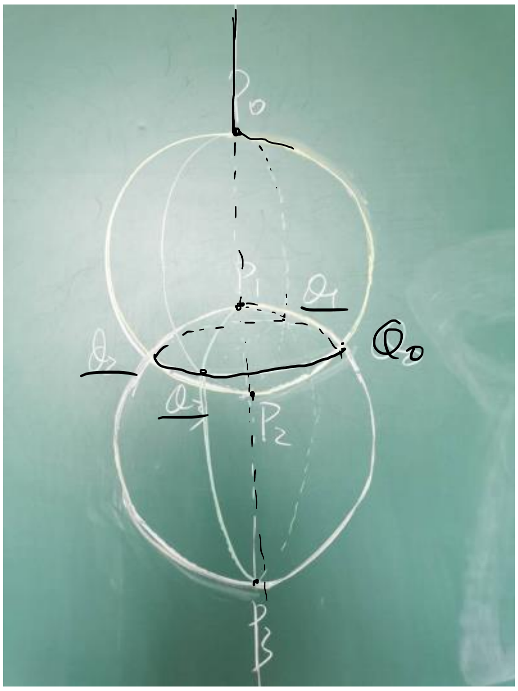
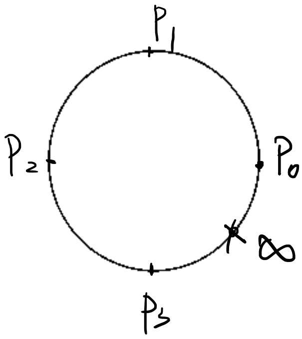
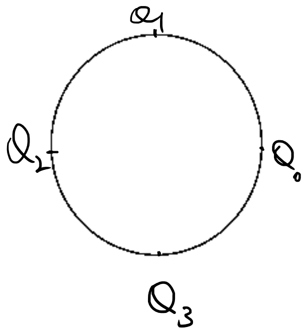


$$S^3 \subset \mathbb{R}^4 \ni (\underline{x_1, x_2}, \underline{x_3, x_4})$$

$2(k+1)$.

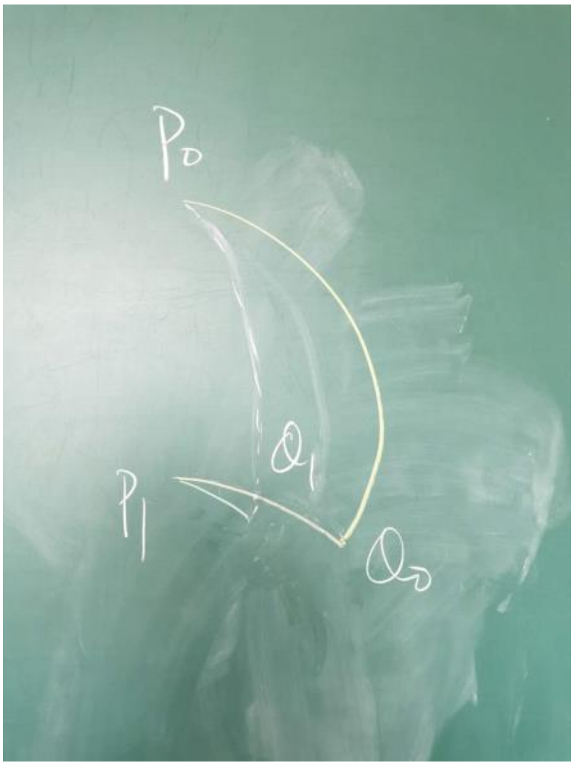


$2m+2$



$\gamma = (\cos t, \sin t, 0, 0)$. $\tilde{\gamma} = (0, 0, \cos t, \sin t)$

$$R_\gamma(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, -x_4)$$

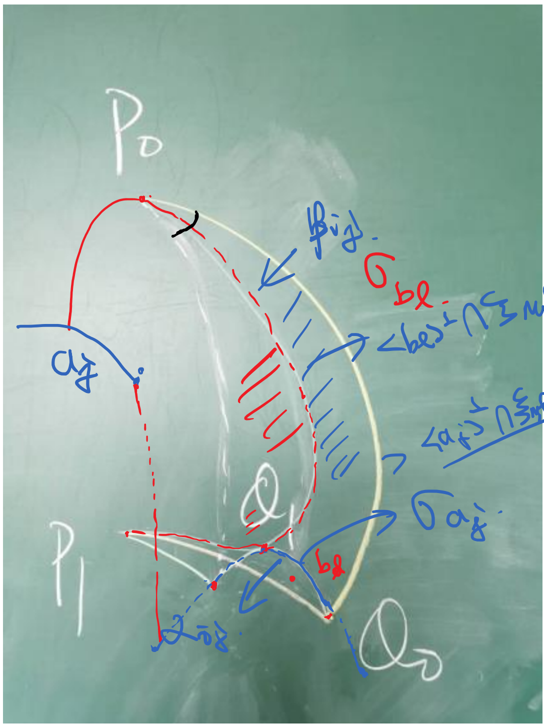


Morrey.

[illegible] (Meeks-Yau 1982 Math Z.)

$\exists$ minimal area disk Σ_{ij}

$\partial \Sigma_{ij} = P_i Q_j P_{i+1} Q_{j+1}$



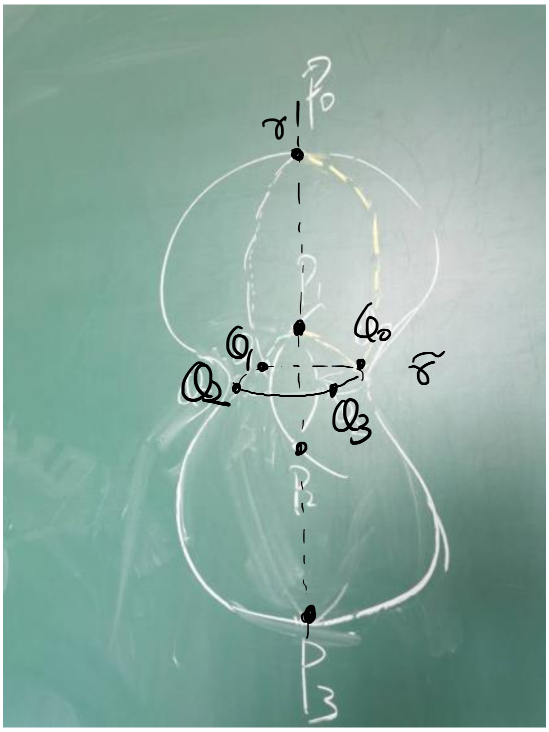
Schwarz reflection thm.

$\{P_i, Q_j\}$ inv. under reflection γ_{kl}

$\Rightarrow \Sigma = \bigsqcup_{(i,j)\text{ even}} \Sigma_{ij}$
(i,j) odd.

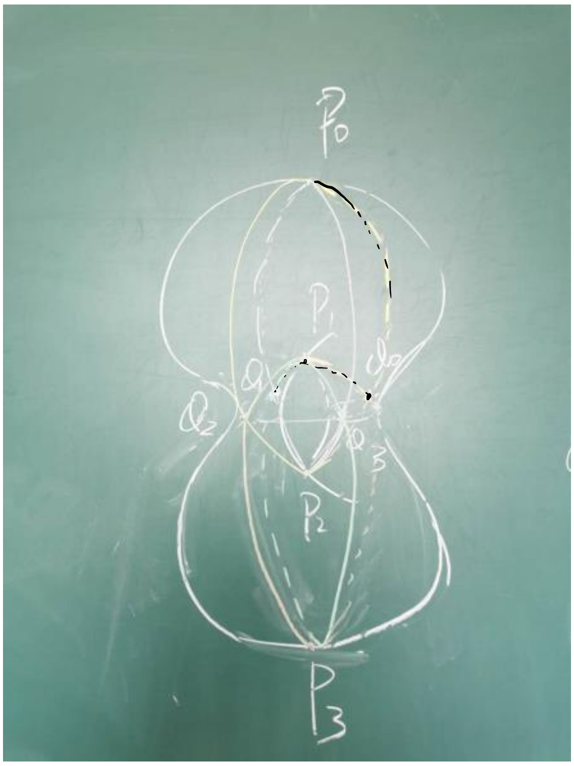
S^3

f_1 $\quad$ S_{ij}^+



$\gamma \cap \Sigma = \{P_j\}$ $2(k+1)$

$\tilde{\gamma} \cap \Sigma = \{Q_j\}$ $2(m+1)$



$\underline{P_0Q_0P_2Q_2}$

$\Updownarrow$

geodesic circle

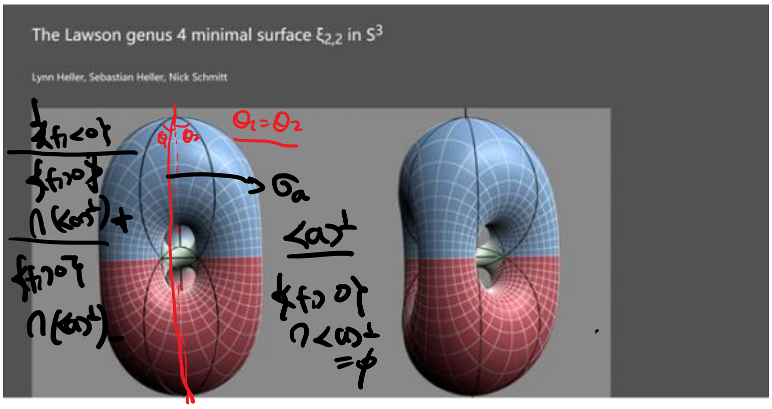
$\gamma_{00} \subset \Sigma$

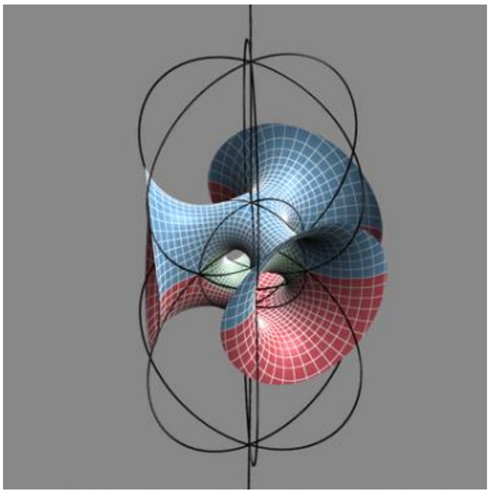
$\underline{P_1Q_0P_3Q_2} \subset \Sigma$

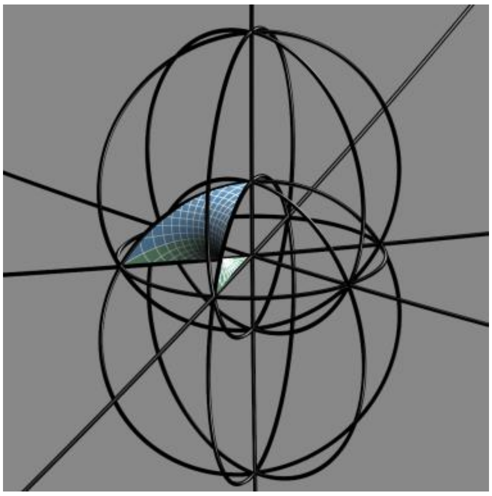
$\underline{P_0Q_1P_2Q_3} \subset \Sigma$

$\underline{P_1Q_1P_3Q_3} \subset \Sigma$

https://www.math.uni-tuebingen.de/user/nick/







(1). geodesic 4-gon $P_iQ_jP_{i+1}Q_{j+1}$. sym.

(2) Σ_{ij} Area maximizer $\Rightarrow$ embedding $\partial\Sigma_{ij} = P_iQ_jQ_{i+1}Q_{j+1}$. $\Sigma_{ij}\cong D$

(3) Schwarz reflections $\Rightarrow$ $\Sigma = \bigcup\limits_{(i,j)\text{ even}} \Sigma_{ij}$

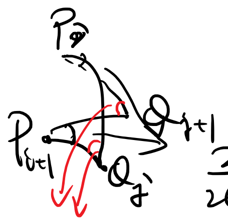
(4) Σ embedding. smooth at $\partial\Sigma_{ij}$ $\left\{\begin{array}{l}\text{odd. density } 1\\ \perp \partial\Sigma_{ij}\end{array}\right.$

(5) $\{P_i\}\{Q_j\}$ Simon Brendle. $\Rightarrow \partial\Sigma_{ij}$ immersion

singularity removing

(6). genus of Σ ?

$\partial\Sigma_{ij} = \Gamma_{ij}$.



$\frac{2\pi}{2(m+1)} = \frac{\pi}{m+1}$.

$\frac{2\pi}{2(k+1)} = \frac{\pi}{k+1}$.

$-\int_{\partial\Sigma_{ij}} k\,dM = 2\pi - \frac{\pi}{m+1} - \frac{\pi}{k+1}$
$- \frac{\pi}{m+1} - \frac{\pi}{k+1}$.

$\# \Sigma_{ij} = 2(k+1)(m+1) \Rightarrow$

$4\pi(g-1) = -\int_\Sigma k\,dM = 2(k+1)(m+1)\cdot\left(2\pi - \frac{2\pi}{m+1} - \frac{2\pi}{k+1}\right)$
$= 4\pi(km-1)$

$\rightarrow g = km$.

(6)'. Another proof: Euler formula.

$\chi(M) = v - e + f$
顶点 边 面.

$v = 2(m+1) + 2(k+1)$.

$f = 2(k+1)(m+1)$.

$e = 2(k+1)(m+1)\cdot 4/2 = 4(k+1)(m+1)$

$\Rightarrow \chi(M) = 2 - 2km = 2-2g \Rightarrow g = km \quad \underline{(m \geq k)}$

<u>Rmk:</u> (1) $k=0 \quad g=0 \quad S^2$

(2) $k=1, m=1 \Rightarrow T^2$ (Clifford torus.

(3) $k=1, m=g \Rightarrow g(\xi) = \underline{g} \in \mathbb{Z}^+$.

(4) $S^3 \setminus \Sigma = S^3_+ \cup S^3_-$ same volume

(5) <u>Karcher–Pinkall–Sterling</u> 88 JDG

genus 3, 5, 6, 7, 11, 19, 73, 601.

$\Rightarrow \exists \Sigma$ embedded minimal $\quad S^3 \setminus \Sigma = S^3_+ \cup S^3_-$

$\text{Vol}(S^3_+) \neq \text{Vol}(S^3_-)$

reflection w.r.t. $\underline{S^2(1) \subset S^3}$

(5). further symmetries of $\underline{\xi_{m,k}}$

reflection w.r.t. $\underline{S^2}$.

<u>§3.</u> λ_1-estimates of $\xi_{m,k}$ (Choe – Soret 2009. <u>Indiana</u>)

1. <u>Thm:</u> $\lambda_1(\xi_{m,k}) = 2$. $\qquad \lambda_1(KPS) = 2$.

Yau's Conj. holds for $\xi_{m,k}$ KPS.

2. <u>Thm:</u> (Courant nodal thm). (M, g). Δ. λ_1 1st eigen-

$\Delta f_1 + \lambda f_1 = 0$. $\quad f_1 \in C^\infty(M)$. Compact Rie. surface

nodal sets $= \{f_1 = 0\}$ 节点

$M \setminus$ nodal set of f_1 has 2 connected domains.

<u>Pf:</u> (1) $\underline{\lambda_0 = 0} \quad f_0 = 1 \qquad \Delta f_0 + \lambda_0 f_0 = 0$.

(2). $f_1 \perp f_0 \iff \int_M f_0 f_1 \, dM = 0$

$f_1 \not\equiv 0 \qquad \int_M f_1 \, dM = 0$.

$\Rightarrow f_1$ changes signs on M. $\quad \{f_1 > 0\} \neq \phi$

$\{f_1 < 0\} \neq \phi$.

$$\lambda_1 = \inf \frac{\quad dM}{\int_M f^2 dM} \qquad \int_M f\,dM = 0$$

Def. $f_1^j = \begin{cases} f_1 & \text{on } \Sigma_j \\ 0 & \text{on } M \setminus \Sigma_j \end{cases}$ $\quad f_1^1 \quad f_1^2 \quad f_1^3$

$\Rightarrow \exists\, a_1, a_2 \in \mathbb{R}$ s.t.

$f = a_1 f_1^1 + a_2 f_1^2$ s.t. $\int f\,dM = 0 \quad f \perp e^1$.

$$\Rightarrow \lambda_1 \le \frac{\int_M |\nabla f|^2 dM}{\int_M f^2 dM} = \frac{\sum_{j=1}^{2} a_j^2 \int_{\Sigma_j} |\nabla f_1^j|^2 dM}{\sum_{j=1}^{2} a_j^2 \int_{\Sigma_j} (f_1^j)^2 dM}$$

On Σ_j $\quad \int_{\Sigma_j} |\nabla f_1^j|^2 dM = \lambda_1 \int_{\Sigma_j} |f_1^j| dM$. $\quad \|$

$\Rightarrow \lambda_1 = \frac{\int_M |\nabla f|^2 dM}{\int_M f^2 dM}$ $\quad$ (f_1 1st eigenfunction)

$\Rightarrow \underline{\Delta f + \lambda f = 0}$ $\quad f$ 1st eigen function?!

But $f|_{\Sigma_3} \equiv 0 \Rightarrow f = 0$ 矛盾.

(Cheng 1978 CMH)

<u>Thm:</u> # nodal domains of $f_j \le j+1$.

$\Delta f_j + \lambda_j f_j = 0 \quad M$ oriented, closed

<u>3.</u> <u>(Ros.)</u> Thm (two-piece properties of embedded minimal surface in S^3)

<u>Rmk:</u> Yau's Conj holds $\Rightarrow y = \begin{pmatrix} y_1 \\ \vdots \\ y_4 \end{pmatrix}$ $\quad y_j \quad M \xrightarrow{y} S^3$

$\Delta y_j + 2 y_j = 0 \quad \lambda_1 = 2.$ $\quad \downarrow$ 1st eigenfunc

$\Rightarrow$ $\underline{\{y_j = 0\}}$ divides M into 2 connected pieces.

($\mathbb{R}^4$ 中任何超平面.)

$\Rightarrow$ If Yau Conj $\checkmark$ $\Rightarrow$ Any hyperplane of $\mathbb{R}^4$ containing origin divides M into 2 pieces. $\quad 0 < \lambda_1 \le \lambda_2 < \lambda_3$

<u>R-s thm:</u> Any hyperplane of $\mathbb{R}^4$ containing origin divides M into 2 connected domains. $\quad \#\{\cdots f_3\} \le 4$

$\exists\, \lambda_j \quad j > 4$

$\#\{f_j\} = 4$

4. Proof (Choe-Soret, Brendle) (Σ, g) $\quad \lambda_1, f_1$ $\quad M$

4. $\boxed{\text{Prop 1. (Choe-Soret. Brendle) } (\Sigma, g) \quad \lambda_1, f_1. \text{ Assume ① } \mu, \tilde f \cdot \ \Delta_\Sigma \tilde f + \mu \tilde f = 0 \text{ ② } \{\tilde f = 0\} \subset \{f = 0\} \text{ Then } \lambda_1 = \mu.}$ $\#\{f_1 = 0\} = 4$

Pf: $\{\tilde f = 0\} \subset \{f = 0\}$. Def $D_+ = \{f > 0\}$ $D_- = \{f < 0\}$.

$$D_+ = (\underbrace{\{\tilde f > 0\} \cap D_+}_{D_{+1}}) \cup \underbrace{\{\tilde f < 0\} \cap D_+}_{D_{+2}}\}.$$

$D_- = D_{-1} \cup D_{-2}$ $\quad D_{+1} \cap D_{+2} = \phi$. $\quad D_{-1} \cap D_{-2} = \phi$

$D_\pm$ connected $\Rightarrow$ $\underline{D_{+1} = D_+}$ or $\underline{D_{+2} = D_+}$

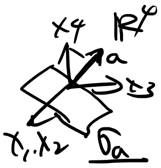
$\underline{D_{-1} = D_-}$ or $D_{-1} = D_-$.

If $\tilde f|_{D_+}$. $\tilde f|_{D_-}$ of same signs

$\Rightarrow$ $\int_\Sigma \tilde f \, dM \neq 0$, but $\int_\Sigma \tilde f \, dM = 0$ $\quad \times$

If $\tilde f|_{D_+}$. $\tilde f|_{D_-}$ of diff. signs $\Rightarrow$

$\int_\Sigma \tilde f f \, dM \neq 0$ $\quad \times$.



$\Rightarrow \lambda = \mu$.

5. Cor: $\Sigma \hookrightarrow S^3$ embedding minimal. $\underline{\sigma_a: S^3 \to S^3}$ reflection of $\langle a \rangle^\perp$. $\sigma_a(\Sigma) = \Sigma$.

$$\sigma_a(y) = y - 2\langle a, y \rangle a$$

λ_1 of Δ_Σ. If $\lambda_1 < 2$, f the eigen

Then. $f = f \circ \sigma$, i.e. f σ-inv.

Pf: If $f - f\circ\sigma \neq 0$, $\underline{\tilde f = f - f \circ \sigma}$.

$\Rightarrow \Delta_\Sigma \tilde f + \lambda_1 \tilde f = 0$ $\quad$ (Σ : σ-inv.)

Def: $y_a = \langle a, y \rangle$ $\Rightarrow \Delta_\Sigma y_a + 2 y_a = 0$ $\quad$ $\underline{\{y_a = 0\}} \subset \underline{\{\tilde f = 0\}}$

$y_a = \langle u_4, x\rangle$, $\Rightarrow \Delta y_a + 2y_a = 0$

$\{y_a>0\} \subset \{f>0\}$

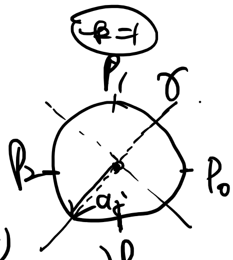
$p \in \{y_a = 0\}$ $\sigma(p) = p \Rightarrow$

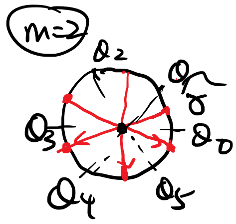
$(f - f\circ\sigma)(p) = f(p) - f(p) = 0$

$\Rightarrow \lambda_1 = 2. \Rightarrow$ Contradiction.

6. Proof of (Choe-Soret)

(Fraendle).





$a_j = \left(\sin\frac{\pi(2j+1)}{2(k+1)}, -\cos\frac{\pi(2j+1)}{2(k+1)}, 0, 0\right) P_3$ $j = 0, \dots, k.$

$b_j = \left(0, 0, \sin\frac{\pi(2j+1)}{2(m+1)}, -\cos\frac{\pi(2j+1)}{2(m+1)}\right)$

Lemma: $\xi_{m,k}$. σ_{a_j}. σ_{b_l} inv.

Lemma: $\{\alpha_{0j}, \beta_{0j}\}$ gives another division of $\xi_{m,k}$

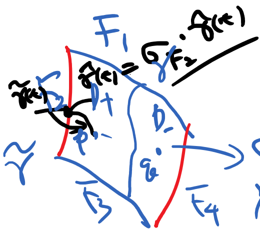
S_{0j}^+. $\Rightarrow$ $\Sigma = \xi_{m,k}$

$= G$ action of $S_{0,0}$

$G = \langle \sigma_{a_j}, \sigma_{b_l} \rangle$

$\lambda_1 < 2$. $f_1 \xRightarrow{\text{Prop}} f_1$ G-inv.

Consider f_1 on $\tilde\gamma$



inv. σ_{a_j} σ_{b_l}.

S_{0j}^+

$p' = \sigma_a(p)$ $\langle a\rangle^\perp = F_4$

$\{f_1 = 0\}$ can not contain Jordan curves.

$\{f_1>0\}$ divide S_{0j}^+ into two parts D_+, D_-

$\{f_1>0\}\{f_1<0\}$

$\{f_1>0\} \cap F_l \neq \emptyset$, 设 $F_l = F_4$

Consider $p \in D_+ \cap (S_{0j}^+)^\circ$ $q \in D_- \cap (S_{0j}^+)^\circ$.

$\tilde p := \sigma_a(p)$. $\langle a\rangle^\perp \cap \Sigma \supset F_4$.

$\{f_1>0\}$ divides Σ into Σ^+, Σ^- two connected

$t_1 > 0 \quad t < 0$ domains.

$\Rightarrow \boxed{p, \hat{p} \in \Sigma^+.} \quad q \in \Sigma^-.$

$\Rightarrow \exists\ \tilde{\gamma} \subset \Sigma^+. \quad \tilde{\gamma}(0) = p, \quad \tilde{\gamma}(1) = \hat{p}.$
C^0

$\Rightarrow \tilde{\gamma} \cap \langle a \rangle^+ = \phi.$

$\underline{\tilde{\gamma} \subset \{f_1 \neq 0\}.}$ {by $D^+ \cap F_4 = \phi$ in S^+_{Tg} }

$\exists\ \rho(t) : [0,1] \to G.$

s.t. $\hat{\gamma}(t) = \underline{\rho(t)}\,\tilde{\gamma}(t) : [0,1] \to S^+_{Tg}$

$\Rightarrow \begin{cases} \hat{\gamma}(t). \text{ disjoint from } F_4. \\ \hat{\gamma}(0) = \hat{\gamma}(1) = p = \hat{\gamma}(1) = \rho(1) \cdot \underline{\tilde{\gamma}(1)} \end{cases}$

$= \rho(1) \cdot (\hat{p})$

$\Rightarrow \rho(1) = \sigma_{F_4} = \sigma_a \quad = \sigma_a(\hat{p})$

But $\rho(t) \in \langle \{\sigma_{a_f}\} \setminus \sigma_a \rangle. \not\ni \sigma_a$

Contradiction.