Isoparametric hypersurfaces & Chern's conjecture

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§1. Def. of Isoparametric hypersurfaces in real space forms from analytic and geometric views.

§2. Isoparametric family in S^n.

§3. Progress on Chern's Conjecture.

§1. Preliminaries.

For Sec $=0$: $\mathbb{R}^n$: Sec $=0$. Standard Euclidean metric

For Sec $=$ Const >0: $S^n(1)$: Sec $=1$. Rie. metric induced from $\mathbb{R}^{n+1}$.

For Sec $=$ Const <0: $\mathbb{H}^n(-1)$: Sec $=-1$. Lorentz metric of signature $(1, n)$:

$$\langle x, y\rangle =: x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}$$

$$\mathbb{H}^n = \{ x \in \mathbb{R}^{n+1}_1 \mid \langle x, x\rangle = -1, \ x_{n+1} \geq 1 \}$$

real space form $\widetilde{M}^n(c)$: Complete, Connected, Simply connected manifold $\widetilde{M}^n$ with Sec $=c$.

i.e. $c=0$ $\quad \widetilde{M}^n = \mathbb{R}^n$

$c=1$ $\quad \widetilde{M}^n = S^n$

$c=-1$ $\quad \widetilde{M}^n = \mathbb{H}^n$

E. Cartan 1938. 1939. 1939. 1940.

H. F. Münzner 1980. 1981.

Def 1.1 (original def of isop. hyp.)

$F: \tilde{M}^{n+1}(c) \to \mathbb{R}$. non constant, smooth function.

F is <u>isoparametric</u>, if $\exists$ smooth functions $a, b: \mathbb{R} \to \mathbb{R}$, s.t.

$$\begin{cases} |\operatorname{grad} F|^2 = b(F) \\ \Delta F = a(F) \end{cases}$$

Def 1.2. <u>Isoparametric family</u> of $\tilde{M}^{n+1}(c)$

= collection of level sets of an isoparametric function on $\tilde{M}^{n+1}(c)$

Def 1.3. The regular level sets of an isoparametric function F on $\tilde{M}^{n+1}$ are called <u>isoparametric hypersurfaces</u>.

Assertion 1: M: a level set of an isop. funct. F on which $\mathrm{grad}\,F \neq 0$.

$\Leftrightarrow$ M has constant principal curvatures.

Thm 1: $F : \widetilde{M}^{n+1} \to \mathbb{R}$: C^∞. $M = F^{-1}(0)$. Suppose $\mathrm{grad}\,F \neq 0$ on M.

$$\Rightarrow \langle AX, Y \rangle = -\frac{H_F(X, Y)}{|\mathrm{grad}\,F|} \quad \text{for } \forall X, Y \in TM. \qquad H_F: \text{Hessian of } F.$$

Pf: $\xi = \dfrac{\mathrm{grad}\,F}{|\mathrm{grad}\,F|}$: a field of unit normals to M.

Corresponding shape operator A: $\langle AX, Y \rangle = -\langle \widetilde{\nabla}_X \xi, Y \rangle$ $\quad \widetilde{\nabla}$: Levi-Civita on $\widetilde{M}$.

Denote $\rho = |\mathrm{grad}\,F|$ $\Rightarrow \mathrm{grad}\,F = \rho\xi$.

$$\Rightarrow \langle \widetilde{\nabla}_X (\rho\xi), Y \rangle = X(\rho)\langle \xi, Y \rangle + \rho \langle \widetilde{\nabla}_X \xi, Y \rangle = -\rho \langle AX, Y \rangle$$

On the other hand,

$$\langle \widetilde{\nabla}_X \mathrm{grad}\,F, Y \rangle = \widetilde{\nabla}_X \langle \mathrm{grad}\,F, Y \rangle - \langle \mathrm{grad}\,F, \widetilde{\nabla}_X Y \rangle = XY(F) - (\widetilde{\nabla}_X Y)(F)$$
$$= H_F(X, Y)$$

$$\Rightarrow \langle AX, Y \rangle = -\frac{1}{\rho} H_F(X, Y)$$

Remark: $H_F(X, Y) = X(YF) - (\tilde{\nabla}_X Y)(F)$

at a critical pt of F, F is constant on M. $\therefore X(YF) = 0$.

but $(\tilde{\nabla}_X Y)(F) = (\nabla_X Y)(F) + \langle AX, Y\rangle \xi F = \langle AX, Y\rangle \rho$. need not be 0.

<u>**Thm 2.**</u> h: mean curvature of the level hypersurface M in Thm 1

$$\Rightarrow \quad h = \frac{1}{n\rho^2}\left(\langle \operatorname{grad} F, \operatorname{grad} \rho \rangle - \rho \Delta F \right)$$

Pf: $x \in M$. $e_1 \dots e_n$: o.n.b. of T_xM.

$$\Rightarrow \quad nh = \operatorname{tr} A = \sum_{i=1}^{n} \langle Ae_i, e_i \rangle \underset{\text{Thm 1}}{=} -\frac{1}{\rho} \langle \tilde{\nabla}_{e_i} \operatorname{grad} F, e_i \rangle$$

$$= -\frac{1}{\rho}\left(\Delta F - \langle \tilde{\nabla}_{\xi} \operatorname{grad} F, \xi \rangle \right)$$

$$= -\frac{1}{\rho}\left(\Delta F - \frac{1}{2\rho^2} \tilde{\nabla}_{\operatorname{grad} F} \underbrace{|\operatorname{grad} F|^2}_{\rho^2} \right)$$

$$= -\frac{1}{\rho}\left(\Delta F - \frac{1}{\rho} \tilde{\nabla}_{\operatorname{grad} F}\, \rho \right)$$

$$= \frac{1}{\rho^2}\left(\langle \operatorname{grad} F, \operatorname{grad} \rho \rangle - \rho \Delta F \right)$$

#.

Thm 3 $F: \widetilde{M}^{n+1} \to \mathbb{R}$: isoparametric function

$\Rightarrow$ each level hypersurface of F has CMC (Constant mean Curvature).

Pf: F isop. $\Rightarrow$ $\exists$ C^∞ function b, a s.t. $\begin{cases} |\operatorname{grad} F|^2 = b(F) \\ \Delta F = a(F) \end{cases}$

On the level hypersurface $M = F^{-1}(0)$,

$$\begin{aligned} nh &= \frac{1}{\rho^2}\left(\langle \operatorname{grad} F, \operatorname{grad} \rho\rangle - \rho \Delta F\right) \\ &= \frac{1}{\rho^2}\left(\langle \operatorname{grad} F, \frac{b'(F)}{2\rho}\operatorname{grad} F\rangle - \rho\, a(F)\right) \\ &= \frac{1}{\rho^2}\left(\frac{\rho^2}{2\rho} b'(F) - \rho\, a(F)\right) \\ &= \frac{b'(F) - 2a(F)}{2\rho} \\ &= \frac{b' - 2a}{2\sqrt{b}}(F) \end{aligned}$$

Similarly, h is Const on any level hypersurface of F #.

the vector field $\xi = \dfrac{\text{grad} F}{|\text{grad} F|}$ is defined on $\mathcal{U} =: \{ p \in \widetilde{M}^{n+1} \mid \text{grad} F \neq 0 \} \underset{\text{open}}{\subset} \widetilde{M}^{n+1}$

<u>Thm 4</u> $F : \widetilde{M}^{n+1} \to \mathbb{R}$, $|\text{grad} F| = \varphi(F)$

$\Rightarrow$ On $\mathcal{U}$, the integral curves of ξ are geodesics in $\widetilde{M}^{n+1}$.

Aim: show $\widetilde{\nabla}_\xi \xi = 0$ on $\mathcal{U}$.

pf: ① $|\xi| = 1 \Rightarrow \langle \widetilde{\nabla}_\xi \xi, \xi \rangle = 0$

② X: v.f. in $\mathcal{U}$, orthogonal to ξ at each pt of $\mathcal{U}$.

$\Rightarrow$ X is tangent to a level surface of F at each pt of $\mathcal{U}$.

$\langle \widetilde{\nabla}_\xi \xi, X \rangle = -\langle \xi, \widetilde{\nabla}_\xi X \rangle = -\langle \xi, \widetilde{\nabla}_X \xi + [X, \xi] \rangle = -\langle \xi, [X, \xi] \rangle$

Notice $[X, \xi](F) = X\xi(F) - \underbrace{\xi X(F)}_{=0} = X\xi(F) = X(|\text{grad} F|) = X(\varphi(F)) = 0$

$\therefore \langle \widetilde{\nabla}_\xi \xi, X \rangle = 0$ #.

Remark: Thm3 + Thm4 $\Rightarrow$ a family of level hypersurfaces of an isoparametric function is a family of parallel hypersurfaces with CMC.

Thm 5 Let $f_t : M^n \longrightarrow \tilde{M}^{n+1}$ $(-\varepsilon < t < \varepsilon)$ be a family of parallel hypersurfaces

Then $f_0 M$ has CPC (constant principal curvatures) $\Longleftrightarrow$ each $f_t M$ has CMC. (constant mean curvatures)

Corollary 1 $F : \tilde{M}^{n+1} \to \mathbb{R}$: isoparametric function.

$\Rightarrow$ each level hypersurface of F has CPC.

Def 1.3' $M^n \subset \tilde{M}^{n+1}$, connected immersed hypersurface.

M^n is called <u>isoparametric</u> if it has CPC.

tube & parallel hypersurfaces :

$f: M^n \to \tilde{M}^{n+k}$ embedded submanifold.

$NM = \{(x, \xi) \mid x \in M, \ \xi \in T_x^{\perp}M\}$: normal bundle of $f(M)$.

BM : unit normal bundle.

$\exp: T\tilde{M} \to \tilde{M}$ exponential map of $\tilde{M}$

$E = \exp|_{NM} : NM \longrightarrow \tilde{M}$: normal exponential map (or end-point map).

$(x, \xi) \mapsto E(x, \xi)$: the pt of $\tilde{M}$ reached by traversing a distance $|\xi|$ along the geodesic $\gamma(t)$ in $\tilde{M}$ with $\gamma(0) = f(x)$, $\gamma'(0) = \xi$.

$p \in \tilde{M}$ is a focal pt of (M, x) with multiplicity m : $p = E(x, \xi)$, $E_*|_{(x,\xi)}$ has nullity $m > 0$.

focal set $= \{\text{focal pts}\}$ has measure 0 in $\tilde{M}$.

Lemma 1: $f: M^n \to \widetilde{M}^{n+k}$ $\quad$ ξ: unit normal vector to $f(M)$ at $f(x)$.

$p = E(x, t\xi)$ is a focal pt of (M, x) of multiplicity $m > 0$

$\Leftrightarrow$ $\exists$ an eigenvalue λ of A_ξ of multiplicity m s.t.

(1) $\lambda = \frac{1}{t}$ if $\widetilde{M}^{n+k} = \mathbb{R}^{n+k}$

(2) $\lambda = \cot t$ if $\widetilde{M}^{n+k} = S^{n+k}$

(3) $\lambda = \coth t$ if $\widetilde{M}^{n+k} = \mathbb{H}^{n+k}$

(1) if $\widetilde{M}^{n+k} = \mathbb{R}^{n+k}$ $\qquad E(x, t\xi) = f(x) + t\xi$

(2) if $\widetilde{M}^{n+k} = S^{n+k}$ $\qquad E(x, t\xi) = \cos\theta\, f(x) + \sin\theta\, \xi(x)$ $\quad t = \cot\theta$

(3) if $\widetilde{M}^{n+k} = \mathbb{H}^{n+k}$ $\qquad E(x, t\xi) = \cosh\theta\, f(x) + \sinh\theta\, \xi(x)$ $\quad t = \coth\theta$

Def. (tube of radius $t>0$ over M)

$$f_t : BM \longrightarrow \tilde{M}$$
$$(x, \xi) \mapsto f_t(x,\xi) = E(x, t\xi)$$

if $(x, t\xi)$ is not a critical pt of E $\Rightarrow$ f_t is an immersion near (x,ξ) in BM.

Lemma 1 $\Rightarrow$ $\forall x \in M$. $\exists$ a nbhd $\mathcal{U}$ of x. s.t. $f_t|_{B\mathcal{U}}$ is an immersion onto a $(n+k-1)$-dim mfd, which is geometrically a tube of radius t over $\mathcal{U}$.

- $k=1$. $f : M^n \to \tilde{M}^{n+1}$ hypersurface.

  BM : double covering of M.

  locally, assume M is orientable,

  $f_t : M^n \to \tilde{M}^{n+1}$: <u>parallel hypersurface</u> of M.

eg. $\tilde{M}^{n+1} = S^{n+1}$

Principal curvatures of a tube:

Lemma 2: $f: M^n \to \widetilde{M}^{n+k}$

ξ: a unit normal to M s.t. $f_t: BM \to \widetilde{M}^{n+k}$ is an immersion at (x,ξ)

$\{X_1 \cdots X_n\}$: basis of T_xM, $A_\xi X_i = \lambda_i X_i$ $\quad i = 1 \cdots n$.

$\Rightarrow$ the shape operator A_t of the tube f_t of radius t over M at (x,ξ) is given by

(1) $\widetilde{M}^{n+k} = \mathbb{R}^{n+k}$

$$A_t\left(\frac{\partial}{\partial t_j}\right) = -\frac{1}{t}\left(\frac{\partial}{\partial t_j}\right), \quad 2 \le j \le k$$

$$A_t(X_i) = \frac{\lambda_i}{1 - t\lambda_i}(X_i), \quad 1 \le i \le n.$$

(2) $\widetilde{M}^{n+k} = S^{n+k}$

$$A_t\left(\frac{\partial}{\partial t_j}\right) = -\cot t\left(\frac{\partial}{\partial t_j}\right), \quad 2 \le j \le k$$

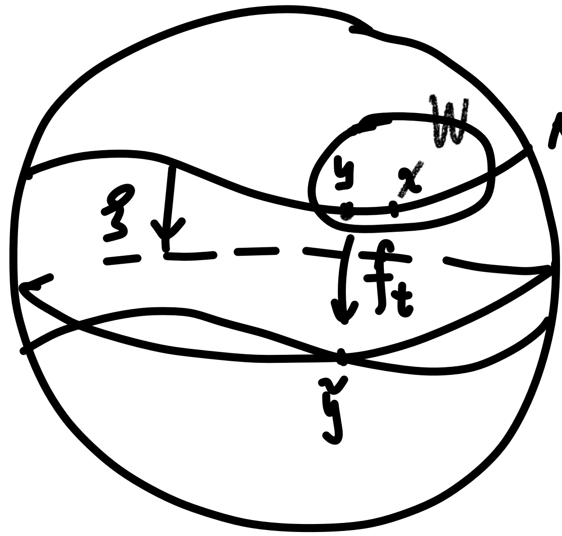
$$A_t(X_i) = \cot(\theta_i - t)(X_i), \quad \text{if } \lambda_i = \cot\theta_i \quad 0 < \theta_i < \pi.$$

(3) $\widetilde{M}^{n+k} = H^{n+k}$

$$A_t\left(\frac{\partial}{\partial t_j}\right) = -\coth t\left(\frac{\partial}{\partial t_j}\right) \quad 2 \le j \le k$$

$$A_t(X_i) = \begin{cases} \coth(\theta_i - t)\, X_i, & \text{if } |\lambda_i| > 1 \text{ and } \lambda_i = \coth\theta_i \\ \pm X_i, & \text{if } \lambda_i = \pm 1 \\ \tanh(\theta_i - t)\, X_i, & \text{if } |\lambda_i| < 1 \text{ and } \lambda_i = \tanh\theta_i. \end{cases}$$

eg. $f: M^n \to S^{n+1}(1)$ principal curvatures $\lambda_i = \cot\theta_i$. $i=1\cdots n$.



$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

M $$\left(0 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n < \pi\right)$$

$M_t = f_t(M)$: principal curvatures

$$\tilde{\lambda}_i = \cot(\theta_i - t) \qquad i = 1\cdots n.$$

pf: Suppose f_t is an imm. in a nbhd W of x. $y \in W$.

$$y \xmapsto{f_t} \tilde{y} = \cos t\, y + \sin t\, \xi(y)$$

$$\xi_y \longmapsto \tilde{\xi}_y = -\sin t\, y + \cos t\, \xi(y) \ : \text{ unit normal to } f_t(M) \text{ at } \tilde{y}.$$

For $X \in T_i(x) = \{X \in T_xM \mid AX = \lambda_i X\}$: $D_X\xi = \tilde{\nabla}_X\xi - \langle X, \xi\rangle x = \tilde{\nabla}_X\xi = -AX = -\cot\theta_i X$

$$(f_t)_* X = (\cos t\, I - \sin t\, A)X = \frac{\sin(\theta_i - t)}{\sin\theta_i} X$$

$$(f_t)_* A_t X = -\tilde{\nabla}_{(f_t)_*X}\tilde{\xi} = -D_{(f_t)_*X}\tilde{\xi} = \sin t\, X + \cos t \cot\theta_i X = \frac{\cos(\theta_i - t)}{\sin\theta_i} X = \cot(\theta_i - t)(f_t)_* X$$

$\therefore$ $A_t X = \cot(\theta_i - t) X$ and multiplicity of $\cot(\theta_i - t)$ = multiplicity of $\cot\theta_i$.

Recall Lemma 1, if $t = \begin{cases} \frac{1}{\lambda} \\ \operatorname{arc\,cot} \lambda \\ \operatorname{arc\,coth} \lambda \end{cases}$ for some λ, then $P = E(x, t\xi)$ is a focal pt.

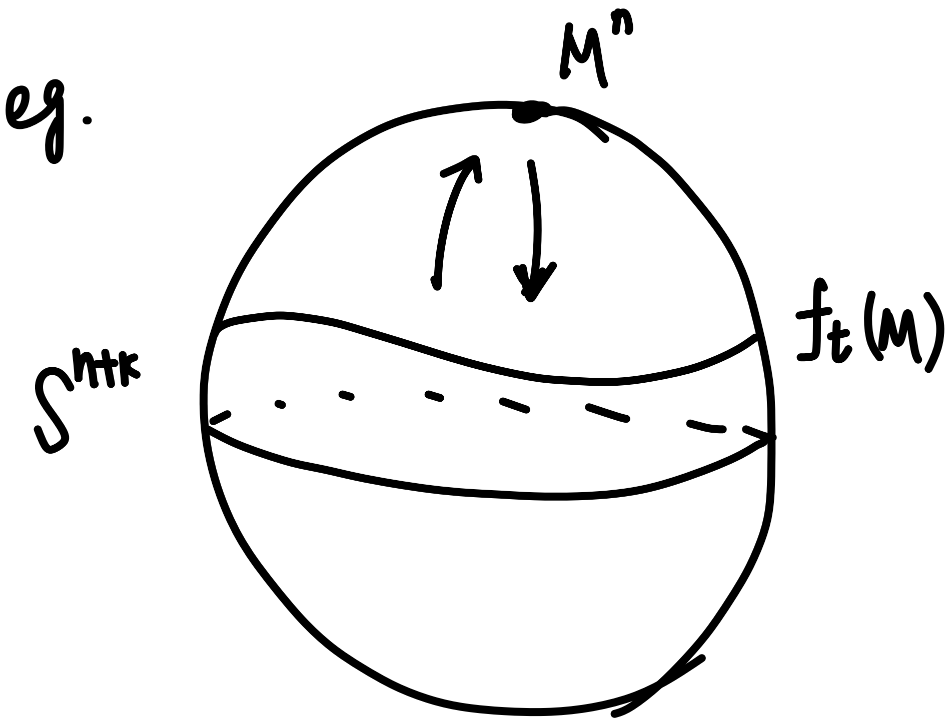
(*)

Lemma 1 + Lemma 2 $\Rightarrow$ if $f: M^n \to \widetilde{M}^{n+k}$, $k>1$

then the points of M are focal points of the tube $f_t(M)$ corresponding to principal curvature

$$\mu = \begin{cases} -\frac{1}{t} & \widetilde{M}^{n+k} = \mathbb{R}^{n+k} \\ -\cot t & \widetilde{M}^{n+k} = S^{n+k} \\ -\coth t & \widetilde{M}^{n+k} = \mathbb{H}^{n+k} \end{cases}$$

eg.



$k>1$. $f_t(M)$ is a tube of M^n.

M^n is a focal set of $f_t(M)$.

$f: M^n \to \tilde{M}^{n+1}(c)$. imm. hyp. Assume $f(M)$ is orientable with global ξ.

If a principal curvature λ has constant multiplicity m on M.

Focal map: $f_\lambda: \mathcal{U} \subset_{\text{open}} M^n \longrightarrow \tilde{M}^{n+1}$ by

(1) $f_\lambda(x) = f(x) + \frac{1}{\lambda}\xi(x)$ $\qquad \tilde{M}^{n+1} = \mathbb{R}^{n+1}$

(2) $f_\lambda(x) = \cos\theta\, f(x) + \sin\theta\, \xi(x)$ $\quad \cot\theta = \lambda$ $\qquad \tilde{M}^{n+1} = S^{n+1}$

(3) $f_\lambda(x) = \cosh\theta\, f(x) + \sinh\theta\, \xi(x)$ $\quad \coth\theta = \lambda$ $\qquad \tilde{M}^{n+1} = H^{n+1}$

$\mathcal{U}$: domain of f_λ.

$(f_\lambda)_*$ has nullity $m > 0$. $\Longrightarrow f_\lambda(\mathcal{U})$: focal set of M.

Next, $f_\lambda(\mathcal{U})$ has manifold structure, thus an $(n-m)$-dim submanifold of $\tilde{M}^{n+1}$.

$f_\lambda(M)$ has manifold structure, its dim $= n-m$.

(1) Ryan 1969 : λ is continuous

(2) Nomizu 1973, Reckziegel 1976. 1979, Singley 1975 :

λ has constant multiplicity $m>0 \Rightarrow \lambda$ is smooth & T_λ is smooth.

$m>1$
(3) T_λ is integrable ($\Rightarrow$ it is an m-dim foliation)

(4) λ is constant along each leaf of principal foliation T_λ.

(5) rank of $f_\lambda = n-m$ on its domain.

$m=1$: (5)' rank $f_\lambda = n-1 \Leftrightarrow \lambda$ is const along each leaf of T_λ.

(6) $f_\lambda : U \underset{\text{open}}{\subset} M \longrightarrow \tilde{M}^{n+1}$ factors through $g_\lambda : U/T_\lambda \to \tilde{M}^{n+1}$. $\dim U/T_\lambda = n-m$

rank $g_\lambda =$ rank f_λ. If M is complete w.r.t. induced metric, U/T_λ is Hausdorff

$\therefore f_\lambda(M)$ is also called focal submfd of M.

Thm 6 $f: M^n \to \tilde{M}^{n+1}$ oriented hypersurface

λ: smooth principal curvature function with const multi $m \geq 1$

λ is const along each leaf of T_λ.

Then λ assumes a critical value along a leaf γ of T_λ.

$\Leftrightarrow$ γ is totally geodesic in M^n.

Corollary 2. $f: M^n \to \tilde{M}^{n+1}$ isoparametric hypersurface

$\Rightarrow$ the leaves of principal foliation T_λ are totally umbilic in $\tilde{M}^{n+1}$, and totally geodesic in M^n, $\forall$ p.c. λ.

Thm 5 Let $f_t : M^n \to \tilde{M}^{n+1}$ $(-\varepsilon < t < \varepsilon)$ be a family of parallel hypersurfaces

Then $f_0 M$ has CPC $\Longleftrightarrow$ each $f_t(M)$ has CMC.

Proof for $\tilde{M}^{n+1} = S^{n+1}$:

" $\Rightarrow$ " Let $\lambda_i = \cot\theta_i$ $i = 1 \dots n$ be principal curvatures of f_0.

Lemma 1 $\Rightarrow$ principal curvatures of $f_t(M)$ are $\cot(\theta_i - t)$.

$\therefore$ $\lambda_i = \text{const}$ $\Rightarrow$ $f_t(M)$ has CPC. thus CMC.

" $\Leftarrow$ " Suppose each $f_t(M)$ has CMC.

$a(t) = \sum_{i=1}^{n} \cot(\theta_i - t)$ is a function of t alone, even though $\theta_i(x)$ depend on x

For $t=0$. $a(0) = \sum_{i=1}^{n} \cot\theta_i(x) = C_1$

$a'(0) = \sum_{i=1}^{n} \csc^2\theta_i(x) = \sum_{i=1}^{n} (1 + \cot^2\theta_i(x))$ is a constant

$\Rightarrow$ $\sum_{i=1}^{n} \cot^2\theta_i(x) = C_2$

Next, $a''(0) = \sum_{i=1}^{n} 2\csc^2\theta_i(x)\cot\theta_i(x) = 2\sum_{i=1}^{n}\cot\theta_i(x) + \cot^3\theta_i(x)$ is const

$\therefore \sum_{i=1}^{n}\cot^3\theta_i(x) = C_3$

. . . . .

$\Rightarrow\quad S_k(x) = \sum_{i=1}^{n}\cot^k\theta_i(x) = \sum_{i=1}^{n}\lambda_i^k(x) = C_k, \qquad 1\le k\le n$

Characteristic polynomial of A:

$$\prod_{i=1}^{n}(x-\lambda_i) = x^n + d_1x^{n-1} + \cdots + d_{n-1}x + d_n$$

By Newton's identities, $d_1, \cdots, d_n$ are polynomials in $S_k(x)$

$\therefore \lambda_i$ is constant $\forall i = 1 .. n$. #.

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isop. hyp   analytic def.   geometric def.

level hyp. of isop. funct.   CPC. hyp.

Conversely, let $f_t : M^n \to \tilde{M}^{n+1}$ $(-\varepsilon < t < \varepsilon)$ be a family of parallel hypersurfaces s.t. $f_0(M)$ has CPC.

Lemma 2 $\Rightarrow$ each $f_t(M)$ has CPC.

Define F by $F(x) = t$ if $x \in f_t(M)$.

- F is a smooth function defined on an open subset of $\tilde{M}^{n+1}$.
- $\operatorname{grad} F = \xi$, $|\xi| = 1$. $\tilde{\nabla}_\xi \xi = 0$
- $\rho = |\operatorname{grad} F| = c$.
- each $f_t(M)$ has CMC.

Thm 2 $\Rightarrow$ ΔF is also constant on $f_t(M)$

$\Rightarrow$ F is isoparametric.

We shall see:

Cartan + Münzner:

$M^n \subset \tilde{M}^{n+1}$: embedded connected isoparametric hypersurface

$\Rightarrow$ M is contained in a unique complete isoparametric hypersurface.

<u>A little history:</u>

$M^n \to \mathbb{R}^{n+1}$: isop. hyp $\Rightarrow$ $M^n \cong \mathbb{R}^n$. S^n or $S^k \times \mathbb{R}^{n-k}$

$n=2$ Somigliana 1918 - 1919

also by Segre 1924 and Levi-Civita 1937.

$\forall n$. Segre 1938

Then 1938~1941. E. Cartan began the study of isop. hyp. in $\tilde{M}^{n+1}(c)$.

Cartan's formula (Cartan's identity)

$M^n \to \widetilde{M}^{n+1}(c)$: isoparametric hypersurface.

$p \in M$. X: unit principal vector at p, λ: the associated principal curvature

For any principal o.n.b $\{e_i\}_{i=1}^n$ satisfying $Ae_i = \mu_i e_i$

$$\sum_{\mu_i \neq \lambda} \frac{c + \lambda\mu_i}{\lambda - \mu_i} = 0.$$

proof. A symmetric $\Rightarrow$ $\nabla_X A$ symmetric $\langle (\nabla_X A)Y, Z\rangle = \langle Y, (\nabla_X A)Z\rangle$

Codazzi eqn: $(\nabla_X A)Y = (\nabla_Y A)X \Rightarrow \begin{cases} \nabla_X Y \in T_\lambda & \text{if } X, Y \in T_\lambda \\ \nabla_X Y \perp T_\lambda & \text{if } X \in T_\lambda, Y \in T_\mu, \mu \neq \lambda. \end{cases}$

g: number of distinct principal curvatures.

- $g > 2$. Y: a second unit principal vector at p with corresponding p.c. $\mu \neq \lambda$.

$X, Y \underset{\text{extend}}{\rightsquigarrow}$ principal v.f. near p.

① By Codazzi eqn, $\langle (\nabla_{[X,Y]}A)X, Y\rangle = (\lambda-\mu)\langle \nabla_X Y, \nabla_Y X\rangle$

② Gauss eqn. $\langle R(X,Y)Y, X\rangle = c+\lambda\mu.$

③ $\langle R(X,Y)Y, X\rangle = \langle \nabla_X Y, \nabla_Y X\rangle + \frac{1}{\lambda-\mu}\langle (\nabla_{[X,Y]}A)X, Y\rangle$

④ for a third unit principal vector Z at p corresponding to p.c. $\nu \neq \lambda, \mu$.

Codazzi eqn $\Rightarrow$ $(\lambda-\nu)(\mu-\nu)\langle \nabla_X Y, Z\rangle\langle \nabla_Y X, Z\rangle = \langle (\nabla_Z A)X, Y\rangle^2$

$$= \langle (\nabla_Z A)Y, X\rangle^2$$

⑤ $\langle \nabla_X Y, \nabla_Y X\rangle = \sum_{\mu_i \neq \lambda,\mu} \langle \nabla_X Y, e_i\rangle\langle \nabla_Y X, e_i\rangle$

$$\Rightarrow \quad c+\lambda\mu = 2\langle \nabla_X Y, \nabla_Y X\rangle = 2\sum_{\mu_i\neq\lambda,\mu}\langle \nabla_X Y, e_i\rangle\langle \nabla_Y X, e_i\rangle = 2\sum_{\mu_i\neq\lambda,\mu}\frac{\langle (\nabla_{e_i}A)Y, X\rangle^2}{(\lambda-\mu_i)(\mu-\mu_i)}$$

$$\Rightarrow \quad \sum_{\mu_j\neq\lambda}\frac{c+\lambda\mu_j}{\lambda-\mu_j} = 2\sum_{\mu_j\neq\lambda}\sum_{\mu_i\neq\lambda,\mu_j}\frac{\langle (\nabla_{e_i}A)e_j, X\rangle^2}{(\lambda-\mu_j)(\lambda-\mu_i)(\mu_j-\mu_i)} = 0$$

• $g=2$. ①+②+③ $\Rightarrow$ $c+\lambda\mu = 2\langle \nabla_X Y, \nabla_Y X\rangle = 0$

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Remark: 1. When $\widetilde{M}^{n+1} = S^{n+1}$, Münzner gave a different proof.

and Cartan's identity $\Longleftrightarrow$ minimality of focal submfds of M^n in S^{n+1}.

Nomizu 1973, also proved the focal submfds are minimal.

2. Berndt found a generalization of Cartan's formula

for real hyp. with CPC in complex space forms 1989

and quaternionic space forms 1991.

3. Nomizu (1973) proved a version of Cartan's formula for isop. hyp.

in Lorentzian forms.

4. Ooguri (2004) found one for equiaffine isop. hyp. in affin. diff. geom.

5. Abe-Hasegawa (2008) extended these results to a more general setting.

6. Koike (2014) found a Cartan type formula for isop. hyp. in symm. space.

Classification of Isoparametric hypersurfaces in $\mathbb{R}^{n+1}$:

$M^n \subset \mathbb{R}^{n+1}$ Connected isop. hyp. $\Rightarrow$ M^n is an open subset of $\mathbb{R}^n$, S^n or $S^k \times \mathbb{R}^{n-k}$

- $g=1$ M^n totally umbilic $\Rightarrow$ M^n is an open subset of $\mathbb{R}^n$ or S^n.
- $g \geq 2$ Choose ξ s.t. at least one principal curvature > 0.

Cartan's identity $\displaystyle\sum_{\lambda_j \neq \lambda_i} m_j \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0$ $\lambda_1 \cdots \lambda_g$ distinct.

Let λ_i be the smallest positive p.c. $\Rightarrow$ $\dfrac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} \leq 0$ $\forall \lambda_j \neq \lambda_i$.

$\Rightarrow$ $\dfrac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0$ $\forall \lambda_j \neq \lambda_i$

$\Rightarrow$ $g \leq 2$ and if $g=2$. $\lambda_1 = 0$ or $\lambda_2 = 0$.

When $g=2$ Suppose $\lambda_1 > 0$, $m_1 = k$ $\lambda_2 = 0$, $m_2 = n-k$.

Then for $t = \frac{1}{\lambda_1}$, the focal submfd $f_t(M)$ has $\dim = n-k$.

$\forall$ unit normal η. A_η has one principal curvature $\dfrac{\lambda_2}{1-t\lambda_2} = 0$.

$\therefore$ $f_t(M) \subset \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$, and M^n is an open subset of tube of radius $\frac{1}{\lambda_1}$ over $\mathbb{R}^{n-k}$

i.e. $S^k(r) \times \mathbb{R}^{n-k}$, radius $r = \frac{1}{\lambda_1}$

Classification of Isoparametric hypersurfaces in $\mathbb{H}^{n+1}$:

- $g=1$. M^n is totally umbilic $\Rightarrow$ M^n is an open subset of a totally geodesic hyperplane, an equidistant hyp. a horosphere or a metric hypersphere in $\mathbb{H}^{n+1}$.

- $g\geq 2$. Choose ξ s.t. at least one p.c. >0.

  $\Rightarrow \exists \lambda_i>0$, s.t. $\nexists$ p.c. lies between λ_i and $\frac{1}{\lambda_i}$.

  $\therefore$ for this λ_i, $\frac{-1+\lambda_i\lambda_j}{\lambda_i-\lambda_j}<0$ unless $\lambda_j=\frac{1}{\lambda_i}$

  $\therefore g\leq 2$. and when $g=2$, $\lambda_1\lambda_2=1$.

- $g=2$. $\lambda_1=\coth\theta$ $m_1=k$. $\lambda_2=\tanh\theta$ $m_2=n-k$.

  if $t=\theta$, the focal submfd $V=f_t(M)$ has $\dim=n-k$

  and $\forall$ unit normal η. A_η has one p.c. $\tanh(\theta-t)=0$.

  $\therefore V\subset\mathbb{H}^{n-k}\subset\mathbb{H}^{n+1}$

  and M^n is a tube of radius $t=\theta$ over $\mathbb{H}^{n-k}$, i.e. $S^k\left(\frac{1}{\sinh^2\theta}\right)\times H^{n-k}\left(\frac{-1}{\cosh^2\theta}\right)$

  Sec.

Isoparametric hypersurfaces $M^n \subset S^{n+1}$:

Cartan's identity $\not\Rightarrow g \leq 2$.

Cartan classified cases $g \leq 3$. and gave examples with $g = 4$.

- $g=1$. M^n is totally umbilic. $\Rightarrow$ M^n is an open subset of a hypersphere.
- $g=2$. $S^p(r) \times S^q(s) \subset \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} = \mathbb{R}^{n+2}$, $r^2+s^2=1$ $r, s > 0$. $p+q=n$.

  (similarly as before, shows that a focal submfd is totally geodesic)
- $g=3$. highlight of Cartan:

  $\lambda_1, \lambda_2, \lambda_3$ have the same multiplicity $m = 1, 2, 4$ or 8.

  M^n is an open subset of a tube of constant radius over a Veronese embedding of $\mathbb{F}P^2$ into S^{3m+1}. $\mathbb{F} = \mathbb{R}$ $\mathbb{C}$ $\mathbb{H}$ $\mathbb{C}a$

  ($m = 1.$ $2.$ $4.$ 8)

  The focal set of M^n consists of two antipodal embeddings of $\mathbb{F}P^2$.

  Up to congruence, $\forall m$, $\exists!$ such family of isop. hyp.

More generally, Cartan showed that any isop. hyp. with g distinct principal curvatures of the same multiplicity can be defined by the equation

$$F = \cos gt \quad (\text{restricted to } S^{n+1}).$$

where F is a homogeneous polynomial of degree g on $\mathbb{R}^{n+2}$ satisfying

$$\begin{cases} |\operatorname{grad} F|^2 = g^2 |x|^{2g-2} \\ \Delta F = 0 \end{cases}$$

In 1980, 1981, Münzner generalized this result.

- $g=4$. Cartan produced examples with $m=1$ or 2.

  and claimed without proof that these are the only examples of isop. hyp. with $g=4$ and the same multiplicity.

  $m=1$ Takagi 1976 ✓

  $m=2$ Ozeki-Takeuchi 1976 ✓

Fact: no other values of m, by Grove-Halperin 1987.

All of Cartan's examples are homogeneous: an orbit of a point under an appropriate closed subgroup of $SO(n+2)$.

Cartan asked:

1. $\forall g \in \mathbb{N}_+$, does there exist an isoparametric family with g distinct principal curvatures of the same multiplicity?

   No! Münzner $\Rightarrow$ g can only be 1. 2. 3. 4. 6.

2. Does there exist isoparametric family of hypersurfaces with $g > 3$ s.t. principal curvatures do not all have the same multiplicity?

   Yes. Nomizu generalized Cartan's example with $g=4$. $m=1$ to those with multiplicities $1.m$

3. Does every isop. family of hypersurfaces admit a transitive group of isometries?

   No. Ozeki-Takeuchi 1975 constructed inhomog isop. hyp. with $g=4$.

   $\rightsquigarrow$ Ferus-Karcher-Münzner 1981 greatly generalized O-T's examples.

§2. Isoparametric family in S^{n+1}

Review: $M^n \subset S^{n+1} \subset \mathbb{R}^{n+2}$ connected. oriented hyp. ξ: u.n.v.f.

$\nabla \quad \tilde{\nabla} \quad D$

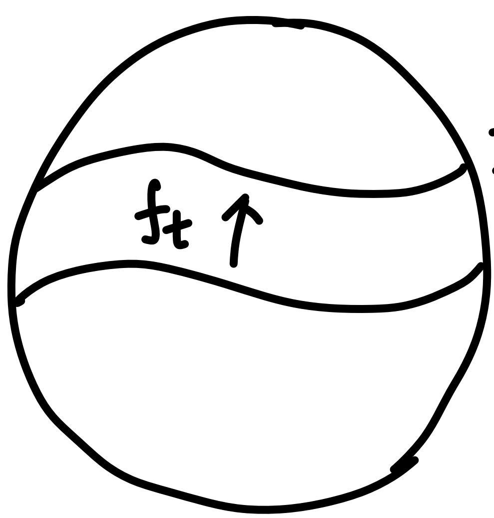
Assume M has g distinct principal curvatures at each point.

$\lambda_i = \cot\theta_i \qquad 0<\theta_1<\theta_2<\dots<\theta_g<\pi$

multiplicity of λ_i : m_i

principal distribution : $T_i(x) = \{X \in T_xM \mid AX = \lambda_i X\} \qquad A = A_\xi$

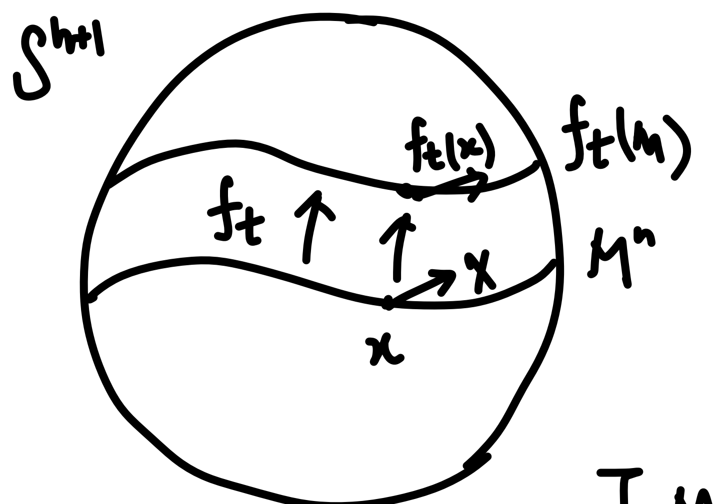
Fact: T_i is a foliation of M with leaves of $\dim = m_i$.



parallel hypersurface $f_t: M^n \to S^{n+1}$

$x \mapsto \cos t\, x + \sin t\, \xi(x)$

- f_t is an imm. at $x \in M \iff \cot t \neq \lambda_i \quad \forall i = 1 \dots g$.



$\forall X \in T_xM$,

$$(f_t)_* X = \cos t\, X + \sin t\, D_X \xi = \cos t\, X - \sin t\, AX = (\cos t\, I - \sin t\, A) X$$

if $X \in T_{\lambda_i} \Rightarrow (f_t)_* X = \dfrac{\sin(\theta_i - t)}{\sin\theta_i} X$

$$T_xM = \bigoplus_i T_{\lambda_i}(x)$$

$\Rightarrow$ $(f_t)_*$ is injective on T_xM unless $t = \theta_i \pmod{\pi}$ for some i

in other words, f_tM is an imm. hyp. if $t \neq \theta_i \pmod{\pi}$ $\forall i$.

Thm 7 $M^n \subset S^{n+1}$ connected isop. hyp. having g distinct principal curvature $\lambda_i = \cot\theta_i$ $(0 < \theta_1 < \cdots < \theta_g < \pi)$ with multiplicity m_i.

If $t \neq \theta_i \pmod{\pi}$ $\forall i$, then the parallel hypersurface f_tM has principal curvatures $\tilde{\lambda}_i = \cot(\theta_i - t)$ with the same multiplicity m_i with λ_i. Moreover, $\forall i$, the principal foliation $T_{\tilde{\lambda}_i}$ is the same as T_{λ_i} on M^n.

Remark: From Münzner's theory, it follows that if $M^n \subset S^{n+1}$ is an embedded isop. hyp, then each parallel isop. hyp. $f_t M$ is also embedded.

Focal submanifold

$\forall\, x \in M,\quad T_x M = T_i(x) \oplus T_i^{\perp}(x) \qquad T_i^{\perp}(x) = \bigoplus_{j \neq i} T_j(x)$

- $t = \theta_i$: $(f_t)_* = 0$ on $T_i(x)$ and $(f_t)_*$ is injective on $T_i^{\perp}(x)$

Consider a map
$$h: M^n \longrightarrow S^{n+1}$$
$$x \longmapsto -\sin t\, x + \cos t\, \xi(x)$$

$$\left.\begin{aligned} &\langle h(x), f_t(x) \rangle = 0 \quad \Rightarrow \quad h(x) \in T_{f_t(x)} S^{n+1} \\ &\langle h(x), X \rangle = 0 \quad \forall X \in T_{\lambda_i}^{\perp}(x) \end{aligned}\right\} \Rightarrow h(x) \text{ is normal to the focal}$$

Submfd $V_i = f_{\lambda_i}(M)$ at $p = f_t(x)$.

Next, we use h to find the shape operator of a normal to V_i :

the leaves of T_{λ_i} on M are open subsets of an m_i-sphere in S^{n+1}.

$\Rightarrow$ $\forall\, p \in V_i$, $\quad C = f^{-1}(p) \underset{\text{open}}{\subset} S^{m_i} \subset S^{n+1}$.

and $\quad h|_C : C \longrightarrow S_p^+ V_i$

$\Rightarrow$ At $x \in C$. $\quad T_x C = T_{\lambda_i}(x)$

$\Rightarrow$ $\forall\, X \in T_{\lambda_i}(x)$, $X \neq 0$, $\quad h_*(X) = -\sin t X - \cot AX = -\frac{1}{\sin\theta_i} X \neq 0$.

i.e. $\operatorname{rank} h_* = m_i$ on C.

$\Rightarrow$ h is a local diffeomorphism.

<u>Thm 8</u> $\quad M^n \subset S^{n+1}$ connected isop. hyp. $\quad \lambda_i = \cot\theta_i \quad 0 < \theta_1 < \cdots < \theta_g < \pi$

$V_i = f_{\lambda_i}(M)$: focal submfd. $\quad \eta$: unit normal to V_i at $p \in V_i$.

Suppose $\eta = h(x)$ for some $x \in f_t^{-1}(p)$. Then $A_\eta X = \cot(\theta_j - \theta_i) X$ for $X \in T_j(x)$, $j \neq i$.

Thm 8 $M^n \subset S^{n+1}$ connected isop. hyp. $\lambda_i = \cot\theta_i$ $0 < \theta_1 < \cdots < \theta_g < \pi$

$V_i = f_{\lambda_i}(M)$: focal submfd. η: unit normal to V_i at $p \in V_i$.

Suppose $\eta = h(x)$ for some $x \in f_t^{-1}(p)$. Then $A_\eta X = \cot(\theta_j - \theta_i) X$ for $X \in T_j(x)$, $j \neq i$.

Corollary 3 $M^n \subset S^{n+1}$ connected isop. hyp. $\lambda_i = \cot\theta_i$ $0 < \theta_1 < \cdots < \theta_g < \pi$

$V_i = f_{\lambda_i}(M)$: focal submfd.

$\Rightarrow$ $\forall \eta$: unit normal to V_i at p. A_η has p.c. $\cot(\theta_j - \theta_i)$ $(j \neq i)$ with multi. m_j.

proof. Thm 8 $\Rightarrow$ Coro 3 holds on $h(C) \underset{\text{open}}{\subseteq} S_p^\perp V_i$

Consider the characteristic poly. $P_u(\eta) = \det(A_\eta - uI)$ as a function of η on $T_p^\perp V_i$

A_η is linear in η $\Rightarrow$ $\forall u \in \mathbb{R}$, $P_u(\eta)$ is a poly. of deg $n - m_i$ on $T_p^\perp V_i$

$\Rightarrow$ $P_u(\eta)\big|_{S_p^\perp V_i}$ is an analytic function of η.

$P_u(\eta)$ is const on $h(C) \underset{\text{open}}{\subseteq} S_p^\perp V_i$ $\Rightarrow$ $P_u(\eta)$ is const on all of $S_p^\perp V_i$.

$M^n \subset S^{n+1}$ isop. hyp. has CPC. focal submfd $V_i = f_{\lambda_i}(M)$ has CPC.

<u>**Corollary 4**</u> $M^n \subset S^{n+1}$ connected isop. hyp. $\Rightarrow$ the focal submfd V_i of M is a minimal submfd of S^{n+1}, $\forall i$

proof. $\forall\, \eta$: unit normal vector to V_i

$\Rightarrow -\eta$: 〃 〃 〃 〃 〃

By Coro 3, A_η, $A_{-\eta}$ have the same eigenvalues with the same multi.

$\Rightarrow \quad \operatorname{tr} A_{-\eta} = \operatorname{tr} A_\eta$

but $\quad A_{-\eta} = -A_\eta \quad \Rightarrow \quad \operatorname{tr} A_{-\eta} = -\operatorname{tr} A_\eta$

$\Rightarrow \quad \operatorname{tr} A_\eta = 0.$

#

Coro 5 (Cartan's identity)

$M^n \subset S^{n+1}$ connected isop. hyp. $\lambda_i = \cot\theta_i$ $\quad 0 < \theta_1 < \cdots < \theta_g < \pi$, m_i.

$$\Rightarrow \forall\, i = 1, \cdots, g \qquad \sum_{j \neq i} m_j \frac{1+\lambda_i\lambda_j}{\lambda_i - \lambda_j} = 0$$

proof. $V_i = f_{\lambda_i}(M)$

Coro 3 $\Rightarrow$ $\forall\, \eta$: unit normal to V_i

$$\operatorname{tr} A_\eta = \sum_{j \neq i} m_j \cot(\theta_j - \theta_i) = \sum_{j \neq i} m_j \frac{1 + \cot\theta_i \cot\theta_j}{\cot\theta_i - \cot\theta_j} = \sum_{j \neq i} m_j \frac{1+\lambda_i\lambda_j}{\lambda_i - \lambda_j}$$

Coro 4 $\Rightarrow$ $\displaystyle\sum_{j \neq i} m_j \frac{1+\lambda_i\lambda_j}{\lambda_i - \lambda_j} = \operatorname{tr} A_\eta = 0$ #

Remark: 1. Cartan's identity $\Leftrightarrow$ minimality of focal submfds.

2. Similar occasions happen in $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$.

Restrictive Conditions for principal Curvatures and their multiplicities:

Thm 9 $M^n \subset S^{n+1}$ Connected isop. hyp. $\lambda_i = \cot\theta_i$ $\quad 0 < \theta_1 < \cdots < \theta_g < \pi$, m_i.

$\Rightarrow$
1. $\theta_i = \theta_1 + (i-1)\frac{\pi}{g}$, $1 \le i \le g$
2. $m_i = m_{i+2}$ (subscripts mod g)
3. $\forall x \in M$, there are $2g$ focal points of (M, x) along the normal geodesic to M through x, and they are evenly distributed at intervals of length $\frac{\pi}{g}$.

proof. • $g=1$. ✓

• $g=2$. $\quad V_1 = f_{\lambda_1}(M) \iff t=\theta_1$

$\Rightarrow$ p.c. of V_1 : $\cot(\theta_2-\theta_1)$ $\quad \forall \eta$: unit normal to V_1

$A_{-\eta} = -A_\eta \Rightarrow -\cot(\theta_2-\theta_1) = \cot(\theta_2-\theta_1) = 0$

$\Rightarrow \theta_2-\theta_1 = \frac{\pi}{2}$

- $g \geq 3$. Fix i, $1 \leq i \leq g$, $V_i = f_{\lambda_i}(M) \leftrightsquigarrow t = \theta_i$

  $\forall \eta$: unit normal to V_i, principal curvature set $= \{\cot(\theta_j - \theta_i) \mid j \neq i\}$

  $A_{-\eta} = A_\eta \Rightarrow \{\cot(\theta_j - \theta_i) \mid j \neq i\} = \{-\cot(\theta_j - \theta_i) \mid j \neq i\}$

- $2 \leq i \leq g-1$, the largest in $\{\cot(\theta_j - \theta_i) \mid j \neq i\}$ is $\cot(\theta_{i+1} - \theta_i) \leftrightsquigarrow m_{i+1}$

  the largest in $\{-\cot(\theta_j - \theta_i) \mid j \neq i\}$ is $-\cot(\theta_{i-1} - \theta_i) \leftrightsquigarrow m_{i-1}$

  $\Rightarrow \theta_{i+1} - \theta_i = \theta_i - \theta_{i-1}$, $m_{i+1} = m_{i-1}$.

- $i = 1$. the largest in $\{\cot(\theta_j - \theta_1) \mid j \neq 1\}$ is $\cot(\theta_2 - \theta_1) \leftrightsquigarrow m_2$

  the largest in $\{-\cot(\theta_j - \theta_1) \mid j \neq 1\}$ is $-\cot(\theta_g - \theta_1) \leftrightsquigarrow m_g$

  $\Rightarrow \theta_2 - \theta_1 = \theta_1 - \theta_g + \pi = \theta_1 - (\theta_g - \pi)$, $m_2 = m_g$

Denote $\theta_2 - \theta_1 = \delta$, $\Rightarrow \theta_g - \theta_1 = (g-1)\delta = \pi - (\theta_2 - \theta_1) = \pi - \delta \Rightarrow \delta = \frac{\pi}{g}$.

$m_1, m_2, m_1, m_2, m_1, \cdots, \underset{\underset{m_2}{\parallel}}{m_g}$ $\Rightarrow$ if g is odd. then $m_1 = m_2 = \cdots = m_g$

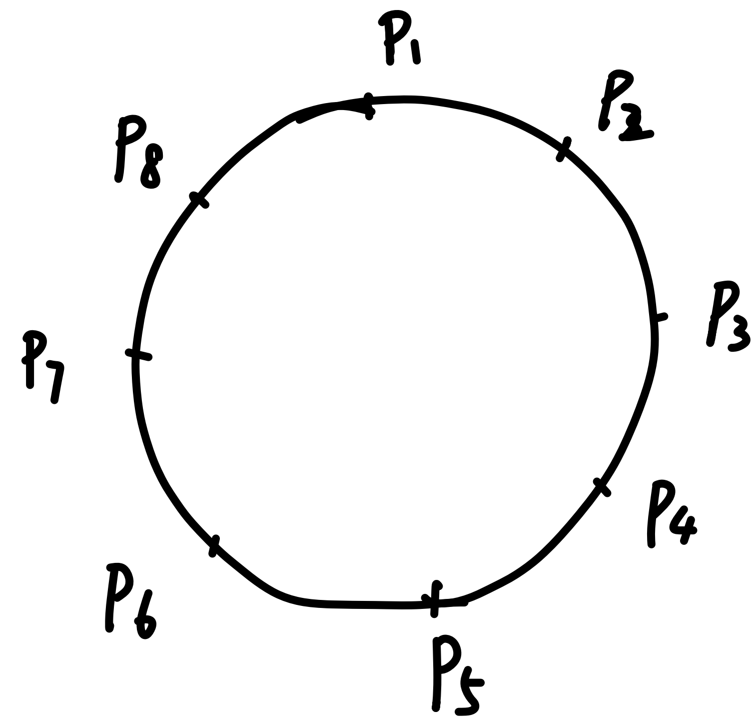
if g is even, $\exists$ at most two distinct multiplicities.

$\forall x \in M$, each $\cot\theta_i$ of M gives rise to 2 antipodal focal points.

$\therefore$ there are $2g$ focal points of (M, x) along the normal geodesic,

and they are evenly distributed at intervals of length $\frac{\pi}{g}$.

eg. $g = 4$



P_i and P_{i+4} are determined by the same p.c. $\lambda_i = \cot\theta_i$, $1 \le i \le 4$.

Coro 6 $V_i \leftrightarrow \lambda_i$ is minimal.

proof. $\forall \eta$, $\quad \operatorname{tr} A_\eta = \sum_{j \neq i} m_j \cot(\theta_j - \theta_i) = \sum_{k=1}^{g-1} m_k \cot \frac{k}{g}\pi = -\sum_{k=1}^{g-1} m_k \cot\left(\pi - \frac{k}{g}\pi\right)$

$$= -\sum_{j=1}^{g-1} m_{g-j} \cot \frac{j}{g}\pi = -\sum_{j=1}^{g-1} m_j \cot \frac{j}{g}\pi = -\operatorname{tr} A_\eta$$

$\Rightarrow \operatorname{tr} A_\eta = 0 \Rightarrow V_i$ is minimal

Classification of isoparametric hypersurfaces with $g=2$

Using Thm 8 & Thm 9, we can derive this classification due to Cartan.

Example (a tube over a totally geodesic submfd of S^{n+1}).

$\mathbb{R}^{n+2} = \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$. $\quad p+q=n$.

$$S^p(r) \times S^q(s) = \{ (x.y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} \mid |x|^2 = r^2 \quad |y|^2 = s^2, \quad r^2+s^2=1 \}$$
$$\subset S^{n+1}(1) \subset \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} = \mathbb{R}^{n+2}$$

- $S^p(r) \times S^q(s)$ is a tube over the totally geodesic p-sphere $V = \{(u.v) \mid v=0\} = S^p(1) \times \{0\}$.

$(u.0) \in V$. $\Rightarrow$ every unit normal to V in S^{n+1} at $(u.0)$ has the form $(0.v)$, $v \in S^q$.

$$\Rightarrow \quad BV = \{ ((u.0), (0.v)) \mid u \in S^p. \ v \in S^q \}$$

$$\Rightarrow \quad f_t((u.0),(0.v)) = \cos t\,(u.0) + \sin t\,(0.v) = (\cos t\, u, \ \sin t\, v) \in S^p(\cos t) \times S^q(\sin t)$$

Conversely, if $(x.y) \in S^p(r) \times S^q(s)$, then $(x.y) = f_t((u.0), (0.v))$, $u = \frac{x}{r}$. $v = \frac{y}{s}$. $t = \cos^{-1} r$

<u>Thm 10</u> $M^n \subset S^{n+1}$ Connected Isop. hyp. with $g=2$. $\Rightarrow M^n \underset{\text{open}}{\subset} S^p(r)\times S^q(s) < S^{n+1}(1)$

pf: $\lambda_1 = \cot\theta_1$ $\quad \lambda_2 = \cot\theta_2$ $\quad 0<\theta_1<\theta_2<\pi$.

$m_1 = q$ $\quad m_2 = p$.

$V_1 = f_{\lambda_1}(M)$ focal submfd $\leftrightarrow \lambda_1$

Thm 8 $\Rightarrow$ V_1 is a p-dim submfd of S^{n+1}

s.t. $\forall \eta$: unit normal to V_1, A_η has one p.c. $\cot(\theta_2-\theta_1)$

Thm 9 $\Rightarrow$ $\theta_2-\theta_1 = \frac{\pi}{2}$ $\Rightarrow \cot(\theta_2-\theta_1)=0$.

$\Rightarrow$ V_1 is a p-dim totally geodesic submfd of S^{n+1}.

i.e. a p-dim great sphere $S^p(1) \subset \mathbb{R}^{p+1} < \mathbb{R}^{n+2}$

and M^n lies on a tube of radius θ_1 over V_1, which is exactly

$S^p(\cos\theta_1)\times S^q(\sin\theta_1)$ by the previous Example.

Isoparametric function on S^{n+1}

Münzner: $M^n \subset S^{n+1}$ isop. hyp. $\Rightarrow$ M is a regular level hyp. of the isop. function V on S^{n+1}, and $V = F|_{S^{n+1}}$,

$F: \mathbb{R}^{n+2} \to \mathbb{R}$ is a homogeneous polynomial of degree g satisfying certain eqns.

Def. $F: \mathbb{R}^{n+2} \to \mathbb{R}$ is homogeneous of degree g if $F(tx) = t^g F(x)$, $\forall t \in \mathbb{R}$.

Euler's thm $\Rightarrow$ $\langle \operatorname{grad}^E F, x \rangle = g F(x)$

Thm 11 $F: \mathbb{R}^{n+2} \to \mathbb{R}$ homogeneous function of degree g. Then

(1) $|\operatorname{grad}^S F|^2 = |\operatorname{grad}^E F|^2 - g^2 F^2$

(2) $\Delta^S F = \Delta^E F - g(g-1)F - g(n+1)F = \Delta^E F - g(n+g)F$.

Thm 11 $F : \mathbb{R}^{n+2} \to \mathbb{R}$ homogeneous function of degree g. Then

(1) $|\mathrm{grad}^S F|^2 = |\mathrm{grad}^E F|^2 - g^2 F^2$

(2) $\Delta^S F = \Delta^E F - g(g-1)F - g(n+1)F = \Delta^E F - g(n+g)F$.

proof. (1) $\mathrm{grad}^S F = \mathrm{grad}^E F - \langle \mathrm{grad}^E F. x \rangle x = \mathrm{grad}^E F - gFx$ ✓

(2) $\forall X \in T_x S^{n+1}$, $\nabla_X \mathrm{grad}^S F = D_X \mathrm{grad}^S F - \langle D_X \mathrm{grad}^S F, x \rangle x$

$$= D_X (\mathrm{grad}^E F - gFx) - \langle D_X(\mathrm{grad}^E F - gFx), x \rangle x$$

$$= D_X \mathrm{grad}^E F - gFX - \langle D_X \mathrm{grad}^E F, x \rangle x$$

$$\Rightarrow \sum_{i=1}^{n+1} \langle D_{e_i} \mathrm{grad}^E F, e_i \rangle = \Delta^E F - \langle D_x \mathrm{grad}^E F. x \rangle$$

$$= \Delta^E F - D_x \langle \mathrm{grad}^E F, x \rangle + \langle \mathrm{grad}^E F, x \rangle$$

$$= \Delta^E F - (g-1) gF$$

$$\Rightarrow \Delta^S F = \sum_{i=1}^{n+1} \langle \nabla_{e_i} \mathrm{grad}^S F, e_i \rangle = \sum_{i=1}^{n+1} \langle D_{e_i} \mathrm{grad}^E F - gFe_i - \langle D_{e_i} \mathrm{grad}^E F, x \rangle x, e_i \rangle$$

$$= \Delta^E F - (g-1) gF - (n+1) gF$$

#

Examples of isoparametric functions on S^{n+1}:

eg I. height function:

$$F: \mathbb{R}^{n+2} \to \mathbb{R}, \quad p \in \mathbb{R}^{n+2}, \ |p| = 1.$$
$$z \mapsto \langle z. p \rangle$$

F is a homog. poly. of degree $g=1$. and $\begin{cases} |\mathrm{grad}^E F|^2 = 1 \\ \Delta^E F = 0. \end{cases}$

Restrict F to S^{n+1} $\Rightarrow$ $\begin{cases} |\mathrm{grad}^S F|^2 = 1 - F^2 \\ \Delta^S F = -(n+1) F \end{cases}$ isoparametric on S^{n+1}!

the level sets of $F|_{S^{n+1}}$: $M_t = \{ x \in S^{n+1} \mid F(x) = t \}$ $-1 < t < 1$

$\{ M_t, -1 < t < 1 \}$ forms an isop. family of n-spheres orthogonal to p.

focal submfds: $\{p\}$ $\{-p\}$

eg 2. $\mathbb{R}^{n+2} = \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ $\quad p, q > 0 \quad p+q=n$

$F: \mathbb{R}^{n+2} \longrightarrow \mathbb{R}$ $\quad$ F is a homogeneous polynomial of degree $g=2$.

$z = (x, y) \longmapsto |x|^2 - |y|^2$

and $\begin{cases} |\mathrm{grad}^E F|^2 = 4r^2 \\ \Delta^E F = 2(p-q) \end{cases}$ $\quad (r^2 = |x|^2 + |y|^2)$

Restrict F to S^{n+1} $\Rightarrow \begin{cases} |\mathrm{grad}^S F|^2 = 4(1-F^2) \\ \Delta^S F = 2(p-q) - 2(n+2) F \end{cases}$ $\quad \therefore$ isoparametric on S^{n+1}

Let $M_t = \{ z \in S^{n+1} \mid F(z) = \cos 2t \}$ $\quad 0 \le t \le \frac{\pi}{2}$

For $(x, y) \in M_t$ $\Rightarrow \begin{cases} |x|^2 - |y|^2 = \cos 2t \\ |x|^2 + |y|^2 = 1 \end{cases} \Rightarrow \begin{cases} |x|^2 = \cos^2 t \\ |y|^2 = \sin^2 t \end{cases}$

$\therefore$ for $t \in (0, \frac{\pi}{2})$, the level set is $S^p(\cos t) \times S^q(\sin t)$

focal submfds: $M_0 = \{ (x, y) \mid y = 0 \} = S^p \times \{0\}$, $M_{\frac{\pi}{2}} = \{ (x, y) \mid x = 0 \} = \{0\} \times S^q$

Remark: One can obtain the same family of level sets from $G(x, y) = |x|^2$, $\quad F = 2G - 1$ on S^{n+1}

Cartan-Münzner polynomial.

Thm 12 $M^n \subset S^{n+1}$ connected isop. hyp. $\lambda_i = \cot\theta_i$, $0<\theta_i<\pi$, m_i $i=1\cdots g$

$\Rightarrow M^n$ is an open subset of a level set of $F|_{S^{n+1}}$,

where F is a homog. poly. of deg g on $\mathbb{R}^{n+2}$ satisfying

$$\begin{cases} |\mathrm{grad}^E F|^2 = g^2 r^{2g-2} \\ \Delta^E F = c\, r^{g-2} \end{cases}, \quad r=|x|, \quad c=\frac{g^2(m_2-m_1)}{2}.$$

Consequences 1 $f = F|_{S^{n+1}} \Rightarrow \begin{cases} |\mathrm{grad}^S f|^2 = g^2(1-f^2) \\ \Delta^S f = c - g(n+g)f \end{cases} \Rightarrow \mathrm{Im} f \subset [-1,1]$

In fact, $\mathrm{Im} f = [-1,1]$.

① $f \neq$ const on $S^{n+1} \Rightarrow \mathrm{Max} f \neq \mathrm{Min} f$

② $\forall s \in (-1,1)$ $\mathrm{grad}^S f \neq 0 \Rightarrow \begin{cases} \mathrm{Max} f = 1 \quad \mathrm{Min} f = -1. \\ f^{-1}(s) \text{ is a compact hypersurface.} \end{cases}$

Münzner also proved (later) that each level set of f is connected.

$\therefore$ the original connected isop. hyp is contained in a unique compact connected isop. hyp.

Consequence 2: For $s=\pm 1$. $\operatorname{grad}^s f = 0$ on $f^{-1}(s)$ and $M_+ = f^{-1}(1)$, $M_- = f^{-1}(-1)$ are submfds of codim >1 in S^{n+1}.

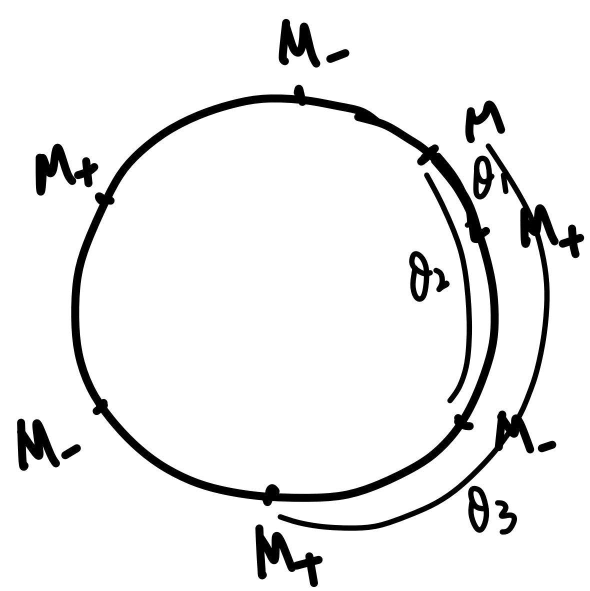
We will show later that M_+, M_- are connected, and they are focal submfds of $f^{-1}(s)$ $-1<s<1$.

$\therefore$ there are only 2 focal submfds regardless of g.

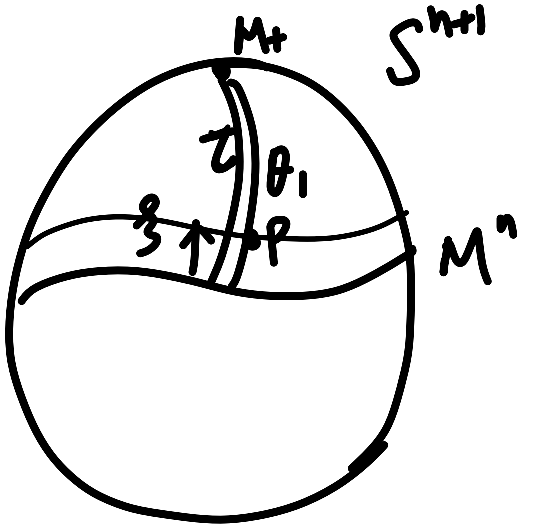
Thm 9 $\Rightarrow$ there are $2g$ focal points evenly distributed along the normal geodesic

We will show later they lie alternately on M_+. M_-.

eg: $g=3$



Münzner's Construction of the polynomial F:



connected ori. isop. hyp.

$\lambda_i = \cot\theta_i \quad m_i$

$0 < \theta_1 < \cdots < \theta_g < \pi$

$NM \cong M \times \mathbb{R}$ trivial.

$E : M \times \mathbb{R} \to S^{n+1}$

$(x, t) \mapsto E(x,t) = f_t(x) = \cos t\, x + \sin t\, \xi(x)$

$\operatorname{rank} E_* = n+1$ if $\cot t \neq \lambda_i \quad \forall i$

For non critical point (x,t) of E, $\exists\, \underset{(x,t)}{\mathcal{U}} \underset{\text{open}}{\subset} M \times R$ s.t. $\mathcal{U} \underset{\text{diffeo}}{\cong} E(\mathcal{U}) = \tilde{\mathcal{U}} \underset{\text{open}}{\subset} S^{n+1}$

Define $\tau : \tilde{\mathcal{U}} \to \mathbb{R}$

$p \mapsto \theta_1 - \pi_2(E^{-1}(p)) = \theta_1 - t$

$f : \tilde{\mathcal{U}} \to \mathbb{R}$

$p \mapsto \cos(g\tau(p))$

On each $f_t M$ in $\tilde{\mathcal{U}}$, $\tau = \text{const}$. $f = \text{const}$.

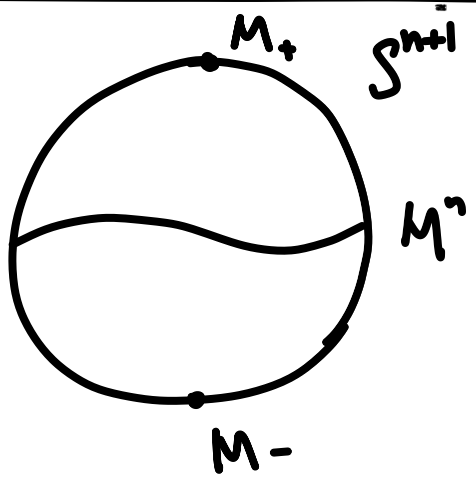
$\begin{cases} M \to M_t \\ \xi \rightsquigarrow -\xi, \end{cases}$ $\tau. f$ the same, $f \rightsquigarrow -f$.

$f \rightsquigarrow F : \mathbb{R}^{n+2} \to \mathbb{R}$

$F(rp) = r^g \cos(g\tau(p))$,

1. F satisfies C-M eqns.
2. F is the restriction of a homog. poly. of deg g to the cone over $\tilde{\mathcal{U}}$. #

Global Structure theorems :



$M^n \subset S^{n+1}$: cpt isop. hyp. g

Then ① M divides S^{n+1} into S_+^{n+1}, S_-^{n+1}, $\partial S_+^{n+1} = \partial S_-^{n+1} = M$

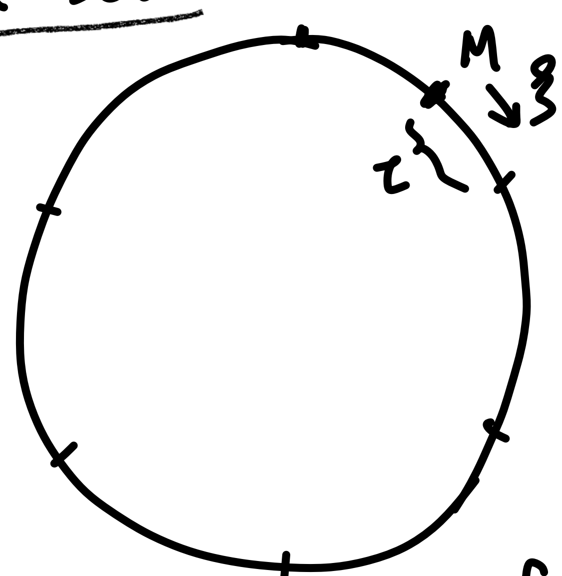
and S_+^{n+1}, S_-^{n+1} are ball bundles over M_+, M_-.

② $g = 1, 2, 3, 4$, or 6.

$F: \mathbb{R}^{n+2} \to \mathbb{R}$: CM-poly of $\deg = g$. $\quad f = F|_{S^{n+1}}$

$\Rightarrow \operatorname{Im} f = [-1, 1]$ $\quad M_t = f^{-1}(t)$ $(-1 < t < 1)$ is an isop. hyp

<u>focal sets</u> :



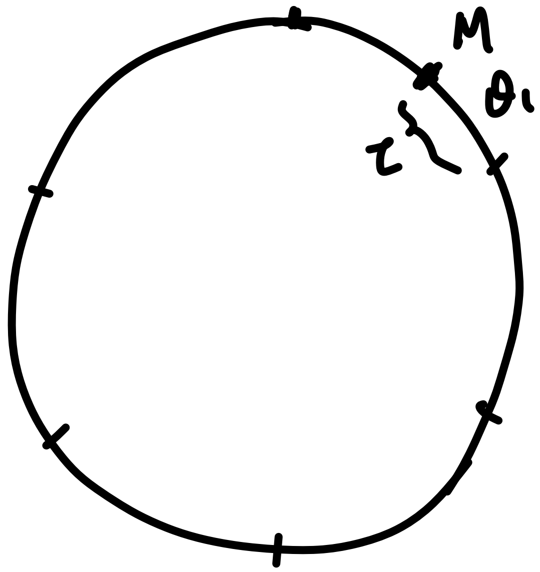
$M = f^{-1}(0)$ $\quad f(p) = \cos(g\,\tau(p))$ $\Rightarrow \tau = \frac{\pi}{2g}$ $(= \theta_1)$

largest p.c. of M : $\cot\theta_1 = \cot\frac{\pi}{2g}$

$\therefore$ the first focal pt of M along ξ direction is

$$f_+(x) = E(x, \tfrac{\pi}{2g}) = \cos\tfrac{\pi}{2g}\, x + \sin\tfrac{\pi}{2g}\, \xi(x)$$

f_+ factors through imm. of M/T_1 into S^{n+1}. $\quad \dim f_+(M) = n - m_1$



Thm 9 $\Rightarrow$ $\theta_g = \theta_1 + \frac{g-1}{g}\pi = \pi - \left(\frac{\pi}{g} - \theta_1\right) = \pi - \frac{\pi}{2g} = -\frac{\pi}{2g} \pmod{\pi}$

$\therefore$ the first focal pt of M in $-\xi$ direction is

$$f_-(x) = E\left(x, -\frac{\pi}{2g}\right) = \cos\left(-\frac{\pi}{2g}\right)x + \sin\left(-\frac{\pi}{2g}\right)\xi(x)$$

the focal map f_- factors through imm. of M/T_g into S^{n+1}.

$\dim f_-(M) = n - m_g = n - m_2$

Thm 13 $F: \mathbb{R}^{n+2} \to \mathbb{R}$ CM-poly $\deg = g$. $f = F|_{S^{n+1}}$.

Then $M_t = f^{-1}(t)$ $(-1<t<1)$, $M_+ = f^{-1}(1)$, $M_- = f^{-1}(-1)$ are connected.

proof $d: M\times\mathbb{R} \to S^{n+1}$ $d(x,\tau) = E\left(x, \frac{\pi}{2g} - \tau\right)$, $f(d(x,\tau)) = \cos g\tau$ on $M\times\mathbb{R}$

$\forall x\in M$. $d|_{\{x\}\times(0,\frac{\pi}{g})}$ is an integral curve of $-\xi$, intersects each M_t, $-1<t<1$ in precisely one point.

$\Rightarrow$ $M\times\left(0,\frac{\pi}{g}\right) \underset{\text{diffeo}}{\cong} f^{-1}(-1,1)$

V analytic $\Rightarrow$ $d\left(M\times\left[0,\frac{\pi}{g}\right]\right)$ is compact, contains dense set $f^{-1}(-1,1)$. $d\left(M\times\left[0,\frac{\pi}{g}\right]\right) \cong S^{n+1}$

$\Rightarrow$ $M_+ = f_+(M)$ $M_- = f_-(M)$ $\Rightarrow$ $S^{n+1} - (M_+ \cup M_-) = f^{-1}(-1,1)$ connected. #

codim $= m_1 + 1 > 1$ codim $= m_2 + 1 > 1$

Consequence : $M^n \subset S^{n+1}$ connected isop. hyp

$\Rightarrow$ M is contained in a unique compact connected isop. hyp

$f^{-1}(t)$ $(-1 < t < 1)$.

<u>Thm 14</u> $M_+ . M_-$: focal submfd in S^{n+1} (M_k) v: a normal to M_k.

$\exp : NM_k \longrightarrow S^{n+1}$ normal exponential map.

Then (1) $f(\exp v) = k \cos(g|v|)$

(2) $B_k = \{ q \in S^{n+1} \mid k f(q) \geq 0 \}$ $(B^\perp M_k . S^\perp M_k)$ bounded unit ball bundle in NM_k

$\Rightarrow$ $\psi_k : (B^\perp M_k, S^\perp M_k) \to (B_k . M)$ is a diffeomorphism.

$\left(M = f^{-1}(0), \quad \psi_k(v) = \exp\left(\frac{\pi}{2g} v\right)\right)$

- $\{\text{normal geodesics to } M_k\} = \{\text{normal geodesics to } M\}$

Thm 15 $M^n \subset S^{n+1}$ compact, connected hyp.

M^n divides S^{n+1} into two ball bundles over submfds M_+, M_-

$\Rightarrow \alpha = \frac{1}{2} \dim H^*(M, R)$ can be only 1, 2, 3, 4 or 6.

($R = \mathbb{Z}$ if M_+, M_- orientable. $R = \mathbb{Z}_2$ otherwise).

Remark: 1. Thm 15 can also apply to more general settings

eg. proper Dupin.

2. In the isoparametric case. $\Rightarrow \alpha = \frac{2n}{m_1 + m_2} = g$.

$\therefore$ g can only be 1, 2, 3, 4, 6.

(Moreover, $H^q(M_\pm)$, $H^q(M)$ are clear).

Examples of Isoparametric Hypersurfaces

Example 1. $F: \mathbb{R}^{n+2} \to \mathbb{R}$, $p \in \mathbb{R}^{n+2}$, $|p|=1$. $f = F|_{S^{n+1}}$ is isoparametric on S^{n+1}!

$(g=1)$ $\quad z \mapsto \langle z.p \rangle$

$M_s = \{ z \in S^{n+1} \mid \langle z.p \rangle = \cos s \}$ $(0 < s < \pi)$ $\quad M_0 = \{p\}$ $\quad M_\pi = -\{p\}$

M_s is an n-sphere of radius $\sin s$ lying in the hyperplane perpendicular to p.

$\Rightarrow$ M_s is the geodesic sphere $S(p, s) = S(-p, \pi - s)$

When $s = \frac{\pi}{2}$, $M_{\frac{\pi}{2}}$ is the unique minimal hypersurface in this isop. family.

Shape operator of M_{s_0}:

$z \in M_{s_0}$, $\quad \tilde{\nabla}_X \operatorname{grad}^S f = -\left(\langle X.p \rangle z + \langle z.p \rangle X \right)$

$$\Rightarrow \langle AX, Y \rangle = -\frac{1}{\rho} H_f (X, Y) = \frac{-1}{\sqrt{1-f^2}} \langle \tilde{\nabla}_X \operatorname{grad}^S f, Y \rangle = \cot s_0 \langle X, Y \rangle$$

$\Rightarrow$ M_{s_0} is totally umbilic with p.c. $\cot s_0$,

and $Sec_{M_{s_0}} = 1 + \cot^2 s_0 = \dfrac{1}{\sin^2 s_0} = Sec_{S^n(\sin s_0)}$

Homogeneity:

$\{M_s\}$ can be realized as the set of orbits of a group action.

$p \in S^{n+1}$. $G_p \subset SO(n+2)$ leaves p fixed.

$\Rightarrow$ G_p is a naturally embedded copy of $SO(n+1)$.

$\forall z \in S^{n+1} \setminus \{\pm p\}$, $G_p z$ has codim $= 1$ in S^{n+1},

$G_p z$ is just the geodesic sphere through z with centers $\pm p$.

Example 2 ($g=2$)

$\mathbb{R}^{n+2} = \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ $p, q > 0$ $p+q=n$

$F: \mathbb{R}^{n+2} \longrightarrow \mathbb{R}$ $f = F|_{S^{n+1}}$ is isoparametric on S^{n+1}. $\begin{cases} |\mathrm{grad}^S f|^2 = 4(1-f^2) \\ \Delta^S f = 2(p-q) - 2(n+2)f \end{cases}$

$z = (x, y) \longmapsto |x|^2 - |y|^2$

$M_s = \{ z \in S^{n+1} \mid F(z) = \cos 2s \} \cong S^p(\cos s) \times S^q(\sin s)$ for $0 < s < \frac{\pi}{2}$

focal submfds: $M_0 = \{ (x, y) \mid y=0 \} = S^p \times \{0\}$, $M_{\frac{\pi}{2}} = \{ (x,y) \mid x=0 \} = \{0\} \times S^q$

$M_{s_0 - t} = f_t(M_{s_0})$ $s_0 - \frac{\pi}{2} < t < s_0$ with $\xi = \dfrac{\mathrm{grad}^S f}{|\mathrm{grad}^S f|}$

At $z = (x, y)$,
$$\tilde{\nabla}_X \mathrm{grad}^S f = 2(1-\cos 2s) X \qquad X \in \mathbb{R}^{p+1} \cap T_z S^{n+1}$$
$$\tilde{\nabla}_Y \mathrm{grad}^S f = -2(1+\cos 2s) Y \qquad Y \in \mathbb{R}^{q+1} \cap T_z S^{n+1}$$

$$T_z M = \{ (X, Y) \mid \langle X, x \rangle = 0, \langle Y, y \rangle = 0 \}.$$

$$\Rightarrow A_\xi X = -\frac{2(1-\cos 2s)}{2\sqrt{1-\cos^2 2s}} X = \underbrace{-\tan s}_{\text{multiplicity} = p} X, \qquad A_\xi Y = \frac{2(1+\cos 2s)}{2\sqrt{1-\cos^2 2s}} Y = \underbrace{\cot s}_{\text{multiplicity} = q} Y$$

Homogeneity :

$z_0 = (x_0 . y_0) \in S^{n+1} \setminus \{M_0 , M_{\frac{\pi}{2}}\}$

$G = SO(p+1) \times SO(q+1) \subset SO(n+2)$

$\mathfrak{g} = \text{Lie } G \qquad \mathfrak{g} = \mathfrak{o}(p+1) \oplus \mathfrak{o}(q+1) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\}$ A. B skew-symm.

$G_{z_0} = SO(p) \times SO(q)$

$\Rightarrow$ each orbit has codim 2 in $\mathbb{R}^{n+2}$, codim 1 in S^{n+1}.

$T_{z_0} M = \{ (Ax_0 . By_0) \mid A \in \mathfrak{o}(p+1), \ B \in \mathfrak{o}(q+1) \}$

$\Rightarrow \xi$: unit normal

$\xi = \pm (\tan s \, x_0, \ -\cot s \, y_0) \qquad s \in (0. \frac{\pi}{2}) \qquad |x_0| = \cos s \quad |y_0| = \sin s$

Take $\xi = (\tan s \, x_0, \ -\cot s \, y_0)$ for $Z = (Ax_0 . By_0)$

$$D_Z \xi = \frac{d}{dt} \begin{bmatrix} \exp tA & 0 \\ 0 & \exp tB \end{bmatrix} \begin{pmatrix} \tan s \, x_0 \\ -\cot s \, y_0 \end{pmatrix} = - \begin{pmatrix} -\tan s \, Ax_0 \\ \cot s \, By_0 \end{pmatrix}$$

M_s is a tube of radius s over $M_0 = S^p \times \{0\}$, Lemma 2 $\Rightarrow \lambda_1 = \tan s, \ \lambda_2 = -\cot s$

homogeneous hypersurface $M^n \subset S^{n+1}$, homog.

rank $=2$

M^n can be characterised as a principal orbit of the isotropy group of some symm. sp. with focal submfds $\leftrightarrow$ singular orbits

- rank 2 symm sp. U/K $\mathfrak{u} = \mathrm{Lie}\, U$. $\mathfrak{k} = \mathrm{Lie}\, K$. $\Rightarrow$ $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition

$z_0 \in \mathfrak{p}$ unit vector. $M = \mathrm{Ad}(K) z_0$: adjoint orbit.

$\mathfrak{k} = \mathfrak{k}_{z_0} \oplus \mathfrak{m}$ $\mathfrak{k}_{z_0} = \{ Y \in \mathfrak{k} \mid [Y, z_0] = 0 \}$ isotropy subalgebra at z_0.

$\Rightarrow$ In $\mathfrak{p}$, $T_{z_0} M = [\mathfrak{m}, z_0]$ $T_{z_0}^{\perp} M = \{ \xi \in \mathfrak{p} \mid [\xi, z_0] = 0 \}$. $A_\xi [\mathfrak{m}, z_0] = -[\mathfrak{m}, \xi]^T$.

- $S^{n+1} \subset \mathfrak{p}$. $K \times \mathfrak{p} \to \mathfrak{p}$. $S^{n+1} \subset \mathfrak{p}$, K acts on S^{n+1} with Cohomogeneity one.

$S^{n+1}/K \cong [-1, 1]$ $M_\pm \leftrightarrow +1, -1$.

$M^n \leftrightarrow$ isotropy subgp. K_0. $M^n \cong K/K_0$

$M_\pm \leftrightarrow$ isotropy subgp. $K_\pm$ $M_\pm \cong K/K_\pm$

$$S^{n+1} = K \times_{K_+} B_+^{m_1+1} \cup_M K \times_{K_-} B_-^{m_2+1}$$

(1972)
R. Takagi–Takahashi's classifications of homog. hyp. in S^{n+1}.

Homogeneous (isoparametric) hypersurfaces in the unit sphere.

g	(m_+, m_-)	(U, K)	K_0	K_+	K_-
1	$n-1$	$(S^1 \times SO(n+1),\ SO(n))$ $n \geq 2$	$SO(n-1)$	$SO(n)$	$SO(n)$
2	(p, q)	$(SO(p+2) \times SO(q+2),$ $SO(p+1) \times SO(q+1))$ $p, q \geq 1$	$SO(p) \times SO(q)$	$SO(p+1) \times SO(q)$	$SO(p) \times SO(q+1)$
3	(1, 1)	$(SU(3), SO(3))$	$\mathbb{Z}_2 + \mathbb{Z}_2$	$S(O(2) \times O(1))$	$S(O(1) \times O(2))$
3	(2, 2)	$(SU(3) \times SU(3), SU(3))$	T^2	$S(U(2) \times U(1))$	$S(U(1) \times U(2))$
3	(4, 4)	$(SU(6), Sp(3))$	$Sp(1)^3$	$Sp(2) \times Sp(1)$	$Sp(2) \times Sp(1)$
3	(8, 8)	(E_6, F_4)	$\mathrm{Spin}(8)$	$\mathrm{Spin}(9)$	$\mathrm{Spin}(9)$
4	(2, 2)	$(SO(5) \times SO(5), SO(5))$	T^2	$SO(2) \times SO(3)$	$U(2)$
4	(4, 5)	$(SO(10), U(5))$	$SU(2)^2 \times U(1)$	$Sp(2) \times U(1)$	$SU(2) \times U(3)$
4	(6, 9)	$(E_6, T \cdot \mathrm{Spin}(10))$	$U(1) \cdot \mathrm{Spin}(6)$	$U(1) \cdot \mathrm{Spin}(7)$	$S^1 \cdot SU(5)$
4	(1, m-2)	$(SO(m+2), SO(m) \times SO(2))$ $m \geq 3$	$SO(m-2) \times \mathbb{Z}_2$	$SO(m-2) \times SO(2)$	$O(m-1)$
4	(2, 2m-3)	$(SU(m+2), S(U(m) \times U(2)))$ $m \geq 3$	$S(U(m-2) \times T^2)$	$S(U(m-2) \times U(2))$	$S(U(m-1) \times T^2)$
4	(4, 4m-5)	$(Sp(m+2), Sp(m) \times Sp(2))$ $m \geq 2$	$Sp(m-2) \times Sp(1)^2$	$Sp(m-2) \times Sp(2)$	$Sp(m-1) \times Sp(1)^2$
6	(1, 1)	$(G_2, SO(4))$	$\mathbb{Z}_2 + \mathbb{Z}_2$	$O(2)$	$O(2)$
6	(2, 2)	$(G_2 \times G_2, G_2)$	T^2	$U(2)$	$U(2)$

Tang–Xie–Yan [redacted] 2012.

H. Takagi (1985): gave equivalent conditions for M to be homog. in terms of derivatives of second fundamental form.

Example 3 ($g=3$) $M^n \subset S^{n+1}$ isop. hyp.

Thm 9 $\Rightarrow m_1 = m_2 = m \Rightarrow n = 3m$

$\begin{cases} \text{(1) level sets of an isop. funct.} \\ \text{(2) tubes over its focal submfd} \\ \text{(3) orbits of certain group action (homogeneous)} \end{cases}$

Cartan (1939) gave the CM-poly. on $\mathbb{R}^{3m+2}$:

$$F(x, y, X, Y, Z) = x^3 - 3xy^2 + \frac{3}{2}x\left(X\bar{X} + Y\bar{Y} - 2Z\bar{Z}\right) + \frac{3\sqrt{3}}{2}y\left(X\bar{X} - Y\bar{Y}\right) + \frac{3\sqrt{3}}{2}\left(XYZ + \overline{ZYX}\right)$$

$x, y \in \mathbb{R}$. $X, Y, Z \in \mathbb{R}^m = \mathbb{R}\ (m=1)$ $\mathbb{C}\ (m=2)$ $\mathbb{H}\ (m=4)$ or $\mathbb{C}a\ (m=8)$

isop. hyp.: $M_t = \{ z \in S^{3m+1} \mid F(z) = \cos 3t \}$ $0 < t < \frac{\pi}{3}$

focal submfds: M_0, $M_{\frac{\pi}{3}}$: a pair of antipodal standard embeddings of $\mathbb{F}P^2$.

M_0: $\mathbb{F} = \mathbb{R}$ standard Veronese surface in S^4.

$\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. parametrization: $X = \sqrt{3}\, v\bar{w}$, $Y = \sqrt{3}\, w\bar{u}$, $Z = \sqrt{3}\, u\bar{v}$

$(u, v, w) \in S^{3m-1} \subset \mathbb{F}$. $x = \frac{\sqrt{3}}{2}\left(|u|^2 - |v|^2\right)$ $y = |w|^2 - \frac{|u|^2 + |v|^2}{2}$

The map is invariant under $(u.v.w) \sim (u\lambda. v\lambda, w\lambda)$ $\quad \lambda \in S^{m-1} \subset \mathbb{F}$

$\Rightarrow$ it is an embedding of $\mathbb{F}P^2$ into S^{3m+1}.

<u>principal curvatures</u>

$\mathbb{F}=\mathbb{R}$. M_0. Thm 8 + Thm 9 $\Rightarrow$ $A_\xi = \begin{pmatrix} \cot\frac{2}{3}\pi & 0 \\ 0 & \cot\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt3} & 0 \\ 0 & \frac{1}{\sqrt3} \end{pmatrix}$

$0<t<\frac{\pi}{3}$ $\quad M_t$ is a tube of radius t over M_0.

Lemma 2 $\Rightarrow$ $A_t = \begin{pmatrix} \cot(\frac{2}{3}\pi - t) & & \\ & \cot(\frac{\pi}{3}-t) & \\ & & -\cot t \end{pmatrix}$

$\operatorname{tr} A_t = 0 \Rightarrow t = \frac{\pi}{6}$

$A_t = \begin{pmatrix} 0 & & \\ & \sqrt3 & \\ & & -\sqrt3 \end{pmatrix}$, $M_{\frac{\pi}{6}}$ is the unique min. isop. hyp in this family.

$\mathbb{F}=\mathbb{C}a$ $\quad$ no corresponding parametrization of $\mathbb{C}aP^2$.

$$V = \{ A \in M_{3\times3}(\mathbb{C}a) \mid \bar{A}^T = A = A^2, \ \operatorname{tr} A = 1 \} \subset S^{25} \subset M_{3\times3}(\mathbb{C}a)$$

Other way to study (minimal) Cartan hyp: Adachi-Maeda (2006), Sanchez (2011), etc

Homogeneity: $\mathbb{F}=\mathbb{R}$, analogous for other cases. Symm. sp. $SU(3)/SO(3)$

$\mathbb{R}^p = M_{3\times3}(\mathbb{R}) \qquad \langle A.B\rangle = \operatorname{tr} AB^t$

$S^4 = \{A \in M_{3\times3}(\mathbb{R}) \mid A=A^t.\ \operatorname{tr}A=0.\ \ |A|=1\}$

$$\begin{array}{ccc} SO(3)\times S^4 & \to & S^4 \\ B \quad A & \mapsto & BAB^t \end{array}$$ isometric & preserves S^4.

$\forall A\in S^4.\ \exists\, U\in SO(3)$ s.t. UAU^t is diagonal.

$\Rightarrow$ every orbit contains a representative of the form B_t:

$$B_t = \operatorname{diag}\sqrt{\tfrac{2}{3}}\left\{\cos\left(t-\tfrac{\pi}{3}\right),\ \cos\left(t+\tfrac{\pi}{3}\right),\ \cos(t+\pi)\right\}$$

- If eigenvalues of B_t are all distinct $\Rightarrow$ dim orbit $=3$. eg $B_{\frac{\pi}{6}} = \operatorname{diag}\{\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\}$

  The isotropy subgroup of $B_{\frac{\pi}{6}}$ is $\operatorname{diag}\{\pm1.\ \pm1.\ \pm1\}. \cong \mathbb{Z}_2\times\mathbb{Z}_2$

  $$\underline{B_{\frac{\pi}{6}} \cong SO(3)/\mathbb{Z}_2\times\mathbb{Z}_2}$$

- If B_t has repeated eigenvalues $\Rightarrow$ dim orbit <3. eg: $B_0 = \operatorname{diag}\{\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\}$

  The isotropy subgroup of $B_0 = S(O(2)\times O(1)) = \left\{\begin{pmatrix} A & 0\\ 0 & \pm1\end{pmatrix} \,\middle|\, A\in O(2),\ \det\begin{pmatrix} A & \\ & \pm1\end{pmatrix}=1\right\}$.

  $$B_0 \cong SO(3)/S(O(2)\times O(1))$$

Examples with $g=4$

1. Nomizu's example with $(g, m_1, m_2) = (4, 1, m-1)$

$m=1$: Cartan.

$$\mathbb{C}^{m+1} = \mathbb{R}^{m+1} \oplus \mathbb{R}^{m+1}$$

$\langle z, w\rangle = \langle x, u\rangle + \langle y, v\rangle$ for $z = x+iy$ $w = u+iv$, $x, y, u, v \in \mathbb{R}^{m+1}$.

$$S^{2m+1} = \{ z \in \mathbb{C}^{m+1} \mid |z| = 1 \}$$

- $m=1$ product of circles in S^3.
- $m \geq 2$. homog. poly : $F(z) = \left| \sum_{k=0}^{m} z_k^2 \right|^2 = \left(|x|^2 - |y|^2 \right)^2 + 4\langle x, y\rangle^2$,

$z = x+iy$

$x = (x_0, \dots, x_m) \in \mathbb{R}^{m+1}$

$y = (y_0, \dots, y_m) \in \mathbb{R}^{m+1}$

$$\Rightarrow \begin{cases} |\mathrm{grad}^E F|^2 = 16 r^2 F \\ \Delta^E F = 16 r^2 \end{cases} \Rightarrow \begin{cases} |\mathrm{grad}^S f|^2 = 16 f(1-f) \\ \Delta^S f = 16 - (16+8m) f \end{cases}$$

$\Rightarrow$ $f = F|_{S^{2m+1}}$ is isoparametric on S^{2m+1}.

Remark: F is not CM-poly.

$\tilde{F} = r^4 - 2F$ is CM-poly with the same level sets as F.

$$\begin{cases} |\mathrm{grad}^E \tilde{F}|^2 = 16 r^6 \\ \Delta^E \tilde{F} = \underline{8(m-2) r^2} \end{cases}$$

$$\left. \begin{array}{l} c = \frac{g^2}{2}(m_2 - m_1) \\ \dim M = 2m = \frac{g}{2}(m_1 + m_2) \end{array} \right\} \Rightarrow \begin{array}{l} m_1 = 1 \\ m_2 = m-1 \end{array}$$

focal submfds

$$\begin{cases} |\mathrm{grad}^S f|^2 = 16 f(1-f) \\ \Delta^S f = 16 - (16+8m) f \end{cases}$$

focal submfds $\Leftrightarrow$ $f = 0, 1.$

- $f = 1 \iff \left|\sum_{k=0}^{m} z_k^2\right| = 1 \iff x /\!/ y \iff z = (x, y)$ lies in $M_0 = \{ e^{i\theta} x \mid x \in S^m \}$

$\dim M_0 = m+1.$

For $x \in S^m$ $\quad T_x M_0 = T_x S^m \oplus \mathrm{Span}\{ix\}$

$T_x^{\perp} M_0 = \{ iy \mid y \in S^m, \langle x, y \rangle = 0 \}$

normal geodesic : $\cos t\, x + \sin t\, iy$

$\Rightarrow$ For $e^{i\theta} x \in M_0$, $\quad T_x^{\perp} M_0 = \{ e^{i\theta} iy \mid y \in S^m, \langle x, y \rangle = 0 \}$

normal geodesic : $\cos t\, e^{i\theta} x + \sin t\, e^{i\theta} iy = e^{i\theta}(\cos t\, x + \sin t\, iy)$

$\Rightarrow$ $V_{m+1,2}$: Stiefel mfd of o.n. pairs (x, y)

$\Rightarrow$ tube M_t of radius t over M_0 is

$$M_t = \{ e^{i\theta} (\cos t\, x + \sin t\, iy) \mid (x, y) \in V_{m+1,2} \}$$

$$f_t : S^1 \times V_{m+1,2} \longrightarrow S^{2m+1} \quad \text{imm.}$$
$$e^{i\theta} \quad (x, y) \longmapsto e^{i\theta}(\cos t\, x + \sin t\, iy)$$

$f_t(e^{i(\theta+\pi)}, (-x, -y)) = f_t(e^{i\theta}, (x, y))$

$M_t = \{ e^{i\theta} (\cos t\, x + \sin t\, iy) \mid (x,y) \in V_{m+1,2} \}$

Substituting into F $\Rightarrow$ $f(z) = (\cos^2 t - \sin^2 t)^2 = \cos^2 2t$

$\therefore$ another focal submfd $\leftrightarrow$ $t = \frac{\pi}{4}$ denote by $M_{\frac{\pi}{4}}$

- $f = 0 \Leftrightarrow |x| = |y|$ $\langle x, y \rangle = 0$ $\Leftrightarrow$ $M_{\frac{\pi}{4}} = \{ \frac{x+iy}{\sqrt{2}} \mid (x,y) \in V_{m+1,2} \}$

  $\dim M_{\frac{\pi}{4}} = 2m-1.$

- isop. hyp. M_t has p.c. $\cot t$, $\cot(t+\frac{\pi}{4})$, $\cot(t+\frac{\pi}{2})$, $\cot(t+\frac{3}{4}\pi)$

  $m-1$ 1 $m-1$ 1

Homogenity: $f_t(e^{i\theta}, (x,y)) = e^{i\theta}(\cos t\, x + \sin t\, iy)$

$\Rightarrow$ M_t admits a transitive group of isometries : $SO(2) \times SO(m+1)$

Remark: 1. Takagi - Takahashi's classification : $(1, k)$ $(2, 2k-1)$ $(4, 4k-1)$ $(2, 2)$ $(4, 5)$ $(6, 6)$

2. Takagi (1976) : if an isop hyp M in S^{2m+1} has p.c. with $(m-1, 1)$

$\Rightarrow$ M is congruent to Nomizu's example.

Isoparametric hypersurfaces with $g=6$

Münzner : $m_1 = m_2 = m$

Abresch : $m = 1$ or 2.

Takagi - Takahashi : found examples with $m=1$ and 2. (homog.)
showed there is only one homog. family in each case.

Peng - Hou : CM-poly of deg $= 6$.

Grove - Halperin, Fang : topology of isop. hyp with $g=6$

Dorfmeister - Neher : isop. hyp. with $g=6$ $m=1$ is homog.
(1985)

Important Open problems in Geometry : 34 : Classification of isop. hyp.s in S^{n+1}
(S. T. Yau. 1990) ($g=4$ & $g=6$ $m=2$)

Miyaoka (1993): $m=1$.

$$\begin{array}{ccc} \tilde{M}^6 & \subset & S^7 \quad \text{isop. } g=6 \ m=1 \\ & & \downarrow \pi : \text{Hopf fibration} \\ M^3 & \subset & S^4 \\ \text{Cartan } g=3 \ m=1 & & \end{array}$$

$\tilde{M}^6 = \pi^{-1}(M^3)$

Miyaoka (2011): $m=2$

$$\begin{array}{ccccc} M^6 & \hookrightarrow & \tilde{M}^{12} & \subset & S^{13} \end{array}$$

Cartan $g=3$ $m=2$ (M^6)

isop: principal G_2 orbit in S^{13}

$$\tilde{M}^{12} \longrightarrow S^6$$

$$\begin{array}{ccc} \mathbb{C}P^2 & \hookrightarrow & \tilde{M}_+ \\ & & \downarrow \\ & & S^6 \end{array} \qquad \begin{array}{ccc} \mathbb{C}P^1 & \hookrightarrow & \tilde{M}_- \\ & & \downarrow \\ & & G_2/SO(4) \end{array}$$

Miyaoka (2013, 2016): Isop. hyp. in S^{n+1} with $g=6$ $m=2$ is homogeneous.

Isoparametric hypersurfaces of OT-FKM type ($g=4$)

Ozeki-Takeuchi (1975, 1976) : analyzing CM-poly F with $g=4$, using certain Clifford algebras.

Condition A + Condition B $\Leftrightarrow$ $(m_1, m_2) = (1, k)$ $(3, 4r)$, $(7, 8r)$

$(3, 4r)$ $(7, 8r)$: first inhomog. isop. examples.

Ferus-Karcher-Münzner (1981): generalized O-T's examples:

Thm $(P_0 \cdots P_m)$ Clifford system on $\mathbb{R}^{2l}$. $m_1 = m \in \mathbb{N}_+$. $m_2 = l-m-1$.

$$F : \mathbb{R}^{2l} \to \mathbb{R}$$
$$x \mapsto F(x) = |x|^4 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2$$

is a CM-poly.

If $m_2 > 0$, then level sets of $F|_{S^{2l-1}}$ form a family of isop. hyp.

(symm.) Clifford system : $\{P_0 \cdots P_m\}$: $P_i \in M_{2l\times 2l}(\mathbb{R})$

$$P_i^t = P_i \quad P_i^2 = I. \quad P_iP_j = -P_jP_i \quad (i\neq j)$$

Clifford algebra C_m : the associative algebra over $\mathbb{R}$, generated by $1, e_1 \cdots e_m$

$$e_i^2 = -1. \quad e_ie_j = -e_je_i \quad (i\neq j) \quad 1\leq i,j\leq m$$

$\dim C_m = 2^m$

Representation of Clifford algebra on $\mathbb{R}^l$: $\{E_1 \cdots E_m\}$ $E_i \in M_{l\times l}(\mathbb{R})$

$$E_i^t = -E_i, \quad E_i^2 = -I \quad E_iE_j = -E_jE_i \quad (i\neq j)$$

$$\boxed{P_0 = \begin{pmatrix} I & \\ & -I \end{pmatrix} \quad P_1 = \begin{pmatrix} & I \\ I & \end{pmatrix} \quad P_{1+i} = \begin{pmatrix} & E_i \\ -E_i & \end{pmatrix} \quad 1\leq i\leq m-1}$$

$\delta(m)$: dim of irreducible C_{m-1} : $l = k\delta(m)$ ($k \in \mathbb{N}_+$ s.t $l-m-1>0$)

m	1	2	3	4	5	6	7	8	$k+8$
C_{m-1}	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$C_{k-1}(16)$
$\delta(m)$	1	2	4	4	8	8	8	8	$16\delta(k)$

$F(x) = |x|^4 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2$ $\qquad f = F|_{S^{2l-1}} \in [-1, 1]$.

$m_1 = m$. $\qquad m_2 = l - m - 1 = k\delta(m) - m - 1$

(m_1, m_2)

k \ $\delta(m)$	1	2	4	4	8	8	8	8	16	32	64	64	...
1	—	—	—	—	(5, 2)	(6, 1)	—	—	(9, 6)	(10, 21)	(11, 52)	(12, 51)	...
2	—	(2, 1)	(3, 4)	(4, 3)	(5, 10)	(6, 9)	(7, 8)	(8, 7)	(9, 22)	(10, 53)	(11, 116)	(12, 115)	...
3	(1, 1)	(2, 3)	(3, 8)	(4, 7)	.	.	.	(8, 15)	.	.	.	(12, 179)	...
4	(1, 2)	(2, 5)	(3, 12)	(4, 11)	.	.	.	(8, 23)	.	.	.	(12, 243)	...
5	(1, 3)	(2, 7)	(3, 16)	(4, 15)	.	.	.	(8, 31)	.	.	.	(12, 307)	...
.	.	.	.	.	.	.	.	.	.	.	.	(.	...
.	.	.	.	.	.	.	.	.	.	.	.	.	...
.	.	.	.	.	.	.	.	.	.	.	.	.	...

$M_+ = f^{-1}(1) = \{x \in S^{2l-1} \mid \langle P_i x, x\rangle = 0$ [redacted] $i = 0, \cdots, m\}$ $\qquad$ Codim $= m_1 + 1 = m+1$

$M_- = f^{-1}(-1) = \{x \in S^{2l-1} \mid \sum_{i=0}^{m} \langle P_i x, x\rangle^2 = 1\}$

$= \{x \in S^{2l-1} \mid \exists P \in \Sigma(P_0 \cdots P_m) \text{ s.t } Px = x\}$ $\qquad$ Codim $= m_2 + 1 = k\delta(m) - m$

M_- is the total space of a $(l-1)$-sphere bundle over m-sphere.

Isoparametric hypersurfaces

g: number of distinct principal curvatures of isoparametric hyp. M^n in S^{n+1}

Münzner (1981)

g can be only $1, 2, 3, 4$ or 6.

- $g \leq 3$: **E. Cartan.**
  $g = 1$: hypersphere;
  $g = 2$: Clifford tori $S^p(r) \times S^{n-p}(s)$ $(0 < p < n, r^2 + s^2 = 1)$;
  $g = 3$: M^n are tubes over minimal Veronese embeddings of $\mathbb{F}P^2$ $(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O})$ into S^{3m+1} $(m = 1, 2, 4$ or $8)$
- $g = 4$: **Abresch**(1983), **Tang**(1991), **Fang**(1999), **Stolz**(1999): multiplicities;
  Cecil-Chi-Jensen (2007), **Immervoll** (2008), **Chi** (2011, 2013, 2016):
  M^n is either of OT-FKM type or homogeneous with (2.2) (4.5)
- $g = 6$: **Dorfmeister-Neher** (1985), **Miyaoka** (2013, 2016):
  M^n must be homogeneous.

[redacted]

OFKM's translation: arXiv: 1112.2780

Applications and related topics of isoparametric functions (hypersurfaces):

C. Peng and Z. Tang [Sci. China, Ser. A, 1996] (Harmonic maps)

H. Ma and Y. Ohnita [Math. Z., 2009; J. Diff. Geom.,2013] (Lagrangian embedding)

J. Ge and Y. Xie [J. Funct. Anal., 2010] (Brézis problem)

Z. Tang, Y. Xie and W. Yan [Comm. Anal. Geom., 2012] (Positive scalar curvature and Schoen-Yau-Gromov-Lawson theory)

Peng - Qian (2017) positive Ricci curvature

R. Miyaoka [Math. Ann., 2012] (Moment maps)

Fujii (2010)

Fujii - Tamaru (2011)

Muto-Ohnita-Urakawa (1984) Kotani (1985)
Muto (1988) Solomon (1990 I. II) ← spectrum of isop. hyp. with $g=3$.

Z. Tang and W. Yan [J. Diff. Geom., 2013] ← first eigenvalue of focal submfds

Z. Tang, Y. Xie and W. Yan [J. Funct. Anal., 2014]

(The first eigenvalue and Yau's conjecture)

C. Qian, Z. Tang and W. Yan [Ann. Glob. Anal. Geom., I and II, 2012, 2013] (New examples of Willmore submanifolds in spheres)

Z. Tang and W. Yan [Adv. Math., 2015]

Q. Li and W. Yan [Sci. China Math., 2015]

(Ricci tensor of focal submanifolds and a problem of Besse)

Ge-Tang (2013)
Qian-Tang (2015) } isop. foliation [redacted] exotic spheres

Tang-Zhang (2020) : minimizing-cone

Tang-Wei-Yan, Tang-Yan : Chern's Conjecture.