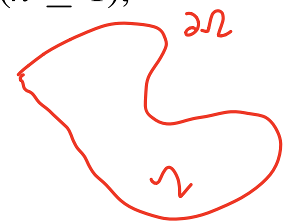
$$= -\nabla_i \nabla_j F + F \underbrace{(h^2)_{ij}} \longrightarrow \sum_k h_i^k h_{kj}$$

④ $\frac{\partial}{\partial t} h_i^j = \frac{\partial}{\partial t}\left(\sum_k h_{ik} g^{kj}\right)$

$$= -\nabla^j \nabla_i F - F (h^2)_i^j$$

We focus on the application of curvature flow in the proof of geometric inequality.

Example: Isoperimetric inequality. For any bounded domain $\Omega \subset \mathbb{R}^{n+1}$ $(n \geq 1)$, there holds



$$\boxed{\mathrm{Area}(\partial\Omega) \geq (n+1)^{\frac{n}{n+1}} \omega_n^{\frac{1}{n+1}} \mathrm{Vol}(\Omega)^{\frac{n}{n+1}}}$$

with equality holds if and only if Ω is a round ball. if $|\Omega| = |B(r)|$

$\Rightarrow |\partial\Omega| \geq |\partial B(r)|$

Volume preserving mean curvature flow

A family of embeddings $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ satisfying

$n=1$: $L^2 \geq 4\pi A$

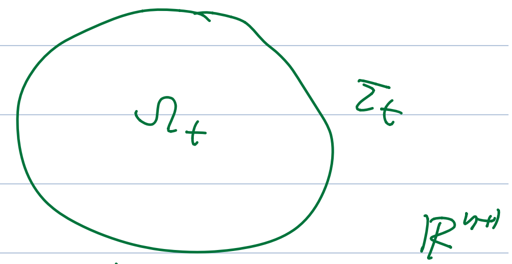
$$\frac{\partial}{\partial t} X(x,t) = \left(\phi(t) - H\right) \nu(x,t) \tag{2}$$

where $\phi(t)$ is chosen to preserve the enclosed volume by $\Sigma_t = X(M,t)$.

$$0 = \frac{d}{dt}|\Omega_t| = \int_{\Sigma_t} \left(\phi(t) - H\right) d\mu_t \Leftrightarrow \quad \phi(t) = \frac{\int_{\Sigma_t} H d\mu_t}{|\Sigma_t|} = \bar{H}$$

$$\frac{d}{dt}|\Sigma_t| = -\int_{\Sigma_t} (H - \bar{H}) H d\mu_t = -\int_{\Sigma_t} (H - \bar{H})^2 \leq 0$$

$$\frac{\partial}{\partial t} X = F \vec{\nu}$$



$$\Rightarrow \quad \frac{d}{dt} |\Omega_t| = \int_{\partial \Omega} F \, d\mu_t$$

proof: $\quad \frac{1}{2} \Delta |X|^2 = (n+1) \quad \left(\because |X|^2 = \sum_{i=1}^{n+1} x_i^2 \right)$

$$\Rightarrow (n+1) |\Omega_t| = \frac{1}{2} \int_{\Omega_t} \Delta |X|^2$$

$$= \frac{1}{2} \int_{\partial \Omega_t} D |X|^2 \cdot \vec{\nu} = \int_{\partial \Omega_t} \langle X, \vec{\nu} \rangle \, d\mu_t$$

$$\Rightarrow (n+1) \frac{d}{dt} |\Omega_t| = \int_{\partial \Omega_t} \left(\langle \partial_t X, \vec{\nu} \rangle + \langle X, \partial_t \vec{\nu} \rangle + \langle X, \vec{\nu} \rangle F H \right) d\mu_t$$

$$= \int_{\partial \Omega_t} (F \underbrace{- \langle X, \nabla F \rangle + \langle X, \vec{\nu} \rangle F H}_{*}) \, d\mu_t$$

note: $\quad \mathrm{div}_{\partial \Omega_t} (F X^T) = \sum_{i=1}^{n} e_i \left(F \langle X, e_i \rangle \right)$

$$= \langle X, \nabla F \rangle + nF + F \langle X, -H \vec{\nu} \rangle$$

$$\Rightarrow (*) = \int_{\partial \Omega_t} \left(-\mathrm{div}_{\partial \Omega_t} (F X^T) + nF \right) = n \int_{\partial \Omega_t} F \, d\mu_t$$

$$\therefore \frac{d}{dt} |\Omega_t| = \int_{\partial \Omega_t} F \, d\mu_t \qquad \square$$

Theorem (Huisken, 1987) $(k_i > 0,\ i = 1 \ldots n)$

If $\Sigma_0 = \partial\Omega_0$ is a convex hypersurface, then the solution $\Sigma_t = X(M, t)$ of (2) exists for all time $t > 0$, Σ_t converges to a sphere $S^n(r)$ as $t \to \infty$.

The isoperimetric ratio

$$\mathcal{I}(\Omega_t) = \frac{|\Sigma_t|}{|\Omega_t|^{\frac{n}{n+1}}}$$

is non-increasing. Applying Huisken's theorem,

$$\mathcal{I}(\Omega_0) \geq \lim_{t\to\infty} \mathcal{I}(\Omega_t) = \mathcal{I}(B(r)) = \omega_n^{\frac{1}{n+1}}(n+1)^{\frac{n}{n+1}}$$

which implies the Isoperimetric inequality (for convex domains).

- F. Schulze 2008. $\partial_t X = -H^k \vec{\nu}$ in $\mathbb{R}^{n+1}$

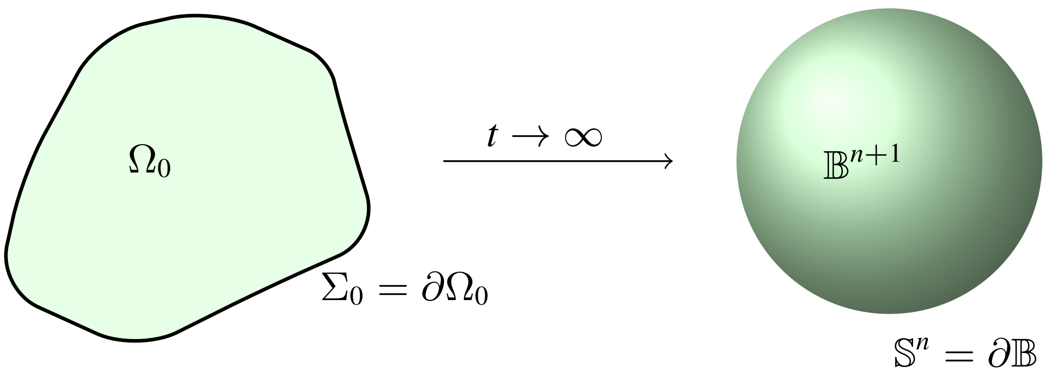
Isoperimetric inequality for bounded Ω

star-shaped ?

The key steps to establish a geometric inequality using curvature flow:

Convex, k-convex

1. Under suitable conditions on the initial data Ω_0, the curvature flow exists and converges to a "model" hypersurface (e.g., round sphere)
2. Find a geometric quantity $\mathcal{I}(\Omega_t)$ (e.g., isoperimetric ratio), which is monotone increasing/decreasing along the flow.
3. Estimate the limit of this geometric quantity $\lim_{t\to\infty} \mathcal{I}(\Omega_t)$.

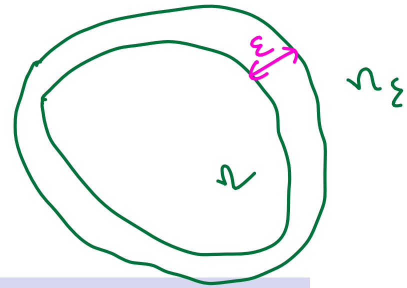


$$\mathcal{I}(\Omega_0) \geq \lim_{t\to\infty} \mathcal{I}(\Omega_t) = \mathcal{I}(Ball).$$

Quermassintegrals

For convex Ω in $\mathbb{R}^{n+1}$ and $\varepsilon > 0$, denote the ε-parallel set of Ω:

$$\Omega_\varepsilon := \Omega + \varepsilon\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1},\ d(x,\Omega) < \varepsilon\}.$$



Steiner formula

The volume of Ω_ε can be expressed as a polynomial of ε of order $n+1$:

$$\mathrm{Vol}(\Omega_\varepsilon) = \frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k} V_k(\Omega)\,\varepsilon^k.$$

$$= \mathrm{Vol}(\Omega) + |\partial\Omega|\,\varepsilon + \frac{n}{2} V_2(\Omega)\,\varepsilon^2 + \cdots + \frac{V_{n+1}}{n+1}\varepsilon^{n+1}$$

Quermassintegrals of Ω are defined as the coefficients of the Steiner formula.

$$V_0(\Omega) = (n+1)|\Omega|$$
$$V_1(\Omega) = |\partial\Omega|$$
$$\vdots$$
$$V_{n+1}(\Omega) = (n+1)|\mathbb{B}| = \omega_n$$

C^2

If $\Sigma = \partial\Omega$ is smooth, then

$$V_k(\Omega) = \int_{\partial\Omega} E_{k-1} d\mu, \quad k = 1, \cdots, n,$$

where

$$E_k = \binom{n}{k}^{-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \kappa_{i_1} \cdots \kappa_{i_k}, \quad k = 1, \cdots, n,$$

is the kth normalized mean curvature of $\partial\Omega$.

Alexandrov (1937) and Fenchel (1936) proved

$k=1, \quad \ell=0$

$\Rightarrow V_1(\Omega) \geq \omega_n^{\frac{1}{n+1}} V_0(\Omega)^{\frac{n}{n+1}}$

area volume

Alexandrov-Fenchel inequalities

For convex bounded domain Ω in $\mathbb{R}^{n+1}$, we have

$$\boxed{V_k(\Omega) \ \geq \ \omega_n^{\frac{k-\ell}{n+1-\ell}} V_\ell(\Omega)^{\frac{n+1-k}{n+1-\ell}}, \quad 0 \le \ell < k \le n}$$

with equality if and only if Ω is a ball, where $\omega_n = |\mathbb{S}^n|$.

Variational formula

If $X : M \times [0, T) \to \mathbb{R}^{n+1}$ satisfies

$$\frac{\partial}{\partial t} X(x,t) = F(x,t)\nu(x,t)$$

then Quermassintegrals have a nice variation equation

$$\boxed{\frac{d}{dt} V_\ell(\Omega_t) = (n+1-\ell) \int_{\Sigma_t} F E_\ell \, d\mu_t, \quad \ell = 0, 1, \cdots, n.}$$

In particular, $\ell = 0, 1$ reduce to the variation formula for volume and area.

Inspired by Huisken's (1987) volume preserving flow, McCoy (2005) introduced the following Quermassintegral preserving flow

$$\frac{\partial}{\partial t} X(x,t) = \left(\phi(t) - \left(\frac{E_k}{E_\ell} \right)^{1/(k-\ell)} \right) \nu(x,t), \quad 0 \le \ell < k \le n \tag{3}$$

with $\phi(t)$ chosen to preserve $V_\ell(\Omega_t)$.

Lem $\frac{\partial X}{\partial t} = F\vec{\nu}$ in $\mathbb{R}^{n+1}$

$$\Rightarrow \frac{d}{dt} V_k(\Omega_t) = (n+1-k)\int_{\partial\Omega_t} F\cdot E_k \, d\mu_t$$

proof: $\because$ $\partial\Omega_t$ is smooth., $\therefore$ $V_k(\Omega_t) = \int_{\partial\Omega_t} E_{k-1}\, d\mu_t$

Recall: $\frac{\partial}{\partial t} h_i^j = -\nabla^j\nabla_i F - F(h^2)_i^j$

$$\Rightarrow \frac{\partial}{\partial t}\sigma_k = \dot{\sigma}_k^{ij}\frac{\partial}{\partial t} h_i^j$$

$$= -\underbrace{\nabla^j(\dot{\sigma}_k^{ij}\nabla_i F)} - F\underbrace{\dot{\sigma}_k^{ij}(h^2)_i^j}_{=\sigma_1\sigma_k-(k+1)\sigma_{k+1}}$$

$$\Rightarrow \frac{d}{dt}\int_{\partial\Omega_t}\sigma_k\, d\mu_t = (k+1)\int_{\partial\Omega_t} F\cdot\sigma_{k+1}\, d\mu_t$$

Note: $E_k = \sigma_k / \binom{n}{k}$ $\Rightarrow$ $\square$

$$\frac{\partial}{\partial t}X(x,t) = \left(\phi(t) - \left(\frac{E_k}{E_\ell}\right)^{1/(k-\ell)}\right)\nu(x,t), \quad 0 \le \ell < k \le n \qquad (3)$$

$$0 = \frac{d}{dt}V_\ell(\Omega_t) = \int_{\partial\Omega_t}\left(\phi(t) - \left(\frac{E_k}{E_\ell}\right)^{\frac{1}{k-\ell}}\right)\cdot E_\ell\, d\mu_t$$

$$\frac{d}{dt}V_k(\Omega_t) = \int_{\partial\Omega_t}\left(\phi(t) - \left(\frac{E_k}{E_\ell}\right)^{\frac{1}{k-\ell}}\right)\cdot E_k$$

$$= -\int_{\partial\Omega_t}\underbrace{\left(\phi(t) - \left(\frac{E_k}{E_\ell}\right)^{\frac{1}{k-\ell}}\right)\cdot\left(\cancel{\phi(t)^{k-\ell}} - \frac{E_k}{E_\ell}\right)}\cdot E_\ell$$

≤ 0

Theorem (McCoy 2005)

If $\Sigma_0 \subset \mathbb{R}^{n+1}$ is convex, then the solution Σ_t converges to a sphere $S^n(r)$ as $t \to \infty$.

The higher order isoperimetric ratio

$$\mathcal{I}_{k,\ell}(\Omega_t) = \frac{V_k(\Omega_t)}{V_\ell(\Omega_t)^{\frac{n+1-k}{n+1-\ell}}}$$

is non-increasing along the flow. McCoy's theorem implies

$$\mathcal{I}_{k,\ell}(\Omega_0) \geq \lim_{t\to\infty} \mathcal{I}_{k,\ell}(\Omega_t) = \mathcal{I}_{k,\ell}(B(r)) = \omega_n^{\frac{k-\ell}{n+1-\ell}}.$$

New proof for the Alexandrov-Fenchel inequalities for convex domains.

In the rest of today's lecture, we sketch the proof of McCoy's theorem.

8:52 — 9:02
休息

Denote

$$F = \left(\frac{E_k}{E_\ell}\right)^{1/(k-\ell)}.$$

Since $\kappa_i > 0$, $i = 1, \cdots, n$ everywhere on Σ_0, we have

$$\frac{\partial F}{\partial \kappa_i} > 0, \quad i = 1, \cdots, n$$

Huisken – Polden 1999

holds on Σ_0. Then a smooth solution Σ_t exists at least for a short time $t \in [0, \varepsilon)$.

Short time existence of the solution can be proved in several way - by writing the evolving hypersurface Σ_t as graphs over the initial hypersurface Σ_0, or over the sphere centered at a point contained in Σ_0 (for star-shaped hypersurfaces), or by considering the Gauss map parametrization (for convex hypersurfaces).

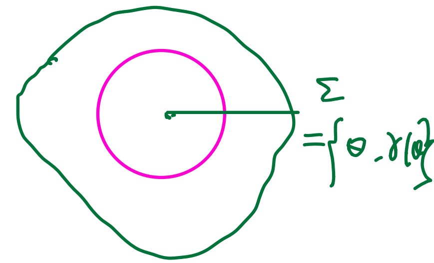
The idea is to break the diffeomorphism invariance and reduce the flow equation to a scalar parabolic PDE.

$$\frac{\partial X}{\partial t} = (\phi(t) - F)\vec{\nu} \quad \text{in } \mathbb{R}^{n+1}$$

$\Longleftrightarrow$ a scalar parabolic PDE

We can write a convex hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ as a graph of radial function r over a sphere $\mathbb{S}^n$ centered at some point x_0

$$\Sigma = \{(\theta, r(\theta)), \quad \theta \in \mathbb{S}^n\}$$



The flow equation is equivalent to a scalar parabolic PDE

$\frac{\partial}{\partial t} r = G(\partial^2 r, \partial r, r, t)$

$$\boxed{\frac{\partial}{\partial t} r = \left(\phi(t) - F(h_i^j)\right) \omega}$$

for the radial function $r(\cdot, t)$ on the sphere $\mathbb{S}^n$, where $\omega = \sqrt{1 + |Dr|^2/r^2}$ and

$$(*) \qquad h_i^j = -\frac{1}{r^2\omega}\left(\sigma^{jk} - \frac{r^j r^k}{r^2\omega^2}\right) r_{ik} + \frac{1}{r^3\omega^3} r^j r_i + \frac{1}{rw}\delta_i^j$$

is the Weingarten matrix of Σ in the coordinates $\{\theta^1, \cdots, \theta^n\}$. Short time existence follows from the theory of parabolic PDE.

$\mathbb{R}^{n+1} = \mathbb{R}^+ \times S^n$. $\qquad dr^2 + r^2 g_{S^n}$

$\Sigma = \{ (\theta, r(\theta)), \ \theta \in S^n \}$

let $\{\theta^1, \dots \theta^n\}$ local coordinates on S^n

$e_i = \partial_i + r_i \partial_r$

$$\Rightarrow \quad g_{ij} = r^2 \sigma_{ij} + r_i r_j \,, \qquad \sigma_{ij} = g_{S^n}(\partial_i, \partial_j)$$

Denote $\omega = \sqrt{1 + |Dr|^2/r^2}$, Then Unit normal $\vec{\nu} = \frac{1}{\omega}\left(\partial_r - \frac{r^i \partial_i}{r^2}\right)$

$$h_{ij} = - \langle \nabla_{e_i} e_j, \vec{\nu} \rangle$$

$$= \frac{1}{\omega}\left(r \sigma_{ij} + \frac{2}{r} r_i r_j - r_{ij} \right)$$

$$\Rightarrow \quad (h_i^j) = \cdots \qquad [\text{exercise}]$$

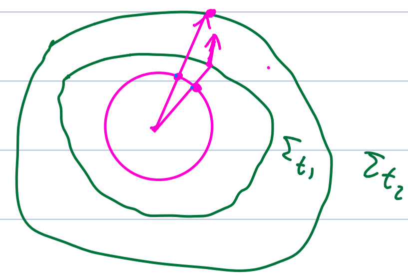
$$\cdot \ \frac{\partial}{\partial t} X = (\phi(t) - F)\, \vec{\nu} \,, \qquad F = F(h_i^j)$$

$$X : S^n \times [0, T) \longrightarrow \mathbb{R}^{n+1}$$

$$X(z, t) = \big(\theta(z,t),\ r(\theta(z,t), t) \big)$$

$$\theta(\cdot, t) : S^n \longrightarrow S^n$$

diffeomorphism



$$\begin{cases} \frac{\partial}{\partial t} X = \left(\frac{\partial}{\partial t}\theta,\ \frac{\partial}{\partial t} r + r_i \cdot \frac{\partial \theta^i}{\partial t} \right) \\ \vec{\nu} = \frac{1}{\omega}\left(\partial_r - \frac{r^i \partial_i}{r^2} \right) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial \theta^i}{\partial t} = \frac{\partial X}{\partial t} \cdot \partial_i = (\phi(t) - F)\, \vec{\nu} \cdot \partial_i \\ \frac{\partial}{\partial t} r + r_i \cdot \frac{\partial \theta^i}{\partial t} = \frac{\partial X}{\partial t} \cdot \partial_r = (\phi(t) - F)\, \vec{\nu} \cdot \partial_r \end{cases}$$

$$\Rightarrow \quad \frac{\partial}{\partial t} r = (\phi(t) - F)\big(\underbrace{\vec{\nu} \cdot \partial_r - (\vec{\nu} \cdot \partial_i) \cdot r^i} \big)$$

$$= (\phi(t) - F)\, \omega \qquad \omega = \sqrt{1 + \frac{|Dr|^2}{r^2}}$$

$$(*) \quad \frac{\partial r}{\partial t} = G(D^2 r, Dr, r, t) \quad \text{on} \quad S^n \times [0, T)$$

$$\left(\frac{\partial G}{\partial r_{ij}} \right) = -\omega \underbrace{\frac{\partial F}{\partial h_j^k}}_{>0} \cdot \left(-\frac{1}{r^2 \omega} \underbrace{\left(\sigma^{kj} - \frac{r^k r^j}{\omega^2 r^2} \right)}_{>0} \right) > 0$$

$\Rightarrow$ (*) is parabolic PDE. $\Rightarrow$ short-time existence

In order to prove the long time existence, it suffices to prove uniform estimates on all derivatives of the solution Σ_t.

1. Estimates up to C^2 order.
2. $C^{2,\alpha}$ estimates for fully nonlinear parabolic PDE due to Krylov and Safonov.
3. $C^{k,\alpha}$ estimate for all $k \geq 2$, by bootstrapping using Schauder theory.

$\kappa_i > 0$, $i = 1 \ldots n$ convexity

Convexity preserved.

By assumption $(h_{ij}) > 0$ at $t = 0$. h_{ij} satisfies

$$\frac{\partial}{\partial t} h_{ij} = \dot{F}^{k\ell}\nabla_k\nabla_\ell h_{ij} + \ddot{F}^{k\ell,pq}\nabla_i h_{k\ell}\nabla_j h_{pq} + \dot{F}^{k\ell}(h^2)_{k\ell}h_{ij} + (\phi(t) - 2F)(h^2)_{ij}.$$

We apply tensor maximum principle to show that $(h_{ij}) > 0$ along the flow for $t > 0$.

$$\frac{\partial}{\partial t} h_{ij} = \nabla_i\nabla_j F + (\phi(t) - F)(h^2)_{ij}$$

$$= \nabla_i(\dot{F}^{k\ell}\nabla_j h_{k\ell}) + \cdots$$

$$= \dot{F}^{k\ell}\nabla_i\nabla_j h_{k\ell} + \ddot{F}^{k\ell,pq}\nabla_i h_{k\ell}\nabla_j h_{pq} + (\phi(t) - F)(h^2)_{ij}$$

Simons identity $\Rightarrow \nabla_i\nabla_j h_{k\ell} = \nabla_k\nabla_\ell h_{ij} + h_{ij}(h^2)_{k\ell} - h_{k\ell}(h^2)_{ij}$

Theorem (Hamilton 1982)

Let M be a closed manifold with smooth time-varying metric $g(t)$, and $\alpha(t)$ be a symmetric $(2,0)$ - tensor field satisfying

$$\frac{\partial}{\partial t}\alpha = \Delta\alpha + \beta.$$

Suppose that $\beta(t) = \beta(\alpha(t), g(t), t)$ is a symmetric $(2,0)$-tensor which satisfies

$$\beta(v,v) \geq 0$$

whenever v is a null eigenvector of α, i.e. $\alpha_{ij}v^j = 0$. If $\alpha(0) \geq 0$, then $\alpha(t) \geq 0$ on M for $0 \leq t \leq T$.

$\alpha(0) > 0 \Rightarrow \alpha(t) > 0$

The idea is essentially to reduce to the scalar case by evaluating the tensor on a suitable vector field.

Let t_0 be the first time there exits a point x_0 and $\alpha(t_0)$ has a null vector $v \in T_{x_0}M$. Extend v in space-time neighborhood of (x_0, t_0) by parallel translation in space along geodesic rays and then taking v to be independent of time. Then we compute

$\Rightarrow \quad v$

$$\begin{aligned} 0 \geq & \frac{\partial}{\partial t}\Big|_{(x_0,t_0)} \alpha(v,v) \\ = & \left(\frac{\partial}{\partial t}\alpha\right)(v,v) + \cancel{2\alpha(\partial_t v, v)} \\ = & (\Delta\alpha)(v,v) + \beta(v,v) \\ = & \Delta(\alpha(v,v)) + \beta(v,v) \geq 0 \end{aligned}$$

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We have almost obtained a contradiction. A strict contradiction can be obtained by adding a positive ε in a standard way.

$\tilde{\alpha} = \alpha + \varepsilon(t+\delta)g$

When $F = H$,

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} + |h|^2 h_{ij} + \underbrace{(\phi(t) - 2H)(h^2)_{ij}}_{\beta}.$$

$\beta(v,v) = 0$

when $h(v,v) = 0$

Hamilton's maximum principle implies that convexity ($h_{ij} > 0$) is preserved.

To deal with more general case with $F = (E_k/E_\ell)^{1/k-\ell}$, we need to apply a generalized tensor maximum principle by Ben Andrews (2007).

Theorem (Andrews 2007)

Let M be a closed manifold with smooth time-varying metric $g(t)$, and $\alpha(t)$ be a symmetric $(2,0)$ - tensor field satisfying

$$\frac{\partial}{\partial t}\alpha = a^{k\ell}\nabla_k\nabla_\ell\alpha + b^k\nabla_k\alpha + \beta,$$

where $(a^{kl}) > 0$ and b^k are smooth. Suppose that $\beta(t) = \beta(\alpha(t), g(t), t)$ is a symmetric $(2,0)$-tensor satisfying

$$\beta(v,v) + \boxed{\sup_{\Lambda_k^p} 2a^{k\ell}(2\Lambda_k^p\nabla_\ell\alpha_{ip}v^i - \Lambda_k^p\Lambda_\ell^q\alpha_{pq})} \geq 0,$$

how to deal with this term

whenever v is a null eigenvector of α. If $\alpha(0) \geq 0$, then $\alpha(t) \geq 0$ on M for $0 \leq t \leq T$.

1. The idea is to extend the null vector v in space in more general way.
2. We can further show that $h_{ij} \geq \varepsilon H g_{ij}$ is preserved. Equivalently, $\kappa_{\min} \geq \varepsilon H$ holds on Σ_t for $t > 0$.

$(h_{ij}) > 0$ at $t=0$

$\therefore \exists\ \varepsilon > 0$ s.t. $h_{ij} \geq \varepsilon H g_{ij}$

Andrews tensor max principle $\Rightarrow$ $h_{ij} \geq \varepsilon H g_{ij}$ is preserved.

how to show $(h_{ij}) > 0$ is preserved

$$\frac{\partial}{\partial t} h_{ij} = \dot F^{k\ell}\nabla_k\nabla_\ell h_{ij} + \ddot F^{k\ell,pq}\nabla_i h_{k\ell}\nabla_j h_{pq} + \cancel{\dot F^{k\ell}(h^2)_{k\ell} h_{ij}} + \cancel{(\phi(t)-2F)(h^2)_{ij}}.$$

Assume at (x_0, t_0). (h_{ij}) has a null vector v

choose local orthonormal frame $\{e_1, e_2, \dots, e_n\}$ s.t.

$v = e_1$. $\Rightarrow$ $h_{11} = 0$ $(h^2)_{11} = 0$

Gradient terms, we need

$$\underbrace{\ddot F^{k\ell,pq}\nabla_1 h_{k\ell}\nabla_1 h_{pq}}_{B} + \underbrace{2\sup_{\Lambda_k^p} \dot F^{k\ell}\left(2\Lambda_k^p\nabla_\ell h_{1p} - \Lambda_k^p\Lambda_\ell^q h_{pq}\right)}_{A} \geq 0$$

$$A = \sup \dot F^k\left(2\Lambda_k^p\nabla_k h_{1p} - (\Lambda_k^p)^2 h_{pp}\right)$$

$$= \sup \dot F^k\left(\frac{(\nabla_k h_{1p})^2}{h_{pp}} - \cancel{\left(\Lambda_k^p - \frac{\nabla_k h_{1p}}{h_{pp}}\right)^2 h_{pp}}\right)$$

$$B = \ddot F^{k\ell}\nabla_1 h_{kk}\nabla_1 h_{\ell\ell} + \sum_{k\neq\ell}\frac{\dot F^k - \dot F^\ell}{\kappa_k - \kappa_\ell}(\nabla_1 h_{k\ell})^2$$

$$\left\{\begin{array}{l}\dot F^k = \dfrac{\partial F}{\partial \kappa_k}\\[2mm] \ddot F^{k\ell} = \dfrac{\partial^2 F}{\partial\kappa_k\partial\kappa_\ell}\end{array}\right.$$

$\because$ $F = \left(\frac{E_k}{E_\ell}\right)^{\frac{1}{k-\ell}}$

$\Rightarrow$ $A + B \geq \frac{2}{F}|\nabla_1 F|^2 \geq 0$

Andrews' thm $\Rightarrow$ $(h_{ij}) > 0$ is preserved.

1. Huisken and Ilmanen (2001): weak solution of inverse mean curvature flow, Riemannian Penrose inequality (conjecture by Penrose);
2. F. Schulze (2006): power of mean curvature flow / Isoperimetric inequality for bounded domain.

   $\partial_t X = -H^k \vec{\nu}$
3. Guan and Li (2009): smooth solution of inverse curvature flow, Alexandrov-Fenchel inequalities for star-shaped and k-convex domains.
4. Simon Brendle, P.-K. Hung, M.-T. Wang (2012): inverse mean curvature flow, Minkowski inequality in Anti-de Sitter-Schwarzchild space.
5. Recent years: Alexandrov-Fenchel type inequalities in hyperbolic space / in sphere / in general ambient space.

Thank you!

- Q & A.