Geometry of exotic spheres / Jianquan Ge (BNU)

2020年7月20日 17:12

§0 Notations

Topological space X, $\mathcal{T} = \{$open subsets / finite $\cap$ & countable $\cup$ are closed operations$\}$

Geometry on exotic spheres

2018.11.22 1st

Jianquan Ge 2020.07.20 2nd.

§1 Introduction.

Exotic spheres are smooth manifolds which are homeomorphic but not diffeomorphic to S^n. (diffeom. $\cong$)

Homotopy spheres are smooth manifolds which are homotopy equivalent to S^n ($\simeq$)

Topological spheres are manifolds which are homeomorphic to S^n. (homeom. $\overset{h}{\cong}$)

Poincaré conjecture: $\Sigma^n \simeq S^n \Leftarrow\Rightarrow \Sigma^n \overset{\text{homeom.}}{\cong} S^n$ one of the biggest Open problems.

$\begin{cases} n \geq 5, & \text{Smale (1961-1962)} \\ n=4, & \text{Freedman (1982)} \\ n=3, & \text{Perelman (2002-2003)} \end{cases}$

$n=1$ or 2, trivial.

$\rightsquigarrow$ exotic spheres.

$\Rightarrow$ $\boxed{\text{Smooth poincare conjecture in dim 4 ???}}$

$\} \ \Sigma^n \overset{\text{homeom.}}{\cong} S^n \overset{n\leq 3}{\Leftarrow\Rightarrow} \Sigma^n \overset{\text{diffeo.}}{\cong} S^n$

(Edwin E. Moise) C^∞-str. unique in dim ≤ 3.

$\to$ Fields medal's works.

Milnor (1956) first found exotic 7-spheres as S^3-bundles over S^4.

Kervaire-Milnor (1963).

Θ_n: group of h-cobordism classes of oriented homotopy n-spheres under connected sum.

bP_{n+1}: subgroup of Θ_n that bound parallelizable manifolds.

Θ_n: finite, abelian; bP_{n+1} cyclic, is trivial or order 2 except in case $n=4k+3$.

$\boxed{n \neq 4: \Theta_n = \{\text{ori. diff. classes of } S^n\}}$

$|bP_{4k}| = 2^{2k-2}(2^{2k-1}-1)B$. where B is numerator of $|4B_{2k}/k|$, B_{2k} is Bernoulli number

$$B_{2k} = \frac{(-1)^{k+1}2(2k)!}{(2\pi)^{2k}}\zeta(2k) \quad \text{where } \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \text{ is the Riemann zeta function.}$$

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1}, \quad |x| < \frac{\pi}{2}$$

Ex. $B_4 = -\frac{1}{30}$. $B=1$.

$|bP_{2k+1}| = 1$ ($n=2k$ even).

$|bP_{4k+2}| = \begin{cases} 1 & n=4k+1 = 1, 5, 13, 29, 61; \boxed{125} \text{ open! related with Kervaire invariant one problem} \\ 2 & \text{otherwise.} \end{cases}$

$n=4k+1 \equiv 1 \pmod 4$.

Whether there's a 126-dim mfd with $KI=1$

Dim n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\lvert\Theta_n\rvert$	1	1	1	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16	523264	24
$\lvert bP_{n+1}\rvert$	1	1	1	1	1	1	28	1	2	1	992	1	1	1	8128	1	2	1	261632	1

§2 Various of constructions of exotic spheres.

1 Milnor spheres $\Sigma^7_{m,n}$ as S^3-bundles over S^4

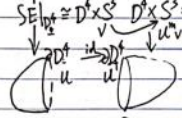
Rank 4 vector bundles over S^4: $E \to S^4$ are characterized by the transition maps $S^3 \to SO(4)$. and thus elements in $\pi_3 SO(4) \cong \mathbb{Z} \oplus \mathbb{Z}$.

$[TS^4]$ $[\eta_H]$

$SE|_{D^4_1} \cong D^4 \times S^3 \quad D^4 \times S^3 = SE|_{D^4_2}$

$v \mapsto u^m v u^n$

$D^4 \xrightarrow{id} D^4$



For $(\overset{2a}{m}, \overset{b}{n}) \in \mathbb{Z} \oplus \mathbb{Z}$, we consider the bundle by using the transition map $f_{(m,n)}: S^3 \to SO(4)$ by $u \mapsto f_{(m,n)}(u)(v) = u^m v u^n$.

$m+n = \pm 1 \Rightarrow \Sigma^7_{m,n} := SE \simeq S^7$ is a homotopy sphere.

Let $M^8 = DE$ (D^8, D^4) be the disk bundle of E, 3-connected, i.e. $\pi_1 = \pi_2 = \pi_3 = 0$.

$H^4(M) \cong \mathbb{Z} \Rightarrow$ the signature $\sigma(M^8) = \pm 1$.

WLOG, let $m+n=1$, $\sigma(M^8)=1$. $P_1^2[M^8] = 4(m-n)^2 = 4(2m-1)^2$. (Pontrjagin number.)

By Hirzebruch's formula.

$$\sigma(M^8) = \frac{1}{45}(7P_2[M^8] - P_1^2[M^8])$$

we have

$$P_2[M^8] = \frac{4(2m-1)^2+45}{7}$$

Once M^8 be smooth. $P_2[M^8]$ should be an integer.

$X \xrightarrow{f} Y$ continuous, C^0

if $f^{-1}(U) \subset X$ open for any open $U \subset Y$

f is a homeomorphism ($\overset{h}{\cong}$) if f is 1-1 and f^{-1} is C^0.

Topological manifold M^n

1) top. sp. + Hausdorff

$U \cap V = \phi$

2) 2nd countable

basis for the topology $\mathcal{T}$.

3) locally Euclidean

$\forall p \in U \xrightarrow[(\overset{h}{\cong})]{\varphi} \varphi(U) \overset{\text{open}}{\subset} \mathbb{R}^n$

$\Rightarrow \varphi_1 \circ \varphi_2^{-1}: \varphi_2(U_1 \cap U_2) \xrightarrow{C^0} \varphi_1(U_1 \cap U_2)$

Differentiable manifold M^n (or. smooth, C^∞)

1) + 2) + 3) + (*) $\varphi_1 \circ \varphi_2^{-1}$ C^∞

$M^m \xrightarrow{f} N^n$ is C^∞, smooth

$U \subset M$, $\varphi: U \to \mathbb{R}^n$; $f(U) \subset N$, $\psi: f(U) \to \mathbb{R}^n$

if $\psi \circ f \circ \varphi^{-1}: \varphi(U) \xrightarrow{C^\infty} \psi f(U)$, $\varphi(U) \subset \mathbb{R}^m$, $\psi f(U) \subset \mathbb{R}^n$

f is a diffeomorphism ($\cong$) if f is 1-1 and f^{-1} is C^∞.

($\Rightarrow m=n$ if $\overset{h}{\cong}$ or $\cong$)

$\sigma(M^8) = \frac{1}{45}(7p_2[M^8] - p_1^2[M^8])$

we have
$$p_2[M^8] = \frac{4(m-n)^2+45}{7}$$
Once M^8 be smooth, $p_2[M^8]$ should be an integer.
However, certain choices of (m,n), like $(m,n)=(2,-1)$, give no integral $p_2[M^8]$.
$\Rightarrow$ M^8 admits no smooth structure.
$\Rightarrow$ $\Sigma^7_{m,n}$ is an exotic sphere. (otherwise, if $\exists f: S^7 \xrightarrow{\text{diff}} \Sigma^7$, then $M^8 = DE \cup_f D^8$, C^∞)

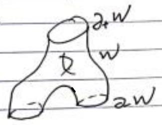
Eells-Kuiper's μ-invariant.
$$\mu(\Sigma^7_{m,n}) = \frac{1}{2^7 \cdot 7}\left(p_1^2[M^8] - 4\,\mathrm{sign}(M^8)\right) \pmod{\mathbb{Z}} \in \mathbb{Q}/\mathbb{Z}$$
$= k(k-1)/56 \Rightarrow$ 10 Milnor spheres (including standard)
$\Rightarrow \mu: \Theta_7 \xrightarrow{\cong} \mathbb{Z}_{28}$. 4 non-Milnor spheres ignoring orientations
$k\Sigma^7_{2,1} \longmapsto k/28$ $(k=2,5,9,12)$

Ref. 1) Milnor, On manifolds homeomorphic to the 7-sphere, Ann. Math. 1956.
2). Eells-Kuiper, An invariant for certain smooth manifolds, Ann. Mat. Pura Appl. 1962.
3). Farrel-Su yang, Introduction to Manifold topology book.

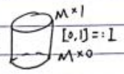
2. twisted spheres. $\Sigma^n_{[f]} := D^n \cup_f D^n$, where $f \in \mathrm{Diff}^+(S^{n-1})$, $[f] \in \pi_0 \mathrm{Diff}^+(S^{n-1})$
$\Gamma_n := \{\Sigma^n_{[f]}\}$
Fact. $\Theta_n \cong \Gamma_n$ $(n>5)$
$n \le 3$. Moise's thm. $\exists!$ C^∞-structure
$n=4$. Cerf's thm: $\Sigma^4_{[f]} \overset{\text{diff}}{\cong} S^4$ & $\Theta_4 = 1$ by Kervaire-Milnor
$n \geq 5$. Generalized Poincaré conjecture GPC. proven by Smale.

Defn. A cobordism W is a cpt mfd with boundary, together with a decomposition of ∂W into 2 non-empty disjoint open sets $\partial_- W$ and $\partial_+ W$.

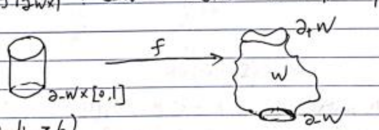
An h-cobordism W is a cobordism s.t. both inclusions $\partial_- W \hookrightarrow W$ and $\partial_+ W \hookrightarrow W$ are homotopy equivalences.



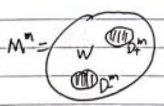
e.g. $W = M \times [0,1]$



h-cobordism Theorem (Smale)
Let W^n be a C^∞, 1-connected, h-cobordism $(n \geq 6)$. Then
$\exists$ a diffeomorphism $f: \partial_- W \times I \xrightarrow{\cong} W$, s.t. $f|_{\partial_- W \times 0} = id$.
In particular, $f|_{\partial_- W \times 1}: \partial_- W \xrightarrow{\cong} \partial_+ W$ a diffeomorphism.



Proof of GPC (in dim ≥ 6).
Let $M^m \simeq S^m$ $(m \geq 6)$ be a homotopy sphere.
Let $W^m := M^m - \mathring{D}^m_- - \mathring{D}^m_+$, a cobordism from $\partial_- W = \partial_0 D^m_- = S^{m-1}_-$ to $\partial_+ W = \partial D^m_+ = S^{m-1}_+$



Since $H_i(W, \partial_- W) \overset{\text{excision}}{\cong} H_i(M - \mathring{D}^m_+, D^m_-) = 0$ $\forall i$.
$\therefore (W; \partial_- W, \partial_+ W)$ is an h-cobordism.
By h-cobordism thm, $\exists$ a diffeomorphism $f: W \to \partial_- W \times [0,1] = S^{m-1} \times [0,1]$
$f|_{\partial_- W} = id$
$\therefore f$ extends to a diffeomorphism $\tilde{f}: W \cup_{id} D^m_- \to D^m$
Now extend $f: \partial_+ W \to S^{m-1}$ by the Alexander trick to a homeomorphism $D^m_+ \to D^m$, give a homeomorphism $M = \underbrace{(M - \mathring{D}^m_+)}_{M - \mathring{D}^m_+} \cup D^m \xrightarrow{f} D^m \cup D^m = S^m$

Corollary Σ^n an exotic sphere, $n \geq 6$ then
$\Sigma^n = D^n \cup_f D^n$, $f: S^{n-1} \xrightarrow{\text{diff}} S^{n-1}$.

3. Kervaire cohere

if f is 1-1 and f^{-1} is C^∞.
$(\Rightarrow m = n$ if $\overset{\Delta}{\cong}$ or $\cong)$

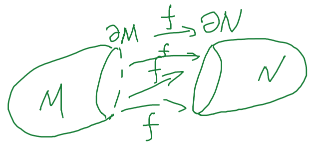
$M \underset{f_0}{\overset{f_1}{\rightrightarrows}} N$ are C^0 (resp. C^∞ embeddings)
$(f_0 \simeq f_1)$ homotopic if $\exists F: M \times [0,1] \to N$
(resp1. pseudo-isotopic) s.t. $F_0 = f_0$, $F_1 = f_1$
(resp2. isotopic) (resp1. F C^∞)
(resp2 F C^∞, F_t embedding)

$M \simeq N$ homotopy equivalent
if $\exists M \underset{g}{\overset{f}{\rightleftarrows}} N$ st $\begin{cases} f \circ g \simeq id_N \\ g \circ f \simeq id_M \end{cases}$.

homotopy group homotopy class
$\pi_k(M) := \{[f] \mid f: S^k \to M\}$
$\pi_0(M) = \{$connected components of $M\}$
$\pi_1(M) =$ fundamental group of M.
$\pi_n(S^n) \cong \mathbb{Z}$, $n \geq 1$.
$[f] \longleftrightarrow \deg(f)$

$\forall f \in \mathrm{Diff}^+(S^{n-1})$, $\deg f = 1$
$\Rightarrow f \simeq id_{S^{n-1}}$
$\Rightarrow \Sigma^n_{\{f\}} = D^n \cup_f D^n \simeq D^n \cup_{id} D^n = S^n$
$\{f\}$ isotopy class
$\Gamma^n := \{\Sigma^n_{\{f\}}\}$

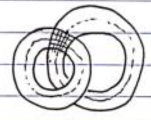
Lem. $M \cup_f N$ $\exists!$ C^∞ str. ind. of choices of $f: \partial M \cong \partial N$ in its (pseudo)-isotopy



$\Gamma^n \cong \pi_0 \mathrm{Diff}^+(S^{n-1})$ $\forall n \neq 5$
Since for $n > 5$, exact seq.
$$\pi_0 \mathrm{Diff}^+(D^n) \to \pi_0 \mathrm{Diff}^+(S^{n-1}) \to \Gamma_n \to 0.$$
$\| \leftarrow$ Cerf pseudo-isotopy thm
0

$\Sigma^n = D^n \cup_f D^n$, $f: S^{n-1} \overset{\text{diff.}}{\cong} S^{n-1}$.

3. Kervaire sphere

Plumbing  $D(TS^1) \,\square\, D(TS^1) = M^2$

$M^{2n} := D(TS^n) \,\square\, D(TS^n)$

Lemma $\overset{n \geq 3}{}$ $\partial M^{2n} \overset{\text{homeom.}}{\cong} S^{2n-1} \iff n$ odd.

Proof:

(1) $\partial M = S(TS^n) \bigcup_{S^{n-1}\times S^{n-1}} D^n \times S^{n-1}$.

van-Kampen thm $\Rightarrow \pi_1(\partial M) = 0$.

(2). $\to H_i(\partial M) \to H_i(M) \to H_i(M, \partial M) \to H_{i-1}(\partial M) \to \cdots$

$H_i(M,\partial M) \overset{\text{P.D.}}{\cong} H^{2n-i}(M)$

$\Rightarrow M \simeq S^n \vee S^n$.

$\therefore\ 0 \to H_n(\partial M) \to H_n(M) \xrightarrow{j_*} H_n(M, \partial M) \to H_{n-1}(\partial M) \to 0$

$H_n(M) \cong \mathbb{Z}^2$, $H_n(M,\partial M) \overset{\text{P.D.}}{\cong} H^n(M) \cong \mathrm{Hom}(H_n(M), \mathbb{Z}) \cong \mathbb{Z}^2$

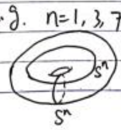
$\forall x, y \in H_n(M)$, $\langle D j_*(x), y \rangle = x \cdot y$ intersection number

$\therefore$ Under the natural basis $\{e_1, e_2\}$ of $H_n(M)$, and the dual basis $\{e_1^*, e_2^*\}$ of $H^n(M)$, j_* is represented by $\begin{pmatrix} e_1\cdot e_1 & e_1 \cdot e_2 \\ e_2 \cdot e_1 & e_2 \cdot e_2 \end{pmatrix}$. $e_i \cdot e_i = \chi(TS^n) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$.

n odd: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; n even: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\therefore j_*$ is an isomorphism $\iff n$ odd.

e.g. $n = 1, 3, 7$. TS^n trivial. $\therefore M^{2n} = S^n \times S^n - \mathring{D}^{2n}$



$\partial M \cong S^{2n-1}$ (diffeom.)

$n=5$, $\partial M^{10} = \Sigma^9 \overset{\text{diff.}}{\not\cong} S^9$. (Kervaire sphere).

$N^{10} = M \cup_{id} D^{10}$ is not hmtp eqiv. to a closed C^∞-mfd.

In general, for $k = ?$ $\partial M^{4k+2} = \Sigma^{4k+1} \overset{\text{diff}}{\cong} S^{4k+1}$

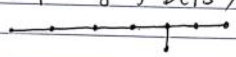
Browder: if $k \neq 2^s - 1$, then $\Sigma^{4k+1} \not\cong S^{4k+1}$

Hill-Hopkins-Ravenel (2009): $\Sigma^{4k+1} \not\cong S^{4k+1}$ if $k \neq 0, 1, 3, 7, 15$ and possiblly $\boxed{k \neq 31 \quad \Sigma^{125}}$ open.

$\Sigma^1, \Sigma^5, \Sigma^{13}, \Sigma^{29}, \Sigma^{61}$

Rk.

$n = 2k$ even. plumbing of $D(TS^{2k})$ according to the Dynkin diagram of E_8:



Lemma. $M^{4k} = D(TS^{2k}) \,\square_{8}\, D(TS^{2k})$ has $(\partial M)^{4k-1} \overset{\text{homeom.}}{\cong} S^{4k-1}$ $(k \geq 2)$.

pf. (1) Van-Kampen thm $\Rightarrow \pi_1(\partial M) = 0$

(2) $M \simeq \bigvee_8 S^{2k}$

$\Rightarrow\ 0 \to H_{2k}(\partial M) \to H_{2k}(M) \xrightarrow{j_*} H_{2k}(M, \partial M) \to H_{2k-1}(\partial M) \to 0$

$H_{2k}(M) \cong \mathbb{Z}^8$, $H_{2k}(M,\partial M) \overset{\text{P.D.}}{\cong} H^{2k}(M) \cong \mathrm{Hom}(H_{2k}(M), \mathbb{Z}) \cong \mathbb{Z}^8$

Under the natural basis $\{e_i \mid i=1,\cdots,8\}$ of $H_{2k}(M)$ and dual basis $\{e_i^*\}$ of $H^{2k}(M)$

j_* is represented by

$$A = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}$$

Facts: (1) $\det A = 1$ $\therefore A \in GL_8(\mathbb{Z})$

$j_*: H_{2k}(M) \xrightarrow{\cong} H_{2k}(M, \partial M)$

(2) A is positive definite

$\therefore \operatorname{sign} A = 8$.

Question $\partial M^{4k} \cong S^{4k-1}$ (diffeom.) ?

If yes, then let $X^{4k} = M^{4k} \cup_{id} D^{4k}$, X^{4k} is an almost parallelizable closed mfd

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$\overset{\shortparallel}{0} \leftarrow$ Cerf pseudo-isotopy thm

$f_0 \overset{\text{PI}}{\simeq} f_1 \iff f_0 \overset{\text{isotopic}}{\simeq} f_1$

in $\mathrm{Diff}^+(S^{n-1})$ $n > 5$

(in fact for all $n \neq 5$)

Open for $n = 5$, i.e., $\mathrm{Diff}^+(S^4)$

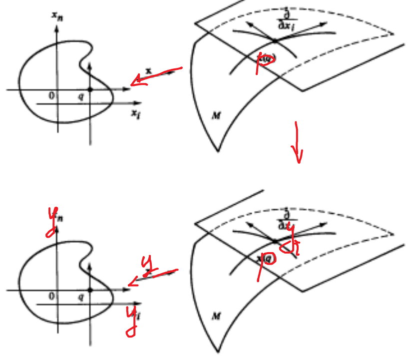
even not known about π_0.

$TM^n = \bigcup_{p \in M} T_pM$ tangent bundle

$T_pM = \mathrm{Span}\left\{ \frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p \right\}$

$(p, x_1, \cdots, x_n)$ local coordinates

$(p, y_1, \cdots, y_n)$



$\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) = \left(\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}\right)\left(\frac{\partial y_i}{\partial x_j}\right)$

$U_x \cap U_y \xrightarrow{J_{xy}} GL(n)$

$E = \bigcup_{p \in M^n} E_p$ vector bundle

E_p vector space of $\dim = r$

call E v.b. of rank r over M

$\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$

$(p, v) \in E \iff p \in M \ \&\ v \in E_p$

$\downarrow \pi$

$p \in M$

(*) $\pi^{-1}(U) \cong U \times \mathbb{R}^r$

<u>Question</u> $\partial M^{4k} \cong S^{4k-1}$ (diffeom.) ?

If yes, then let $X^{4k} = M^{4k} \cup_{id} D^{4k}$, X^{4k} is an almost prallelizable closed mfd

($\because$ $TD(TS^n)$ is stably trivial, hence trivial)

$\therefore$ $\mathrm{sign}(X)$ is divisible by $a_k \cdot b_k \cdot j_k$.

$$\overset{\shortparallel}{\mathrm{sign}} \text{ of } (\underset{H_{2k}(M) \times H_{2k}(M) \to \mathbb{Z}}{\underset{\shortparallel}{H_{2k}(X) \times H_{2k}(X) \to \mathbb{Z}}}) = \mathrm{sign}(A) = 8 \ \text{↯} \quad \therefore \partial M \overset{\text{diff.}}{\not\cong} S^{4k-1}$$

$k \geq 1$.

<u>Example</u>. $k=1$. $\partial M^4 = P^3$ Poincaré homology sphere.

$\pi_1(P) = \tilde{A}_5$ binary icosahedral group (二元二十面体群)

$H_*(P) \cong H_*(S^3)$.

Freedman: $\exists N^4 \simeq *$, top. mfd, $\partial N = P^3$.

$\therefore$ $X^4 = M \cup_P N$ a closed top. mfd, 1-connected, almost parallelizable

$\mathrm{sign}(X) = 8$. (the E_8-mfd)

Rohlin's Thm $\Rightarrow$ X^4 is not smoothable.

$Y^4 = X^4 \# X^4$. $\mathrm{sign}(Y) = 16$, but Y is not smoothable by

<u>Donaldsen's Thm</u>: $\forall M^4$, closed, C^∞, intersection form $S(M)$ is positive definite.

then $S(M) \sim \mathrm{diag}(1, \cdots, 1)$. ~~Here~~

Here $S(Y)$ is even, hence $S(Y) \sim \mathrm{diag}(1, \cdots, 1)$ which is odd.

But the Kummer surface

$V = V(z_0^4 + \cdots + z_3^4 = 0) \subset \mathbb{C}P^3$ is a C^∞-mfd.

$V \overset{\text{homeom.}}{\cong} X^4 \# X^4 \# \underset{3}{\#}(S^2 \times S^2)$ (Freedman) + arithmetic.

Rk. Other constructions using plumbing with surgery would give exotic spheres.

$\iota: S^n \hookrightarrow M^{n+m}$ with trivial normal bundle

extend. $\rightsquigarrow$ $e: S^n \times D^m \hookrightarrow M^{n+m}$.

cutting out interior of the image(e) and stitching $D^{n+1} \times S^{m-1}$ in

4. Brieskorn sphere

Let $P(a_0, \cdots, a_{n+1})$ be the complex polynomial

$$P(a_0, \cdots, a_{n+1}) = z_0^{a_0} + \cdots + z_{n+1}^{a_{n+1}}.$$

Define the Brieskorn manifold $B(a_0, \cdots, a_{n+1}) = P^{-1}(0) \cap S^{2n+3} \subset \mathbb{C}^{n+2}$

$\dim B(a_0, \cdots, a_{n+1}) = 2n+1$, is $(n-1)$-connected smooth manifold.

For n even and $d \equiv 3$ or $5 \bmod 8$, $B(d, 2, \cdots, 2)$ is a <u>Kervaire sphere</u>.

every

$\exists$ a cohomogeneity-one action by $SO(n+1) \times S^1$.

with singular orbits have codim 2 and $2n$.

$$SO(n+1) \times S^1 \ni (A, e^{i\theta}) \cdot (z_0, \cdots, z_{n+1}) = (e^{2i\theta} z_0, A(e^{di\theta} z_1, \cdots, e^{di\theta} z_{n+1}))$$

$$bP_8 = \Theta_7 = \{B(6k-1, 3, 2, 2, 2) \mid k = 1, \cdots, 28\} \cong \mathbb{Z}_{28}.$$

In general, $bP_{4m} = \{B(6k-1, 3, 2, \cdots, 2) \mid k \geq 1\}$ for $k = 1, \cdots, |bP_{4m}|$.

(*) $\pi^{-1}(U) \cong U \times \mathbb{R}^r$

$p \in U \cap V \subset M$ $\quad (p, u) \overset{g_{UV}}{\downarrow} (p, v)$

$\pi^{-1}(V) \cong V \times \mathbb{R}^r \quad g_{UW}$

$\downarrow g_{VW}$

$$U \cap V \xrightarrow{C^\infty} GL(r) \qquad W \times \mathbb{R}^r$$

tansition map. $g_{VW} \circ g_{UV} = g_{UW}$

Given a Euclidean metric on $E \to M$, we say

disk bundle $DE := \{(p, v) \in E \mid |v| \leq 1\}$

sphere bundle $SE := \{(p, v) \in E \mid |v| = 1\}$

$\shortparallel$

$\partial(DE)$

§3 Curvatures on exotic spheres. Ref: ~~Joachim-Wraith~~, Bull. AMS 2008

§3.1 Scalar curvature

Kazdan-Warner (1985) divide C^∞ closed manifolds into three classes:

(1). $\exists$ Riemannian metric of non-negative scalar curvature and positive at least at one point
i.e. $K \geq 0$ & $K(p_0) > 0$. for some point p_0.

(2). $\exists$ Riemannian metric of non-negative scalar curvature but do not belong to (1).
i.e. $K \equiv 0$ (strongly scalar flat)

(3). others.

Classification: ($\dim M = n \geq 3$).

Positive $M \in (1) \Leftrightarrow \forall f \in C^\infty(M)$, $\exists$ Riem. metric g s.t. $K_g = f$

flat. $M \in (2) \Rightarrow f \in C^\infty(M)$ $\exists g$ st $K_g = f \Leftrightarrow f(p_0) < 0$ at some $p_0 \in M$ or $f \equiv 0$.

"negative" $M \in (3) \Rightarrow f \in C^\infty(M)$ $\exists g$ s.t. $K_g = f \Leftrightarrow f(p_0) < 0$ at some $p_0 \in M$.

Examples (1) $\exists M$ with $K > 0$. e.g. S^n
(2) $\exists M$ with $K \equiv 0$ but no metric with $K > 0$. e.g. T^n. Gromov-Lawson. Schoen-Yau.
(3) $\exists M = T^4 \# K3$-surface.

Question: Which one of these three classes does an exotic sphere Σ^n ($n \neq 4$) belong to?
completely anwsered! α-invariant
a generalization of $\hat{A}$-genus in dim $= 4k$.

$$\hat{A}(TM, \nabla^{TM}) = \det\left(\left(\frac{\frac{\sqrt{-1}}{4\pi} R^{TM}}{\sinh(\frac{\sqrt{-1}}{4\pi} R^{TM})}\right)^{\frac{1}{2}}\right) \in \Omega^*(M).$$

$\hat{A}$-genus of M: $\hat{A}(M) = \int_M \hat{A}(TM, \nabla^{TM})$.

Thm (Atiyah-Hitchin-Lichnerowicz-Singer).
Let M be a compact spin manifold with $K > 0$, then $\hat{A}(M) = 0$ in $\dim = 4k$; $\alpha(M) = 0$

Thm (Weiping Zhang Ann. Math 2017) {generalized.
Let M be a compact spin mfd with a foliation of $K^F > 0$ (positive leafwise scalar curv)
$\alpha(M^n) = \frac{3+(-1)^k}{4} \hat{A}(M^n)$ if $n = 8k + 4i$, $i = 0, 1$

spin bordism ring

Atiyah-Bott-Shapiro homomorphism

For closed spin mfds M^n. $\alpha(M) = \text{image}\left(\alpha : \Omega_n^{spin} \longrightarrow KO^{-n}(pt)\right)$

For exotic spheres; $\alpha(\Sigma^n) \begin{cases} = 0 & \text{if } n \not\equiv 1 \text{ or } 2 \pmod 8 \\ \in \mathbb{Z}_2 & \text{if } n \equiv 1 \text{ or } 2 \pmod 8 \end{cases}$

geometrically defn

Fact: $\exists !$ spin structure on Σ^n. $(w_1 = w_2 = 0)$.

$n \equiv 1 \pmod 8$: the real spinor bundle $S \to \Sigma^n$ $\exists$ complex structure J s.t.
the Dirac operator D action on $\Gamma(S)$ preserves J thus $\operatorname{Ker} D$ complex

$n \equiv 2 \pmod 8$: the real spinor bundle $S \to \Sigma^n$ $\exists$ quaternionic structure J_1, J_2, J_3 s.t.
the Dirac operator D action on $\Gamma(S)$ preserves J_1, J_2, J_3 thus $\operatorname{Ker} D$ quaternionic

Then. $\alpha(M) = \begin{cases} \dim_{\mathbb{C}} \operatorname{Ker}(D) \pmod 2 & \text{if } n \equiv 1 \pmod 8 \\ \dim_{\mathbb{H}} \operatorname{Ker}(D) \pmod 2 & \text{if } n \equiv 2 \pmod 8 \end{cases}$

Hitchin (1974). Lichnerowicz-formula (Bochner-Weitzenböck formula)
$D^2 = \nabla^* \nabla + \frac{1}{4} K$
$\Rightarrow$ If $K > 0$, then $D^2 > 0 \Rightarrow \operatorname{Ker}(D) = 0 \Rightarrow \alpha(M) = 0$.

Adams (1966) & Milnor (1965) $n \equiv 1$ or $2 \pmod 8$, $n \geq 9$.
$\alpha : \Theta_n \longrightarrow \mathbb{Z}_2$ is a surjective group homomorphism
$\therefore \exists\ \Sigma^n$ ($n \equiv 1, 2 \bmod 8$, $n \geq 9$) with $\alpha(\Sigma^n) = 1$ nontrivial ~~(half of Θ_n)~~.
~~In particular, denote $\Sigma^n \in \Theta_n$ be the "smallest" element~~.
$\therefore \Sigma^n \notin (1)$. do not admit positive scalar curvature.

Futaki (1993). A closed 1-conn. M^n $(n \geq 5) \in (2)$ strongly scalar flat if and only if
$\alpha(M) \neq 0$ and $M = M_1 \times \cdots \times M_r$
where each M_i is Ricci-flat Kähler mfd or a Riem. mfd with Spin(7)-holonomy.
$\hookrightarrow H^2(M_i) \neq 0$ $(\dim_{\mathbb{R}} M_i$ even$)$. $\hookrightarrow \dim M_j = 8$.
$\therefore \Sigma^n \notin (2)$ do not admit strongly scalar flat metrics.
$\therefore \Sigma^n$ $(n \equiv 1, 2, \bmod 8, n \geq 9)$ with $\alpha(\Sigma^n) = 1$ belongs to (3).

Stolz (1992). A closed 1-connected spin mfd M^n $(n\geq 5)$ with $\alpha(M)=0 \Rightarrow M\in(1)$.
i.e. M admits positive scalar curvature.

use surgery result for scalar curvature by Gromov-Lawson and Schoen-Yau.

SY: A closed M^n $(n\geq 5)$ is obtained by a sequence of surgeries of codim ≥ 3 (i.e. $m\geq 3$).
1979 from another closed mfd N with $K_N>0$. Then M admits positive scalar curv. $K_M>0$.

GL: By the bordism result, any 1-connected closed non-spin mfd M^n $(n\geq 5)$ admits positive scalar curv
1980 since it can be obtained by a sequence of surgeries as SY from some mfds with standard $K_N>0$.

However, homotopy spheres Σ^n are spin mfds $(n\geq 3)$.

In the spin case, the bordism result implies:
A 1-connected closed spin mfd M^n $(n\geq 5)$ admits positive scalar curvature $K_M>0$ if it is spin bordant to some closed spin mfd N with $K_N>0$.

Stolz used stable homotopy theory to show that any closed 1-conn. spin M^n with $\alpha(M)=0$ is spin bordant to the total space N of a bundle with $\mathbb{H}P^2$-fibre and structure group $\mathrm{Isom}(\mathbb{H}P^2)$ (the group of isometries of $\mathbb{H}P^2$).

Given such a $\mathbb{H}P^2$-fibre bundle, one can use O'Neill formulas to produce a metric of positive scalar curvature $K_N>0$ on the total space N from the standard PSC metric on $\mathbb{H}P^2$ and an arbitrary Riem. metric on the base.

Produce: $\begin{cases} g_t|_{\mathcal{V}} = t\, g|_{\mathcal{V}} & \text{- vertical (fibre)} \\ g_t|_{\mathcal{H}} = g|_{\mathcal{H}} & \text{- horizontal (base)} \\ g_t(\mathcal{V},\mathcal{H})=0. \end{cases}$

M Riem submersion with totally geodesic B fibres.

Rk: All sphere bundles over cpt mfds admit positive scalar curvatures.
2) $Ric_F>0, Ric_B>0 \Rightarrow Ric_t>0$, at $t\ll 1$.

$\Rightarrow K_t = \frac{1}{t}K_F + K_B\cdot\pi - t|A|^2$, $\boxed{\begin{cases} Ric_t(U) = Ric_F(u)+t^2|AU|^2 \\ Ric_t(X) = Ric_B(X) - 2t|Ax|^2 \end{cases}}$ $Ric_t(X,U) = t\langle(\check{\delta}A)(X),U\rangle$ $=0$ if Yang-Mills

K_F, K_B: scalar curvature of fibre F and base B respectively.

Hence, If $K_F>0$, F,B compact, let $t\to 0+$, $K_t>0$ for sufficiently ~~large~~ small t.

$\therefore \alpha(M)=0 \xrightarrow[\text{surgery result}]{GL \,\&\, SY} \exists$ positive scalar curvature on M from a $K_N>0$ spin bordant $\mathbb{H}P^2$-bundle.

Cor: 1) $\{$closed 1-connected $\{M^n$ $(n\geq 5)$ non-spin, or spin with $\alpha(M)=0\}\}$
$=$ class (1) $=\{$positive scalar curvature mfds$\}$.

2). $M^n\notin(1) \Leftrightarrow M$ is spin and $\alpha(M)\neq 0$.

$S^n \subset N^{n+m}$ normal b trivial

$S^n\times D^m \subset N^{n+m}$

$\partial(S^n\times D^m) = S^n\times S^{m-1} = \partial(D^{n+1}\times S^{m-1})$

$\tilde{N} := (N\setminus S^n\times D^m)\cup D^{n+1}\times S^{m-1}$

$T_{E_1}E_2 := \mathscr{H}D_{\mathscr{V}E_1}\mathscr{V}E_2 + \mathscr{V}D_{\mathscr{V}E_1}\mathscr{H}E_2,$
$A_{E_1}E_2 := \mathscr{H}D_{\mathscr{H}E_1}\mathscr{V}E_2 + \mathscr{V}D_{\mathscr{H}E_1}\mathscr{H}E_2,$

Classification of exotic spheres w.r.t. positive scalar curvature.

$\Theta_n \ni \Sigma^n \in \begin{cases} (1) & \text{if } n \not\equiv 1,2 \pmod 8, \text{ or } n \equiv 1,2 \pmod 8 \text{ and } \alpha(\Sigma^n)=0; \\ (3). & \text{if } n \equiv 1,2 \pmod 8,\ n \geq 9, \text{ and } \alpha(\Sigma^n)=1 \ (\text{i.e. } \alpha^{-1}(1) \ni \Sigma). \end{cases}$

$(n \neq 4)$.

Nonempty (Recall that α is surjective).

Generalization of Weiping Zhang (Ann. Math. 2017).

Thm1

Let M be a compact spin mfd with a foliation $\mathcal{F}$ (integrable subbundle $F \subset TM$)
if scalar curvature of leafes are positive. $K^F(x) = K^{\mathcal{F}_x}(x) > 0$. (positive leafwise scalar curv)
then $\alpha(M)=0$ and as a corollary, M admits $K_M > 0$ when $\dim M \geq 5$.

Cor2 $\nexists$ foliation (T^n, F) on any torus T^n s.t. positive leafwise scalar curvature $K^F > 0$.

Thm2 Let F be a spin integrable subbundle of TM for a cpt oriented mfd M
If $K^F > 0$ then ~~$\alpha(M)=0$~~. $\hat{A}(M) = 0$

Question1. For a DDBD mfd $E_\psi := E_+ \cup_\psi E_-$, where $\psi: \partial E_+ \xrightarrow[\cong]{\text{diff}} \partial E_-$
constructed from a DDBD of $S^n \cong E_+ \cup_\varphi E_- = E_\varphi$. $\varphi: \partial E_+ \xrightarrow[\cong]{\text{diff}} \partial E_-$,
determine to which classes of (1), (2), (3) E_ψ belongs.
i.e. Find more positive scalar curvature E_ψ;
Are there strongly scalar flat E_ψ? (No if $E_\pm = D^n$)

Rk. $D(E_+)$ or $D(E_-)$ double mflds for classical isop. shown $K>0$ by Tang-Yan
$Ric > 0$ Peng-Qian

Rk. Jianqing Yu - Weiping Zhang (2018).
Let F be an oriented flat vector bundle over a closed spin mfd M with $K_M > 0$
Then $\langle \hat{A}(TM)\, e(F), [M] \rangle = 0$ where $e(F)$ is the Euler class of F.

Question2. For a singular (Riemannian) foliation $(M, \mathcal{F})$ on a closed 1-connected spin mfd M, if scalar curvature of leaves are positive $K^{\mathcal{F}} > 0$ (positive leafwise scalar)
then $\alpha(M) = 0$ and thus M^n admits positive scalar curvature $K_M > 0$ $(n \geq 5)$.
The case SRF of codim 1, i.e. $M = E_\psi$, may help to answer Q1 (partly) by a suitable choices of $E_\pm$.

Defn. Let $F \subset TM^n$ (closed) be an integrable subbundle. g^F be a Euclidean metric on F and
k^F be leafwise scalar curvature. For a covering $\widetilde{M} \xrightarrow{\pi} M$, lift $(\widetilde{F}, g^{\widetilde{F}}) = (\pi^* F, \pi^* g^F)$
(M, F) is called an enlargeable foliation if $\forall \varepsilon > 0$, $\exists$ covering $\widetilde{M} \to M$ & C^∞ map $f: \widetilde{M} \to S^n(1)$
s.t. $f \equiv$ const near infinity and has non-zero degree; $|f_*(X)| \leq \varepsilon |X|$, $\forall X \in \Gamma(\widetilde{F})$.

When $F = TM$ and M is spin, (M, F) is enlargeable $\Leftrightarrow$ M is enlargeable mfd
in the sense of Gromov-Lawson.

Thm (Weiping Zhang 2018) generalization of GL.
Let (M, F) be an enlargeable foliation. Then
(i) if TM spin then NO g^F with $K^F > 0$ over M
(ii) if F spin then NO g^F with $K^F > 0$ over M.

Question3. Generalize Zhang's result to enlargeable singular (Riemannian) foliations.

§3.2. Ricci curvatures

Lohkamp (Ann. Math. 1994). Any M^n $(n \geq 3)$ admits a complete metric with $Ric_M < 0$.

$\therefore$ $\forall$ exotic sphere Σ^n admits $Ric_\Sigma < 0$ metric.
However, for positive Ricci curvature, there exist Σ^n with no such metrics
by the PSC classification in §3.1.

The only difference between PRC $(Ric_M > 0)$ and PSC $(K_M > 0)$ is the Myers' Thm: $\# \pi_1(M) < \infty$
Compact mfd M with positive Ricci curvature has finite fundamental group.

Ehrlich (1976) (claimed by Aubin 1970.)
M with $Ric_M \geq 0$ and $Ric_M|_{p_0} > 0$ at one point $p_0 \in M$.
$\Rightarrow$ the metric can be deformed to one with $Ric_M > 0$ everywhere.

[illegible] If g admits an isometric group action the deformed metric also

M with $Ric_M \geq 0$ and $Ric_M|_{p_0} > 0$ at one point $p_0 \in M$.

$\Rightarrow$ the metric can be deformed to one with $Ric_M > 0$ everywhere.

In fact, stronger: If g_0 admits an isometric group action, the deformed metric also admits an isometric action from the same group.

Rk: If No $Ric|_{p_0} > 0$, then the conclusion cannot be obtained.

e.g. a K3 surface supports a Ricci flat metric but no metric of positive Ricci, scalar, curvature

Question: Whether a 1-connected M^n with $Ric_M \geq 0$ and $K_M > 0$ must always

Open. admit a Ricci positive metric?

Rk: Bach-Hsiang (1987). $\dim = 4m+1 \geq 9$ / Grove-Verdiani-Wilking-Ziller (2006) ≥ 5 } Don't admit invariant metric of $\begin{cases} Sec > 0 \\ Sec \geq 0 \end{cases}$.

1) Cheeger (1973): Kervaire spheres $\Sigma^{4m+1} = B(d, 2, \cdots, 2)$ ($n = 2m$, $d \equiv 3, 5 \bmod 8$).

$\exists$ cohomogeneity-one metric with positive Ricci curvature.

1) construct: $Ric \geq 0$ metric on both disk bundles by DDBD of cohom. 1 action.

which is isometric to a product near the boundaries $\partial E_\pm$

and invariant under the cohom. 1 action.

$\Rightarrow$ The union $E_\varphi = E_+ \cup_\varphi E_-$ admits $Ric_{g_0} \geq 0$ invariant metric

2). Ehrlich result follows from the fact $Ric_{g_0} > 0$ in some nbhd.

2) Hernández-Andrade (1975) Kervaire spheres $\Sigma^{4m+1} = B(d, 2, \cdots, 2)$ and $\Sigma^{4m-1} \subset bP_{4m}$

$bP_{4m} = B(6k-1, 3, 2, \cdots, 2)$, $k = 1, 2, \cdots, |bP_{4m}|$.

i.e

bP_{4m+2} and bP_{4m} are Brieskorn mfds $\subset S^{2n+3} \subset \mathbb{C}^{n+2}$.

Consider the induced (deformed) metric on $\Sigma^{2n+1} \subset S^{2n+3}$ However, too complicated to show $Ric > 0$.

Then replace the zero-locus of P by an explicit nearby variety diff. to the original.

Now the intersection with the sphere (also replaced by an ellipsoid) is ~~Hermitian-ortho~~ -orthogonal.

$\Rightarrow$ given $a_0, \cdots, a_m \geq 2$, $\exists$ integer $N(a_0, \cdots, a_m)$ s.t. $\forall p \geq N$

the Brieskorn mfd $B(a_0, \cdots, a_m, 2, \cdots, 2)$ with p copies of 2 after a_m

has a metric of $Ric > 0$.

$\therefore$ Infinitely many exotic Σ^{4m-1} with $Ric > 0$.

3) Fibre bundle metric.

Vilms (1970) A cpt fibre bundle with a Lie structure group G, $F \hookrightarrow M \xrightarrow{\pi} B$, ($\cancel{Ric_F > 0}$, $Ric\, B > 0$)

G acts on F as isometries, a principal connection on the associated principal bundle

$\Rightarrow$ $\exists$ a canonical metric on M s.t. π is Riem. submersion with totally geodesic fibres.

$\therefore$ $Ric_F > 0$, $Ric\, B > 0$ $\Rightarrow$ $\exists$ $Ric_M > 0$.

$\therefore$ Milnor spheres $\exists$ $Ric_{\Sigma^7} > 0$. Moreover, some $\Sigma^{15} = S^7$-bundles over S^8,

$\therefore$ $\exists$ $Ric_{\Sigma^{15}} > 0$. (Poor 1975; Nash 1979)

4). surgery with positive Ricci curvature.

Sha-Yang (1991) Assume in a tube $\overset{S^n\times D^m}{\cong}$ of $S^n \hookrightarrow M^{n+m}$ with trivial normal bundle the metric $g_0 \overset{isom.}{\cong} g_{S^n(r_1)} \times g_{D^m(r_2)}$ with $n \geq 1$, $m \geq 3$, $\frac{r_1}{r_2} \ll 1$. $\Rightarrow$ surgery preserve positive Ricci curvature.

Wraith (1998). Under the same metric assumptions as above, surgery works if (under tighter dim conditions) different trivialisations of the normal bundle are used.

$\Rightarrow$ $\boxed{\text{All } bP_{n+1} \ni \Sigma^n \text{ admit Ricci positive metrics.}}$ Wraith (1997)

plumbing surgeries. Notice (n odd. i.e. $n = 4m+1$) (Kervaire) or $4m-1$). all done.

$M^{4m} = D(TS^{2m}) \,\square\, D(TS^{2m})$. $\partial M \cong \Sigma^{4m-1} = S(TS^{2m}) \cup_{S^{2m}\times S^{2m-1}} D^{2m}\times S^{2m-1}$

effect of plumbing on the boundary ∂M is precisely a surgery of dim $= 2m$, codim $= 2m$.

In fact, more obtained by Wraith (1997) on $\Theta_n \setminus bP_{n+1}$:

Every mfd arising as the boundary ∂M of a plumbing of disc bundles over spheres according to a simply connected graph (e.g. E_8) admits a Ricci positive metric.

Ex.

$E_+ \in KO(S^4)$ generator, D^5-bundle over S^4; E_-: the nontrivial D^4 bundle over S^5.

Then $M^9 = E_+ \,\square\, E_-$

$\partial M \cong \Sigma^8 \in \Theta_8 \setminus bP_9$. (the unique exotic 8-sphere)

admits a Ricci positive metric.

This is the only known example of an exotic sphere not bounding a parallelisable mfd that admits Ricci positive metrics.

Open Question: Whether Which or $\Theta_n \setminus bP_{n+1} \ni \Sigma^n$ with $K_M > 0$ admit Ricci positive metric?

5). Grove-Ziller (2002). A closed mfd M with a cohomogeneity-one action supports an invariant metric with positive Ricci curvature if and only if $\pi_1(M)$ is finite.

$\therefore$ Certain exotic spheres, like Kervaire spheres Σ^{4m+1} (Cheeger) or Milnor spheres Σ^7 admit $Ric > 0$ metrics. Any others?

6) Sasaki-Einstein metric with PSC

(M, g) Riem. mfd, its Riemannian cone metric $(M \times \mathbb{R}^+, r^2 g + dr^2)$, $r > 0$.
M^n (n odd) with a 1-form θ is contact iff $r^2 d\theta + 2r\, dr \wedge \theta$ is symplectic.
A contact Riemannian mfd (M, g, θ) is Sasakian, if
Riem. cone $(M \times \mathbb{R}^+, r^2 g + dr^2)$ is Kähler mfd with Kähler form $r^2 d\theta + 2r\, dr \wedge \theta$.

Rk. (M^n, g) is Sasakian if holonomy group of $(M \times \mathbb{R}^+, r^2 g + dr^2)$ reduces to a subgroup of $U(\frac{n+1}{2})$.

Let J be the complex str. on the Riem. cone $(M \times \mathbb{R}^+, r^2 g + dr^2)$.
$\xi := J(\frac{\partial}{\partial r})$: the Reeb vector field of the contact str. (M, g, θ).
~~the~~ is a unit Killing vector field satisfying
$K_{sec}(\xi, X) = 1$. sectional curvature of every plane containing ξ is 1.

(M, g) is called 3-Sasakian if holonomy group of $(M \times \mathbb{R}^+, r^2 g + dr^2)$ reduces to a subgroup of $Sp(\frac{n+1}{4})$. $\Rightarrow n \equiv 3 \mod 4$.
the Riem. cone $(M \times \mathbb{R}^+, r^2 g + dr^2)$ is hyper-Kähler.
$\rightsquigarrow \xi_i := J_i(\frac{\partial}{\partial r})$, J_1, J_2, J_3 Reeb v.f. of contact strs (M, g, θ_i) $i = 1, 2, 3$.
o.n. Killing vector fields which give rise to a locally defined free isometric $SU(2)$-action on M.

Open question: Whether there exist 3-Sasakian metrics on exotic spheres Σ^{4k+3}?
(Rk: bP_{2n} admit Sasakian PRC.)
~~Facts:~~ Defn: Sasaki-Einstein if
the Riem. cone of a Sasaki ~~Einstein~~ mfd is Ricci flat.

Facts: $\boxed{\text{Every 3-Sasakian manifold is Einstein with PSC}}$.
The Riem. cone of 3-Sasakian mfd is hyperKähler which are Ricci-flat.
and thus Calabi-Yau mfds, and ~~get~~ (M, J_1) is holomorphically symplectic.
(M compact) equivalent (equipped with a holo. symp. form).

Examples of 3-Sasakian: S^{4k-1}, $\mathbb{RP}^{4k-1}$.   Any compact hyperKähler 4-manifold is either a K_3 surface or T^4.
(All S^{2n+1}, $S^2 \times S^3$ (with a homogeneous metric).
Sasaki showed Brieskorn mfds admit Sasakian structures.)

Boyer - Galicki - Nakamaye (2003).

Defn. A Sasakian mfd M is said to be positive if its basic first Chern class $c_1(\mathcal{F}_\xi)$ can be represented by a basic positive definite $(1,1)$-form.

Using Theorem of El Kacimi - Alaoui : If $c_1(\mathcal{F}_\xi)$ is represented by a real basic $(1,1)$-form ρ^T, then it is the Ricci curvature form of a unique transverse Kähler form ω^T in the same basic cohomology class as $d\theta$.

Prop. If $(M^{2n+1}, g, \xi \text{ or } \theta)$ is a complete, positive Sasakian structure on then M is compact and $b_1(M) = b_1^B(\mathcal{F}_\xi) = 0$. (the 1st Betti number)

Thm. Let $(M^{2n+1}, g, \xi \text{ or } \theta)$ be a compact, positive Sasakian structure. Then M admits a Sasakian str. $(g', \xi', \theta', \Phi')$ with positive Ricci curvature.

i.e. closed M^{2n+1} admits a positive Sasakian str $\Rightarrow$ M admits PRC, $Ric > 0$.

Cor On all Σ^{2n+1} that bound parallelisable mfds there exist Sasakian metric of PRC.
i.e. bP_{4m} and bP_{4m+2} $\subset$ PRC.
$B(6k-1,3,2,\cdots,2)$ Kervaire $B(d,2,\cdots,2)$. New proof of result of Wraith.

Boyer - Galicki - Kollár (Thomas) (2005). Ann. Math. 2005.
All Kervaire spheres bP_{4m+2}, and $bP_8 = \Theta_7$, $bP_{12} = \Theta_{11}$, $bP_{16} \subsetneq \Theta_{15}$ admit an Einstein metric with PSC. (In fact Sasakian-Einstein)

Conj (BGK). All bP_{4m} (odd dim) admit an Einstein metric with PSC. (Sasakian-Einstein).
(All odd dim homotopy spheres that bound parallelisable mfds) $= bP_{2m}$.

Question Are there Non-Einstein gradient Ricci soliton (shrinking) on bP_{2m}?

Question ? give examples of compact, non-Einstein, non-Kähler gradient shrinking Ricci solitons.

$$Ric + \nabla^2 f = \rho g \quad \begin{cases} \text{shrinking} & \rho > 0 \\ \text{steady} & \rho = 0 \\ \text{expanding} & \rho < 0 \end{cases}$$

$\rho = 0$, $\rho < 0$: in compact case $\Rightarrow$ must be Einstein.

§3.3 Sectional curvature

Hadamard-Cartan thm. $\exp: T_pM \to M$ has maximal rank everywhere on a Riem mfd M^n with $\mathrm{Sec}_M \le 0$, and hence exp is a covering map and $T_pM \cong \mathbb{R}^n$ is the universal covering space.

Cr. No homotopy sphere can admit non-positive sectional curvature.

Fundamental Question. Are there exotic spheres support a metric with $\mathrm{Sec} > 0$ everywhere?

Rk: By $\frac{1}{4}$-pinching thm of Brendle-Schoen, exotic spheres cannot admit $\frac{1}{4} < \mathrm{Sec} \le 1$.

1) Gromoll-Meyer sphere $\Sigma^7_{2,-1} \in \Theta_7$ the generator of $\Theta_7 = bP_8 = \mathbb{Z}_{28}$ (1974).

Totaro (2002): It is the only exotic sphere that can be realised as a biquotient of Lie groups.

Consider the action of $\underset{Sp(1)}{S^3 \times S^3}$ on $Sp(2)$ given by

$$(q_1, q_2)\cdot A = \begin{pmatrix} q_1 & 0 \\ 0 & q_1 \end{pmatrix} A \begin{pmatrix} \bar{q}_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$Sp(n) = \{A \in GL(n, \mathbb{H}) \mid A\bar{A}^t = I = \bar{A}^t A\}$

$\dim = n(2n+1)$

$sp(n) = \{A + \bar{A}^t = 0\}$

The action is free, and so is its restriction on the diagonal $D \cong S^3$ in $S^3 \times S^3$. $Sp(2)/S^3 \times S^3 \cong S^4$.

The Gromoll-Meyer sphere $\Sigma^7_{GM} := Sp(2)/D$.

$$Sp(2)/S^3\times S^3 \xrightarrow{\overset{\text{diff}}{\cong}} S^4$$

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \longmapsto (2\bar{b}d, |b|^2 - |d|^2)$$

$$\begin{array}{ccc} S^3\times S^3/D & \cong & S^3 \\ \downarrow & & \downarrow \\ Sp(2)/D & = & \Sigma^7 \\ \downarrow & & \downarrow \\ Sp(2)/S^3\times S^3 & \cong & S^4 \end{array}$$

S^3-bundles over S^4

$(m,n) \in \mathbb{Z} \oplus \mathbb{Z} \cong \pi_3 SO(4)$. $f_{(m,n)}: (\mathbb{R}^4 - 0) \times S^3 \to (\mathbb{R}^4 - 0) \times S^3$

$(u, q) \mapsto \left(\frac{u}{|u|^2}, \left(\frac{u}{|u|}\right)^m q \left(\frac{u}{|u|}\right)^n\right)$

S^4

$D^4 \times S^3 \cup D^4 \times S^3$

$p^m q\, p^n$

$\left.\begin{array}{l} m+n=1 \\ (m-n)^2 \not\equiv 1 \mod 7 \end{array}\right\} \Rightarrow \Sigma^7_{(m,n)}$ exotic sphere.

Thm. $\Sigma^7_{GM} = \Sigma^7_{(2,-1)}$. admits $\mathrm{Sec} \ge 0$ using bi-invariant metric of $Sp(2)$ and the action of $S^3\times S^3$ is isometric.

$\exists$ effective isometric action of $O(2) \times SO(3)$.

In a nbhd, there're planes with $\mathrm{Sec} = 0$.

$$K(X,Y)\circ\pi = \tilde{K}(\tilde{X}, \tilde{Y}) + \frac{3}{4}|[\tilde{X}, \tilde{Y}]^{\nu}|^2$$

horizontal lift

G cpt Lie gp

$K(X,Y) = \frac{1}{4}|[X,Y]|^2$

the connected sums of two rank-one symmetric spaces with any orientations.

Almost ← Only exceptions: Cheeger (1973), Grove-Ziller (2000)

~~of Berger sphere~~ e.g. $\mathbb{CP}^4 \# \mathbb{HP}^2$. Milnor spheres, S^3-bundles over S^4. $Sec \geq 0$ (infinitely many)

Rk. All known examples of closed 1-conn. mfd with $Sec > 0$ are biquotients.

~~Defn. A biquotient $G/\!/H = G/H$ with $H \to G \times G$ acts freely on G by $g_1 g g_2^{-1}$.~~

Defn. A closed mfd is called a biquotient if it is diffeomorphic to $K \backslash G / H$ for some compact Lie group G with closed subgroups K and H such that K acts freely on G/H.

Every biquotients has a Riem. metric of $Sec \geq 0$.

Totaro (2002): 2-connected biquotients in any given dimension are finitely many.

Wilhelm (2001). deform GM metric s.t. almost positive sectional curv. $Sec > 0$ a.e. with effective isometric $SO(3)$ action.

Eschenburg (2002). a family of metrics on Σ^7_{GM} with $Sec > 0$ a.e.

Idea: consider bi-invariant metrics on $Sp(2) \times S^3 \times S^3$, and then to quotient by $S^3 \times S^3$ diagonal action of $S^3 \times S^3$. $\Rightarrow$ quotient $\cong Sp(2)$.

{metric is normal homogeneous and thus $Sec \geq 0$.

choose suitable bi-inv. metrics,

deduce well-defined metric on Σ^7_{GM} with $Sec > 0$ a.e.

and the points possessing zero curvature planes form a hypersurface.

Eschenburg (1982 Inv. Math). All known $Sec > 0$ mfds can be obtained by the Idea above. the first inhomogeneous PSecC mfds.

2). Grove-Ziller's metrics (cohomogeneity-one action).

10 Milnor spheres, all S^3-bundles over S^4, (some S^7-bundles over S^8 (with some Σ^{15})) ?

↳ (effective isometric $O(2) \times SO(3)$ action) $\exists$ infinitely many inequiv., isom, almost-free actions of $SO(3)$ (free)

$M/G = S^1$ or $[a, b]$ $\longleftrightarrow$ $M = E_+ \cup_P E_- = E_P$ DDBD. over singular orbits

all orbits principal, $\exists$ inv. metric with $Sec \geq 0$, but no PRC by the Myers Thm, since $\# \pi_1 M = \infty$.

Each orbit (principal or singular) is a homogeneous space equipped with homogeneous metric.

S^3-bundles over S^4 associated to $SO(4)$ P-bundles over S^4 ($S^3 \times S^3$)

cohom. 1 action from $SO(4) \times SO(3)$ with codim 2 singular orbits

Thm: $\forall$ cohomogeneity-one mfd with codim 2 singular orbits admits an invariant metric with $Sec \geq 0$.

all sphere bundles over S^4, S^5 $K \geq 0$

88/144 $S^3 \to M \to S^4$ $K \geq 0$

Grove-Verdiani-Wilking-Ziller (2006) ← cohomogeneity-one action by $SO(2m+1)\times S^1$

Kervaire spheres $\Sigma^{4m+1} = B(d,2,\cdots,2)$ in dim ≥ 5 admits NO invariant metrics of $Sec \geq 0$.

Weiss (1993). Σ^7_{GM} NO quarter-pinched metric. (In fact, all Σ^n by Brendle-Schoen)

3). Goette-Kerin-Shankar (2017) Ann. Math. 2020.

All $\Sigma^7 \in \Theta_7$ admit an $SO(3)$-invariant metric of $Sec \geq 0$.

$\Rightarrow$ 4 non-Milnor spheres admit $Sec \geq 0$.

Idea: generalise Grove-Ziller's S^3-bundles P^{10} over S^4 (4-parameter family)

to 6-parameter family of cohomogeneity 1 mflds $P^{10}_{a,b}$ with free dom S^3 actions

with both singular orbits of codim 2.

diffeomorphism type → $M^7_{a,b} = P^{10}_{a,b}/S^3$

Eells-Kuiper invariant $\mu(M^7_{a,b})$ & $\mathfrak{t}$-invariant of Crowley.

Recall. action $G\times M \to M$ is said to be of cohomogeneity one if

$\exists$ an orbit of codim 1, or equivalently, if $\dim(M/G) = 1$.

M is called a cohomogeneity-one (G-)manifold.

M closed $\xrightarrow{\pi} M/G = S^1$ or $[-1,1]$ $\quad \pi^{-1}(\pm 1) = G/K_\pm$ singular orbits

$\#\pi_1(M) = \infty$. $\quad \pi^{-1}(t) = G/H$ principal orbits

$t \in (-1,1)$.

singular isotropy groups

principal isotropy group

$$\begin{tikzcd} & G & \\ K_- \arrow[ur, "j_-"] & & K_+ \arrow[ul, "j_+"'] \\ & H \arrow[ul, "i_-"] \arrow[ur, "i_+"'] & \end{tikzcd}$$

$$\boxed{M = (G\times_{K_-} D^{l_-}) \cup_{G/H} (G\times_{K_+} D^{l_+})}$$

$K_\pm/H = S^{l_\pm - 1}$ normal spheres of $G/K_\pm$.

Extension: If $\exists$ cpt Lie group L and homomorphisms $\varphi_\pm : K_\pm \to L$ s.t. $\varphi_-\circ i_- = \varphi_+ \circ i_+$

then cohom-1 $(G\times L)$-mfld P with group diagram

$$\begin{tikzcd} & G\times L & \\ K_- \arrow[ur, "(j_-\text{,}\varphi_-)"] & & K_+ \arrow[ul, "(j_+\text{,}\varphi_+)"'] \\ & H \arrow[ul, "i_-"] \arrow[ur, "i_+"'] & \end{tikzcd}$$

induce a principal L-bundle $L \to P \to M$

($\{e\}\times L \subset G\times L$ action free.) with codim of singular orbits equal in M and P.

$$\begin{tikzcd} & S^3 & \\ Pin(2) \arrow[ur, dash] & & Pjn(2) \arrow[ul, dash] \\ & Q \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd} \qquad L = S^3\times S^3 \xrightarrow{\text{extend}} \qquad \begin{tikzcd} & S^3\times S^3\times S^3 & \\ Pin(2)_{\underline{a}} \arrow[ur, dash] & & Pjn(2)_{\underline{b}} \arrow[ul, dash] \\ & \Delta Q \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

$Q = \{\pm 1, \pm i, \pm j, \pm k\}$ $\quad$ ΔQ is the diagonal embedding of Q in $S^3\times S^3\times S^3$.

$Pin(2) = \{e^{i\theta} \mid \theta\in\mathbb{R}\} \cup \{e^{i\theta}j \mid \theta \in \mathbb{R}\}$ $\quad$ $Pin(2)_{\underline{a}} = \{(e^{ia_1\theta}, e^{ia_2\theta}, e^{ia_3\theta})\} \cup \{(e^{ia_1\theta}j, e^{ia_2\theta}j, e^{ia_3\theta}j)\}$

$Pjn(2) = \{e^{j\theta} \mid \theta\in\mathbb{R}\} \cup \{e^{j\theta}i \mid \theta\in\mathbb{R}\}$.

$a_i, b_i \equiv 1 \mod 4$ to ensure $\Delta Q \subset Pin(2)_{\underline{a}}, \ldots$

$S^4 = D^2(\mathbb{RP}^2) \cup D^2(\mathbb{RP}^2)$

as a cohomog. 1 S^3-mfld

with $K_- = Pin(2)$, $K_+ = Pjn(2)$, $H = Q$.

If $a_1 = b_1 = 1 \Rightarrow P^{10}_{\underline{a},\underline{b}}$ is $S^3\times S^3$-principal bundles over S^4.

reduces to GZ construction.

$\{1\}\times\Delta S^3 \subset \{1\}\times S^3\times S^3$ acts freely on the left of $P^{10}_{\underline{a},\underline{b}}$.

$\Rightarrow (\{1\}\times\Delta S^3)\backslash P^{10}_{\underline{a},\underline{b}} \cong S^3$-bundles over S^4

$= P^{10}_{\underline{a},\underline{b}} \times_{S^3\times S^3} S^3$

give 10 Milnor spheres.

$\gcd(a_1, a_2 \pm a_3) = \gcd(b_1, b_2 \pm b_3) = 1 \Rightarrow \{1\}\times\Delta S^3$ acts freely on $P^{10}_{\underline{a},\underline{b}}$

$\Rightarrow M^7_{\underline{a},\underline{b}} := \{1\}\times\Delta S^3 \backslash P^{10}_{\underline{a},\underline{b}}$

$\exists\ \Sigma^n$ with NO $K\geq 0$ $n\equiv 1,2 \bmod 8$

$\Theta_n \setminus bP_{n+1}$

Dim n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
order Θ_n	1	1	1	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16	523264	24
bP_{n+1}	1	1	1	1	1	1	28	1	2	1	992	1	1	1	8128	1	2	1	261632	1

$K>0$ & $Ric>0$ $\leftarrow$ $Sec\geq 0$

(bP_{4k+2}, bP_8, bP_{12}, bP_{16} Einstein)

Conj. bP_{4k} Einstein ? Q: $bP_{n+1} \subset Sec\geq 0$? $\exists ?\ \Sigma^7$ $Sec>0$?

§4 Isoparametric foliations on exotic spheres.

Fundamental construction theorem.

Theorem (Qian-Tang, 2015)

On a DDBD mfld ($\operatorname{codim} M_\pm \geq 2$) — disk bundles of rank ≥ 2

$$N = D(\nu M_-) \cup D(\nu M_+) \cong E_- \cup_\varphi E_+ =: E_\varphi \qquad \varphi: \partial E_- \xrightarrow{\cong} \partial E_+$$

with the canonical singular foliation $\mathcal{F}_\varphi$. $\Rightarrow$ $\exists$ metric g_φ s.t. $(E_\varphi, g_\varphi, \mathcal{F}_\varphi)$ isop.

To classify foliated diffeomorphism classes of isop. foliations, we only need to
(map leaves to leaves)
classify DDBD structures $E_+, E_-, \varphi \in \mathrm{Diff}(\partial E_- \to \partial E_+)$.
$(E_\varphi, \mathcal{F}_\varphi)$.

Now forget metric.

thm (Ge-Tang, 2013). There are NO isop. fol on exotic Σ^4 (if exist).

Thm (Ge, 2016) On any exotic n-sphere Σ^n $(n>4)$

$$\{[(\Sigma^n, \tilde{\mathcal{F}})]\} \xleftrightarrow{1-1} \{[(S^n, \mathcal{F})]\}$$

$[\,,\,]$ denotes fol. diff. class.

$$(\Sigma^n_\phi = E_- \cup_{d_\phi \circ \varphi} E_+,\ \tilde{\mathcal{F}}_{d_\phi \circ \varphi}) \xleftarrow{1-1} (S^n = E_- \cup_\varphi E_+,\ \mathcal{F}_\varphi)$$

$\cong\downarrow$ $\qquad\qquad$ $\downarrow\cong f$ $\qquad$ $f|_{(E_-)} = E$

$$(\Sigma^n_\phi = E_- \cup_{d_\phi \circ \psi} E_+,\ \tilde{\mathcal{F}}_{d_\phi \circ \psi}) \xleftarrow{1-1} (S^n = E_- \cup_\psi E_+,\ \mathcal{F}_\psi)$$

$\Theta_n \cong \Gamma_n \cong$

DDBD-twisted sphere $\qquad$ Smale's twisted sphere

$$E_{d_\phi \circ \varphi} = E_- \cup_{d_\phi \circ \varphi} E_+ = D^n \cup_\phi D^n = \Sigma^n_\phi,\ \phi \in \mathrm{Diff}^+(S^{n-1}).$$

(WLOG, $\phi: S^{n-1} = D_-^{n-1} \cup_{id} D_+^{n-1} \circlearrowleft$ with $\phi|_{D_-^{n-1}} = id$.

$\partial E_+ = D_+^{n-1} \cup M'$ def. $d_\phi: \partial E_+ \xrightarrow{\cong} \partial E_+$ s.t. $d_\phi|_{D^{n-1}} = \phi|_{D_+^{n-1}}$, $d_\phi|_{M'} = id$.)

Question: isotopy thm on disk $\rightsquigarrow ? \rightarrow$ isotopy thm on disk bundles ?

M^{n+1}

$E_- \quad \partial E_- \xrightarrow{\varphi} \partial E_+ \quad E_+ \qquad M^{n+1}$

$$\begin{tikzcd}
 & G & \\
K_- \arrow[ur, "j_-"] & & K_+ \arrow[ul, "j_+"'] \\
 & H \arrow[ul, "i_-"] \arrow[ur, "i_+"'] &
\end{tikzcd}$$

$B_- = G/K_-$ $\qquad$ $B_+ = G/K_+$

$S^{\ell_-} = K_-/H$ $\qquad$ $S^{\ell_+} = K_+/H$

a manifold

$$M = G \times_{K_-} D^{\ell_-+1} \cup_{G/H} G \times_{K_+} D^{\ell_++1}$$

$$G/H \cong G \times_K K/H \leftarrow G \times K/H$$

$k\cdot(g, \bar{k}H) = (gk^{-1}, k\bar{k}H)$

$gH \mapsto (g, H)$

$gkH \leftarrow (g, kH)$

G $\quad \underline{\langle , \rangle}$, $\times$ $\lambda \langle , \rangle|_{K/H}$ on $G \times K/H$

$G \times \sqrt{\lambda}\, K/H$

$Q_\lambda|_m = \langle , \rangle$, $\quad Q_\lambda|_p = \frac{\lambda}{\lambda+1}\langle , \rangle$

$H \subset K \subset G$

$\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$

$m = \mathfrak{k}^\perp$, $p = \mathfrak{h}^\perp \cap \mathfrak{k}$

$T_H G/H \cong m + p \qquad T_H K/H \cong p$

$G \quad Q$ bi-inv.

Def Q_a: $\langle , \rangle|_m = Q$, $\langle , \rangle|_{\mathfrak{k}} = aQ$

$Q_a \rightsquigarrow G/H \qquad \langle , \rangle|_m = Q,\ \langle , \rangle|_p = \frac{\lambda}{\lambda+1} aQ$

If $a = \frac{\lambda+1}{\lambda} > 1 \Rightarrow Q_a|_{G/H} = Q|_{G/H}$

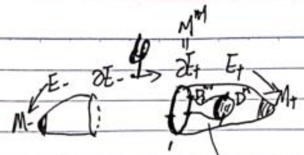
$Q_a|_G$

$K(X, Y) \geq 0$ iff $a \leq 1$ in general

if $a \leq \frac{4}{3}$ in p abelian

$K/H = S^1 \quad Q \quad dr^2 + f(r)^2 d\theta^2 \qquad f$ odd, $f'(0) = 1$

$$(G \times_K D^2) \cup_{G/H} (G \times_{K_+} D^2)$$



$$\partial(E_+\setminus D^n) = \overset{M^{n-1}}{\overset{\|}{\partial E_+}} \sqcup S^{n-1}$$

$$\underset{d\phi}{\circlearrowleft} \qquad \underset{\phi}{\circlearrowleft}$$

$D_+^{n-1}\times[0,1] \subset E_+\setminus D^n$ $\qquad$ $E_+' = E_+\setminus D^n$

extend $\phi|_{D_+^{n-1}}$ to Ψ_ϕ s.t. $\Psi_\phi \in \mathrm{Diff}(E_+\setminus D^n)$

with $\Psi_\phi|_M = d\phi$, $\Psi_\phi|_{S^{n-1}} = \phi$.

double mfld $D(E_+) = E_+ \cup_{id} E_+$ is a S^{n-1}-bundle over M_+ bounding W_+^{n+1}

$$\partial W_+ = D(E_+) = E_+ \bigcup_M E_+' \bigcup_{S^{n-1}} D^n$$

$$\uparrow \Psi_\phi$$

$$\partial D^{n+1} = S^n = \Sigma_\phi = E_- \bigcup_\varphi E_+' \bigcup_{S^{n-1}} D^n$$

$$W^{n+1} := D^{n+1} \bigcup_{\Psi_\phi} W_+^{n+1}. \quad \text{with} \quad \partial W = (E_- \cup_{\Psi_\phi\circ\varphi} E_+) \sqcup (D^n \cup_{\Psi_\phi} D^n)$$

is a h-cobordism between $E_{d\phi\circ\varphi}$ and Σ^n_ϕ.

$$= (E_- \cup_{d\phi\circ\varphi} E_+) \sqcup (D^n \cup_\phi D^n)$$

$$= E_{d\phi\circ\varphi} \sqcup \Sigma^n_\phi.$$

Partial classification Σ^n $(n\neq 4)$ $\quad E_\pm = D^n$

$n\neq 5.$ $\{[(S^n, \mathcal{F})] \mid M_\pm = pt\}$ Unique $\Leftarrow$ (pseudo-isotopy $\Leftrightarrow$ isotopy) $\mathrm{Diff}(S^{n-1})$

$n=5.$ $\{[(S^5, \mathcal{F})] \mid M_\pm = pt\}$ Unique $\Leftrightarrow$ $\pi_0(\mathrm{Diff}^+(S^4)) = 1$? Open

"Exotic" isop. fol. on S^n $\quad\therefore\quad$ $S^m\times D^k$ admit non-trivial disk bundle str.

not appear on the round metric classically

$(m,k) = (7,4), (8,4), (9,4), (11,4), (11,5), (11,6)$

$S^{11}, S^{12}, S^{13}, S^{15}, S^{16}, S^{17}$ admit isop. fol. with $M_\pm = (S^m, S^{k-1})$

non-equivalent to $g=2$ cases.

More $\Sigma^n\times D^l \cong S^n\times D^l$ $(l\geq n+2)$. $\because TS^n \cong T\Sigma^n$. S^{14} has 15 isop. classes. with $M_\pm = (\Sigma^7, S^6)$.

$$\underbrace{K/H = S^1}_{(G\times_{K_-} D^2)\cup_{G/H}(G\times_{K_+} D^2)} \quad Q \quad \underbrace{dr^2 + f(r)^2 d\theta^2}$$

f odd

$f'(0) = 1$