超曲面上的几何曲率流及应用 - Lecture 2

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1. Curvature flows and Alexandrov-Fenchel inequalities in Euclidean space (Continued).

2. Geometry of Hyperbolic space, Alexandrov-Fenchel inequalities for h-convex hypersurface in Hyperbolic space.

3. Alexandrov-Fenchel inequalities for hypersurfaces with positive sectional curvature, Maximum principle for function on orthonormal frame bundle.
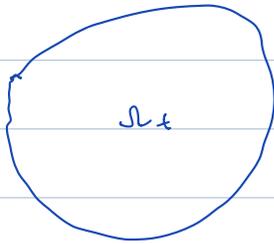
last time

$$\frac{\partial}{\partial t} X = \left(\phi(t) - \left(\frac{E_k}{E_\ell} \right)^{\frac{1}{k-\ell}} \right) \vec{\nu} \quad \text{in } \mathbb{R}^{n+1}$$

$\Sigma_0 \subset \mathbb{R}^{n+1}$ convex $(k_i > 0, \quad i = 1 .. \quad n)$

$\phi(t)$ is chosen s.t $\frac{d}{dt} V_\ell(\Omega_t) = 0$
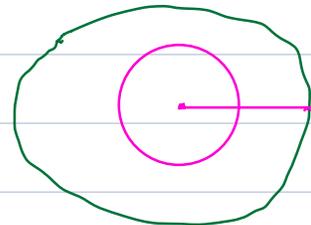
Σ_t

Recal: $V_\ell(\Omega_t) = \displaystyle\int_{\partial \Omega_t} E_{\ell-1} \, d\mu_t$

Ω_t

$\frac{d}{dt} V_k(\Omega_t) \leq 0$

- Short-time existence
- $k_i > 0$
- $k_i \geq \varepsilon H$ is preserved by Andrew's thm

$(\Rightarrow$ If $k = (k_1, k_2, .. \quad k_n)$

$0 < k_1 \leq k_2 \quad \cdots \quad \leq k_n$

$\Rightarrow \quad k_n \leq c \, k_1 \qquad)$

To prove long time existence, we need uniform regularity estimate of Σ_t

flow $\Longleftrightarrow \frac{\partial}{\partial t} r = \left(\phi(t) - F(h_i^j) \right) \cdot \omega$

$\omega = \sqrt{1 + |Dr|^2 / r^2}$

on $S^n \times [0, T)$

$\Sigma = \{ (\theta, r(\theta)) \; \theta \in S^n \}$

$$h_i^j = -\frac{1}{r^2 \omega} \left(\sigma^{jk} - \frac{r^j r^k}{r^2 \omega^2} \right) r_{ik} + \frac{1}{r^3 \omega^3} r^j r_i + \frac{1}{r \omega} \delta_i^j$$

(A) $\boxed{\frac{\partial}{\partial t} r = G[\partial^2 r, \partial r, r, t]}$ on $S^n \times [0, T)$

once we have c^2-estimate. $\Rightarrow$

① (✱) is uniformly parabolic

$$0 < \frac{1}{C}\delta_{ij} \leq \left(\frac{\partial G}{\partial r_{ij}}\right) = \frac{1}{r^2}\left(\frac{\partial F}{\partial h_i^k}\right)\left(\sigma^{jk} - \frac{r^j r^k}{r^2 w^2}\right) \leq C\,\delta_{ij}$$

② $c^{2,\alpha}$-estimate of γ: we need G is concave/convex
w.r.t r_{ij}

$$0 \leq \left(\frac{\partial^2 G}{\partial r_{ij}\,\partial r_{pq}}\right) = -\frac{1}{r^4 w}\left(\frac{\partial^2 F}{\partial h_i^k\,\partial h_p^s}\right)\left(\sigma^{jk} - \frac{r^j r^k}{r^s w^2}\right)\cdot\left(\sigma^{sq} - \frac{r^s r^q}{r^2 w^2}\right)$$

$$\leq 0 \qquad\qquad \geq 0 \qquad\qquad \geq 0$$

Krylov-Safonov / Guji Tian - X.-J. Wang (2013 IMRN)

$$\Rightarrow c^{2,\alpha}\text{-estimate of } \gamma$$

③ Schauder theory $\Rightarrow c^{k,\alpha}$ estimate of γ for all $k \geq 2$

This implies that Σ_t exists for all $t \in [0, +\infty)$

[If not, $T < \infty$. regularity estimate $\Rightarrow \Sigma_t \to \Sigma_T$ smoothly
Short-time existence $\Rightarrow \Sigma_t$ exists for $[0, T+\varepsilon)$]

- convergence to a sphere:

$$k_i \geq \varepsilon H \text{ is preserved}$$
$$\text{We can further prove} \qquad \frac{k_i}{H} \nearrow \frac{1}{n}$$

so key steps : C^2- estimate of Σ_t

<u>C^0- estimate :</u>

$k_n \leq c k_1$ on Σ_t

outer radius   inner radius

Andrews 1994: $\implies \rho_+(\Omega_t) \leq c \rho_-(\Omega_t)$

$\because \quad V_\ell(\Omega_t) \equiv c$

$0 = V_\ell(\Omega_t) \leq V_\ell(B(\rho_+)) = w_n \rho_+^{n+1-\ell} \leq c \, w_n \rho_-^{n+1-\ell}$

$c = V_\ell(\Omega_t) \geq V_\ell(B(\rho_-)) = w_n \rho_-^{n+1-\ell} \geq c \, w_n \rho_+^{n+1-\ell}$

$\implies 0 < \frac{1}{c} \leq \rho_-(\Omega_t) \leq \rho_+(\Omega_t) \leq c$

<u>C^1- estimate</u>

This follows from C^0- estimate + convexity

<u>C^2- estimate</u> $\iff$ curvature estimate

- If we can prove $\left. \begin{array}{c} F \leq C \\ + \quad k_n \leq c \, k_1 \end{array} \right\} \implies 0 < k_i \leq C$

$\left(\because \quad C \geq F = F(k_1, k_2 \cdots k_n) \geq k_1 F(1, \cdots 1) \geq c k_n \right)$

- $F \leq C$: k.Tso's method

$\left(F \leq c \quad \text{as long as} \quad \rho_-(\Omega) \geq c > 0 \right)$

Consider " $Z = \dfrac{F}{u - \delta/2}$ " $u = \langle X - x_0, \vec{v} \rangle$

For $\forall \, t_0 > 0$, $u \geq \bar{r}$ at t_0

we can prove $u \geq \frac{\bar{r}}{2}$ for $t \in [t_0, t_0 + \tau]$

$\frac{\partial}{\partial t} Z \leq \dot{F}^{kl} \nabla_k \nabla_l Z + b^k \nabla_k Z + 2 Z^2 - c Z^3$

dominated term

max principle $\implies Z \leq C$

$\implies F \leq C$

$\square$

Guan-Li 2009: Alexandrov-Fenchel for star-shaped
+ k-convex domain

$$V_k(\Omega) \geq \omega_n^{\frac{k-\ell}{n+1-\ell}} V_\ell(\Omega)^{\frac{n+1-k}{n+1-\ell}}$$

$$0 \leq \ell < k \leq n$$

method: Inverse curvature flow

$$\frac{\partial}{\partial t} X = \frac{E_{k-1}}{E_k} \nu \quad \text{in} \quad \mathbb{R}^{n+1}$$

C. Gerhardt / J. Urbas 1990:

If Σ_0 is <u>star-shaped, k-convex</u> then Σ_t expands to infinity as $t \to \infty$ and $e^{-t} \Sigma_t \longrightarrow S^n(\bar{r})$ smoothly

<u>Check</u>: monotonicity

$$\frac{d}{dt} V_k(\Omega_t) = (n+1-k) \int_{\partial \Omega_t} E_{k-1} d\mu_t = (n+1-k) V_k(\Omega_t)$$

$$\frac{d}{dt} V_{k+1}(\Omega_t) = (n-k) \int_{\partial \Omega_t} \frac{E_{k-1} \cdot E_{k+1}}{E_k} d\mu_t \leq (n-k) \int_{\partial \Omega_t} E_k \, d\mu_t$$

$$= (n-k) V_{k+1}(\Omega_t)$$

$$\therefore \quad I_k(\Omega_t) = \frac{V_{k+1}(\Omega_t)}{V_k(\Omega_t)^{\frac{n-k}{n+1-k}}} \quad \longrightarrow \quad \square$$

We now move to the hyperbolic case.

Hyperbolic space $\mathbb{H}^{n+1}$ is a simply-connected Riemannian manifold with constant negative sectional curvature -1.

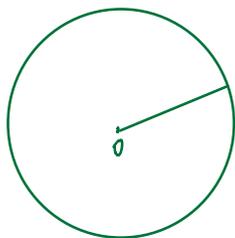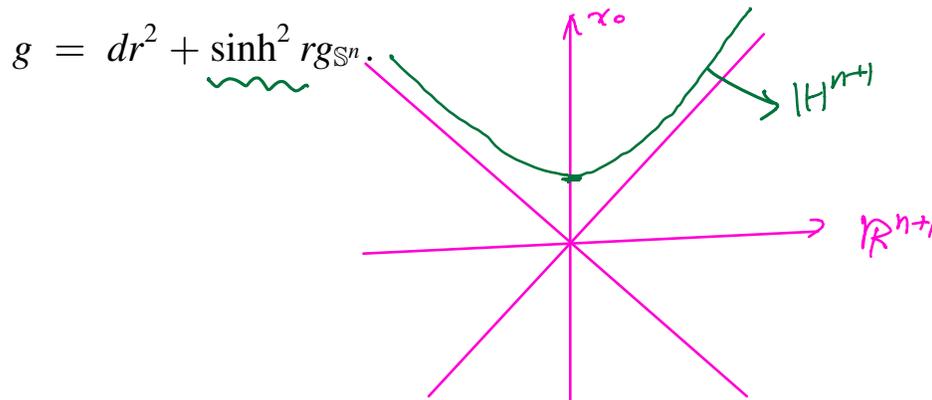1. Hyperboloid model: $\mathbb{H}^{n+1}$ is a hyperboloid in the Minkowski spacetime $\mathbb{R}^{1,n+1}$:

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{1,n+1}, \ \underline{\langle x, x \rangle = -1}, \ \boxed{x_0 > 0}\}$$

   where $\langle x, x \rangle = \underline{-x_0^2} + \sum_{i=1}^{n+1} x_i^2$ for any vector $x = (x_0, \cdots, x_{n+1}) \in \mathbb{R}^{n+2}$.

2. Poincaré disc model.

3. Half-space model. $\left(\mathbb{R}_+^n, \ \frac{1}{x_n^2} \left(dx_1^2 + \cdots + dx_n^2 \right) \right)$

4. We also usually view $\mathbb{H}^{n+1}$ as a Warped product $\mathbb{R}^+ \times \mathbb{S}^n$ equipped with metric

$$g = dr^2 + \underset{\sim}{\sinh^2 r} g_{\mathbb{S}^n}.$$

$$\mathbb{H}^{n+1} = \left(\mathbb{B}^{n+1}(1), \ \frac{4 \, dx^2}{(1 - |x|^2)^2} \right)$$

x_0

$\mathbb{H}^{n+1}$

$\mathbb{R}^{n+1}$

Isoperimetric problem in hyperbolic space has been solved by Schmidt (1940).

▶ For any bounded domain $\Omega \subset \mathbb{H}^{n+1}$,

$$|\Omega| = |B(r)| \quad \Rightarrow \quad |\partial\Omega| \geq |\partial B(r)|$$

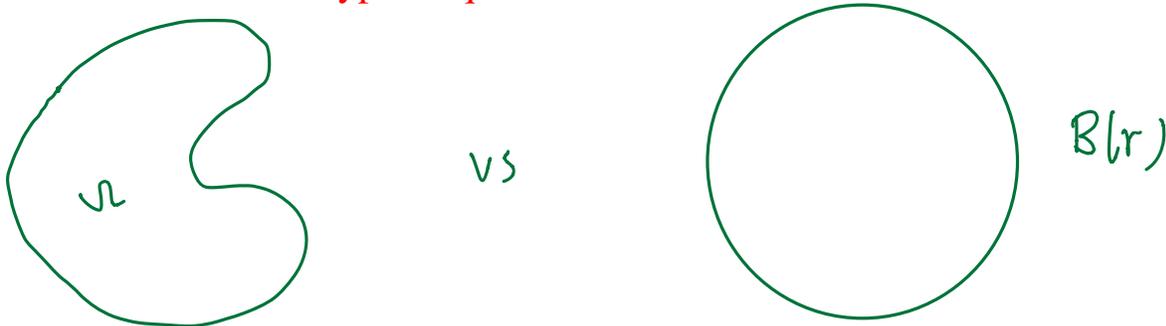with equality holds if and only if $\Omega = B(r)$. In particular, when $n = 1$,

$$L^2 \geq 4\pi A + A^2.$$

▶ We can equivalently formulate this as

$$|\partial\Omega| \geq f_1 \circ f_0^{-1}(|\Omega|)$$

where $f_0(r) = |B(r)|$ and $f_1(r) = |\partial B(r)|$.

Q: Alexandrov-Fenchel type inequalities in $\mathbb{H}^{n+1}$?

Ω vs $B(r)$

Let Ω be a convex domain in $\mathbb{H}^{n+1}$. The quermassintegrals $W_k(\Omega)$ of Ω are defined as

$$W_k(\Omega) = \frac{\omega_{k-1} \cdots \omega_0}{\omega_{n-1} \cdots \omega_{n-k}} \int_{\mathcal{L}_k} \chi(L_k \cap \Omega) dL_k, \quad k = 1, \cdots, n,$$

where $\mathcal{L}_k$ is the space of k-dimensional totally geodesic subspaces L_k in $\mathbb{H}^{n+1}$ and $\omega_n = |\mathbb{S}^n|$, χ is the characteristic function.

If $\partial\Omega$ is C^2, there holds Cauchy-Cronfton formula

$$\left\{ \begin{aligned} W_0(\Omega) &= \text{Vol}(\Omega) \\ W_1(\Omega) &= \frac{1}{n}|\partial\Omega| \end{aligned} \right.$$

> Isoperimetric inequality

$$\bigcirc \quad W_{k+1}(\Omega) = \frac{1}{n-k} \int_{\partial\Omega} E_k - \frac{k}{n-k} W_{k-1}(\Omega), \quad k = 1, \cdots, n-1.$$

$$W_{n+1}(\Omega) = |\mathbb{B}^{n+1}| = \frac{\omega_n}{n+1}.$$

$$\Omega \subset S^n$$

$$W_{k+1}(\Omega) = \frac{1}{n-k} \int_{\partial\Omega} E_k$$

$$+ \frac{k}{n-k} W_{k-1}(\Omega)$$

$\underline{\text{Lem}}$ (Reilly. 1973)

Along the variation $\quad X, \; M^n \times [0, T) \to \mathbb{H}^{n+1}$

$$\frac{\partial}{\partial t} X = F \vec{\nu}$$

$$\Sigma_t = X(M^n, t)$$

$$\Rightarrow \quad \frac{d}{dt} \int_{\Sigma_t} E_k \, d\mu_t = \int_{\Sigma_t} \left((n-k)\, \mathcal{E}_{k+1} + k \, \mathcal{E}_{k-1} \right) F \, d\mu_t$$

休息: $10:51 - 11:00$

Q/A : ① C. Gerhardt 2015: flow in S^n

$$\partial_t X = -F \vec{\nu} \quad \text{in} \quad S^n$$

Σ_0 is convex.

② Makowski - J. Scheuer 15' in S^n

$$\frac{d}{dt} \int_{\Sigma_t} E_k \, d\mu_t = \int_{\Sigma_t} \left((n-k)\, \mathcal{E}_{k+1} + k \, \mathcal{E}_{k-1} \right) F \, d\mu_t$$

<u>proof</u>: $\Sigma_t^n \subset \mathbb{H}^{n+1}$ $\qquad \frac{\partial X}{\partial t} = F\vec{\nu} \in T\mathbb{H}^{n+1}$

$$\frac{\partial}{\partial t} h_{ij} = - \frac{\partial}{\partial t} \bar{g}\left(D_{\partial_i X} \partial_j X, \vec{\nu} \right)$$

$$= - \bar{g}\left(D_{\partial_t X} D_{\partial_i X}(\partial_j X), \vec{\nu} \right) - \bar{g}\left(D_{\partial_i X} \partial_j X, \partial_t \vec{\nu} \right)$$

$$= - \bar{g}\left(D_{\partial_i X} D_{F\vec{\nu}}(\partial_j X), \vec{\nu} \right) + \bar{g}\left(\bar{R}(\partial_i X, \partial_t X) \partial_j X, \vec{\nu} \right)$$

$$\qquad\qquad\qquad\qquad\qquad - \bar{g}\left(D_{\partial_i X}\partial_j X, \partial_t \vec{\nu} \right)$$

$$= - \nabla^2_{ij} F + F (h^2)_{ij} - F \underbrace{\bar{R}_{i\nu\nu j}}_{= F g_{ij}}$$

$$\Rightarrow \quad \frac{\partial}{\partial t} h_i^{\ j} = - \nabla^j \nabla_i F - F (h^2)_i^{\ j} + F \delta_i^{\ j}$$

$$\Rightarrow \quad \frac{\partial}{\partial t} \mathcal{E}_k = \dot{\mathcal{E}}_k^{\ ij} \partial_t h_i^{\ j}$$

$$= - \nabla^j \left(\dot{\mathcal{E}}_k^{\ ij} \nabla_i F \right) - F \dot{\mathcal{E}}_k^{\ ij} (h^2)_i^{\ j} + F \dot{\mathcal{E}}_k^{\ ij} \delta_i^{\ j}$$

$\left(\dot{\mathcal{E}}_k^{\ ij} \right)$ is divergence free, $\quad \begin{cases} \int \dot{\mathcal{E}}_k^{\ ij} (h^2)_i^{\ j} = n E_1 \mathcal{E}_k - (n-k) \mathcal{E}_{k-1} \\ \dot{\mathcal{E}}_k^{\ ij} \delta_i^{\ j} = k \, \mathcal{E}_{k-1} \end{cases}$ $\qquad \square$
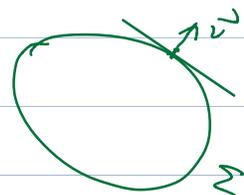
Quermassintegrals in hyperbolic space have a similar variation equation as in Euclidean case:

Lemma (Wang-Xia, 2014)   *Guofang Wang.   Chao Xia*

If the boundary of Ω_t moves outward with speed F

$$\frac{\partial}{\partial t} X(x,t) = F(x,t)\nu(x,t),$$

then

$$\frac{d}{dt} W_\ell(\Omega_t) = \int_{M_t} F\, E_\ell d\mu_t, \quad \ell = 0, 1, \cdots, n.$$

This motivates us to study the volume / quermassintegral preserving flows

$$\frac{\partial}{\partial t} X = \left(\phi(t) - \left(\frac{E_k}{E_\ell} \right)^{1/(k-\ell)} \right) \nu, \quad 0 \le \ell < k \le n \tag{1}$$

in $\mathbb{H}^{n+1}$ with $\phi(t)$ chosen to preserve $W_\ell(\Omega_t)$. We again have $\frac{d}{dt} W_k(\Omega_t) \le 0$ as in Euclidean case.

Cabezas-Rivas and Miquel (2007): Volume preserving mean curvature flow, i.e., $k = 1, \ell = 0$. If the initial domain Ω_0 is h-convex, then the solution Ω_t converges to a geodesic ball.

horospherically convex ($k_i \geqslant 1$)

Failure of convexity: "Convexity $\kappa_i > 0$" may NOT be preserved along the flow (1). The difficulty arises in the evolution of second fundamental form, which has terms arising from the background geometry:

$$\frac{\partial}{\partial t} h_{ij} = -\nabla_i \nabla_j F + F \left((h^2)_{ij} + g_{ij} \right).$$

When $F = \phi(t) - H$,

$$\frac{\partial}{\partial t} h_{ij} = \nabla_i \nabla_j H + (\phi(t) - H) \left((h^2)_{ij} + g_{ij} \right)$$

$$= \Delta h_{ij} + (|h|^2 + n) h_{ij} + (\phi(t) - 2H)(h^2)_{ij} + \phi(t) g_{ij} - 2H g_{ij},$$

where $-2H g_{ij}$ term is a bad term here.

Simons identity:

$$\nabla_i \nabla_j H = \Delta h_{ij} + (|h|^2 + n) h_{ij} - H \left((h^2)_{ij} + g_{ij} \right)$$

If we impose a stronger condition "*h-convexity*", i.e., $\kappa_i \geq 1$, let $S_{ij} = h_{ij} - g_{ij}$,

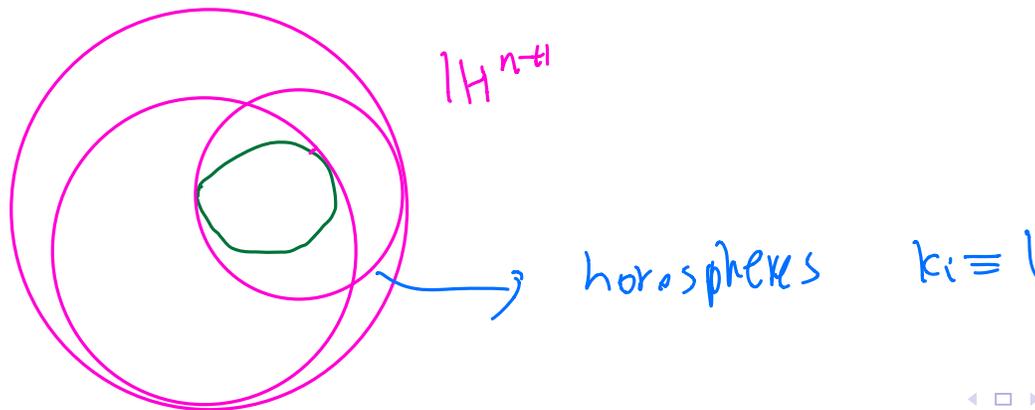$$\frac{\partial}{\partial t}S_{ij} = \Delta S_{ij} + (|h|^2 + n - 2H)S_{ij} + (\phi(t) - 2H)(S^2)_{ij}$$

$$+ (|h|^2 + n - 2H)g_{ij}$$

$$\underbrace{\qquad\qquad} \geq \frac{H^2}{n} + n - 2H \geq 0$$

Hamilton's tensor maximum principle implies that *h-convexity* is preserved.

$$\frac{\partial}{\partial t}X = \left[\phi(t) - \left(\frac{E_k}{E_\ell}\right)^{\frac{1}{k-\ell}}\right]\vec{\nu} \qquad \text{in} \quad \mathbb{H}^{n+1}$$

Andrews tensor max principle $\Rightarrow$ $k_i \geq 1$ is preserved

$\left[\text{Alternative approach} : \text{"constant rank theorem"}\right)$

$\mathbb{H}^{n+1}$

$\longrightarrow$ horospheres $k_i \equiv 1$

Guofang Wang and Chao Xia (2014): if Ω_0 is h-convex, then along the flow

$$\frac{\partial}{\partial t}X = \left(\phi(t) - \left(\frac{E_k}{E_\ell}\right)^{1/(k-\ell)}\right)\nu, \quad 0 \le \ell < k \le n$$

with $\phi(t)$ chosen to preserve $W_\ell(\Omega_t)$, Ω_t is h-convex for all $t > 0$, and Ω_t converges to a geodesic ball $B(r)$ as $t \to \infty$.

Since $W_\ell(\Omega_t)$ is fixed and $W_k(\Omega_t)$ is decreasing,

$$W_\ell(\Omega_0) = W_\ell(B(r)) = f_\ell(r),$$
$$W_k(\Omega_0) \ge W_k(B(r)) = f_k(r).$$

Alexandrov-Fenchel in $\mathbb{H}^{n+1}$ (Wang-Xia, 2014)

If Ω is a smooth, bounded, h-convex domain in $\mathbb{H}^{n+1}$, then

$$\boxed{W_k(\Omega) \ge f_k \circ f_\ell^{-1}(W_\ell(\Omega)), \quad 0 \le \ell < k \le n}$$

with equality if and only if Ω is a geodesic ball, where $f_k(r) = W_k(B(r))$ is an increasing function.

$\underline{Rmk}$. $\quad \Omega \subset \mathbb{H}^{n+1} \quad$ is $\quad$ h-convex . bounded

$$\ell_+(\Omega) \le \ell_-(\Omega) + \ln 2$$

The h-convex condition is convenient for the analysis but strong geometrically. We would like to weaken this assumption to something more closely analogous to convexity in the Euclidean case.

1. $k = 3, \ell = 1$, 2-convex and star-shaped (by Haizhong Li, Changwei Xiong and Yong Wei (2012))

2. $k = n, 0 \le \ell < n$, convex; $k = 2, \ell = 1$, mean convex and star-shaped (by S. Brendle, P. Guan and J. Li).

3. $\ell = 0, 0 < k \le n$, positive sectional curvature (by Ben Andrews, Xuzhong Chen and Yong Wei, 2018).

4. $k = n - 1, \ell = n - (2i + 1)$ $(0 < 2i < n)$, convex (by B. Andrews, Yingxiang Hu and Haizhong Li 2019).

method: $\partial_t X = - \dfrac{E_n}{E_{n-1}} \vec{\nu}$

Q: $(k - 1)$-convex and star-shaped for all $0 \le \ell < k \le n$?

① : $\qquad \dfrac{\partial}{\partial t} X = \dfrac{E_1}{E_2} \vec{\nu} \qquad$ in $\qquad \mathbb{H}^{n+1}$

$\Rightarrow \qquad \dfrac{d}{dt} W_1(\Omega_t) = \displaystyle\int_{\partial \Omega_t} \dfrac{E_1^2}{E_2} \, d\mu_t \ge |\partial \Omega| = n \, W_1(\Omega)$

$$\frac{d}{dt} W_3(\Omega) = \int_{\partial\Omega} \frac{E_1}{E_2} \cdot E_3 \leq \int_{\partial\Omega} E_2 \, d\mu = (n-2) W_3 + 2 W_1$$

$$\Rightarrow \quad I(\Omega) = \frac{W_3(\Omega) - W_1(\Omega)}{W_1(\Omega)^{\frac{n-2}{n}}} \quad \searrow$$

c. Gerhardt 2011.

$$\frac{\partial}{\partial t} X = -\frac{E_{k-1}}{E_k} \, \nu \qquad \text{in} \quad |H^{n+1}$$

k-convex + star-shaped $\implies$ convergence

$\underline{k > 2}$: monotonicity for k-convex, star-shaped ?

If h-convex, Ge-Wang-Wu 13'

$$\int_{\partial\Omega} E_{2k} \, d\mu \geq w_n \left(\left(\frac{|\Sigma|}{w_n} \right)^{\frac{1}{k}} + \left(\frac{|\Sigma|}{w_n} \right)^{\frac{1}{k} \cdot \frac{n-2k}{n}} \right)^k$$

② **Brendle - Guan - Li** (preprint)

$$\frac{\partial}{\partial t} X = \left(\cosh r \cdot \frac{E_{k-1}}{E_k} - \langle \sinh r \, \partial_r, \vec{\nu} \rangle \right) \vec{\nu} \qquad \text{in} \quad |H^{n+1}$$

Minkowski formula: $\Sigma^n \subset |H^{n+1}$

$$\left(|H^{n+1} = |R^+ \times S^n \atop dr^2 + \sinh^2 r \, g_{S^n} \right)$$

$$\int_{\Sigma} \cosh r \cdot E_{k-1} \, d\mu = \int_{\Sigma} \langle \sinh r \, \partial_r, \vec{\nu} \rangle \cdot E_k \, d\mu$$

$$\begin{cases} \dfrac{d}{dt} W_k(\Omega_t) = \int_{\partial\Omega_t} \left(\cosh r \cdot E_{k-1} - \langle \sinh r \, \partial_r, \vec{\nu} \rangle \cdot E_k \right) = 0 \\[4mm] \dfrac{d}{dt} W_{k+1}(\Omega_t) = \int_{\partial\Omega_t} \left(\cosh r \cdot \underbrace{\frac{E_{k-1}}{E_k} \cdot E_{k+1}}_{\leq E_k} - \langle \sinh r \, \partial_r, \vec{\nu} \rangle E_{k+1} \right) \leq 0 \end{cases}$$

$\underline{Q}$: Convergence to a geodesic ball ?

① $k = n$, convex $\checkmark$

② $1 \leq k < n$, k-convex, star-shaped + initial C^1-condition

$$\max_{t=0} |D \ln \cosh r|^2 \leq 12 + 3 \min_{t=0} \sinh^2 r$$

Theorem (Andrews-Chen-W., 2018)

Let $\Omega \subset \mathbb{H}^{n+1}$ be a smooth bounded domain with boundary $\partial\Omega$ having positive sectional curvature. Then for any $n \geq 2$ and $k = 1, \cdots, n$, there holds

$$W_k(\Omega) \geq f_k \circ f_0^{-1}(W_0(\Omega)),$$

with equality if and only if Ω is a geodesic ball, where $f_k(r) = W_k(B(r))$.

The proof again uses the volume preserving curvature flow

$$\frac{\partial}{\partial t}X = \left(\phi(t) - E_k^{1/k}\right)\nu, \quad 1 \leq k \leq n.$$

$\left(\dfrac{E_k}{E_\ell}\right)^{\frac{1}{k-\ell}}$

Key step: "positive sectional curvature" is preserved along the flow.

Gauss equation: $\Sigma^n \subset \mathbb{H}^{n+1}$

$$R_{ijij} = h_{ii}h_{jj} - (h_{ij})^2 + \overline{R}_{ijij}$$

$$k(e_i \wedge e_j) = k_i k_j - 1$$

$\therefore$ PSC $\iff$ $k_i k_j > 1$ for all $i \neq j$

Given a point $x \in M$ and $t \geq 0$, and a frame $\mathbb{O} = \{e_1, \cdots, e_n\}$ for T_xM which is orthonormal with respect to $g(x,t)$, we define

$$\mathbb{O}' = \{e_3, e_4, e_1, e_2 \cdots \ e_n\}$$

$$\boxed{G(x,t,\mathbb{O}) = h_{(x,t)}(e_1,e_1)h_{(x,t)}(e_2,e_2) - h_{(x,t)}(e_1,e_2)^2 - 1.}$$

$G > 0$ at $t = 0$. We consider a point (x_0, t_0) and a frame $\mathbb{O}_0 = \{\bar{e}_1, \cdots, \bar{e}_n\}$ at which a new minimum of the function G is attained, so that we have

$$G(x,t,\mathbb{O}) \geq G(x_0, t_0, \mathbb{O}_0)$$

for all $x \in M$ and all $t \in [0, t_0]$, and all $\mathbb{O} \in F(M)_{(x,t)}$.

The fact that $\mathbb{O}_0$ achieves the minimum of G over the fibre $F(M)_{(x_0,t_0)}$ implies that $\bar{e}_1$ and $\bar{e}_2$ can be rotated to be the eigenvectors of $h_{(x_0,t_0)}$ corresponding to κ_1 and κ_2, where

$$\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$$

are the principal curvatures at (x_0, t_0). Since G is invariant under rotation in the subspace orthogonal to $\bar{e}_1$ and $\bar{e}_2$, we assume $h(\bar{e}_i, \bar{e}_i) = \kappa_i$ and $h(\bar{e}_i, \bar{e}_j) = 0$ for $i \neq j$.

We calculate the evolution equation for G.

Fix x_0

Noting that the frame $\mathbb{O}(t)$ for $T_x M$ defined by

Weingarten operator

$$\frac{d}{dt}e_i(t) = (F - \phi(t))\mathcal{W}(e_i)$$

remains orthonormal with respect to $g(x, t)$ if $e_i(t_0) = \bar{e}_i$ for each i. In fact,

$$\frac{d}{dt}g(e_i, e_j) = \left(\frac{\partial}{\partial t}g\right)(e_i, e_j) + g\left(\frac{d}{dt}e_i(t), e_j(t)\right) + g\left(e_i(t), \frac{d}{dt}e_j(t)\right)$$

$$= (\phi(t) - F)\left(2h(e_i, e_j) - g(\mathcal{W}(e_i), e_j) - g(e_i, \mathcal{W}(e_j))\right) = 0.$$

This yields:

$$\frac{\partial}{\partial t}G\bigg|_{(x_0, t_0, \mathbb{O}_0)} = h_{11}\left(\frac{\partial}{\partial t}h_{22} + 2h\left(\frac{d}{dt}e_2, e_2\right)\right) + h_{22}\left(\frac{\partial}{\partial t}h_{11} + 2h\left(\frac{d}{dt}e_1, e_1\right)\right)$$

$G(x_0, t, \mathbb{O}(t))$

$$- 2h_{12}\frac{\partial}{\partial t}\bigg|_{t=t_0}h(e_1, e_2)$$

$$= \kappa_1\left(\frac{\partial}{\partial t}h_{22} + 2(F - \phi(t))(h_{22})^2\right) + \kappa_2\left(\frac{\partial}{\partial t}h_{11} + 2(F - \phi(t))(h_{11})^2\right)$$
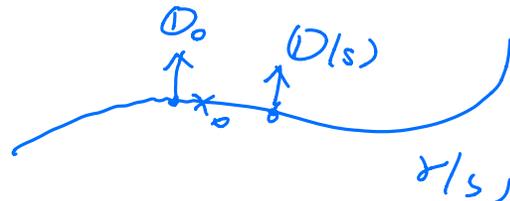
$\dot{f}^k \, \bar{\lambda}_k^2 \, h_{22}$

$\ddot{f}^k \, \bar{\lambda}_k^2 \, h_{11}$

We let γ be any geodesic of $g(t_0)$ in M with $\gamma(0) = x_0$, and define a frame $\mathbb{O}(s) = (e_1(s), \cdots , e_n(s))$ at $\gamma(s)$ by taking $\underline{e_i(0) = \bar{e}_i}$ for each i, and

$$\frac{d}{ds} g_{t_0}(e_i, e_j) = P_{ij} + P_{ji} = 0 \qquad \underline{\nabla_s e_i(s) = \Gamma_{ij} e_j(s)}$$

$\mathbb{O}_0 \uparrow \quad \mathbb{O}(s) \uparrow$

x_0

$\gamma(s)$

for some constant antisymmetric matrix Γ.

Since $(x_0, \mathbb{O}_0)$ is a minimum point of G at time t_0, we have

$$0 = \frac{d}{ds} G(\gamma(s), t_0, \mathbb{O}(s))\Big|_{s=0} = \kappa_2 \nabla_s h_{11} + \kappa_1 \nabla_s h_{22}.$$

Then we compute $\underline{\text{second derivative}}$

$$0 \le \frac{d^2}{ds^2} G(\gamma(s), t_0, \mathbb{O}(s))\Big|_{s=0} = \kappa_2 \nabla_s^2 h_{11} + \kappa_1 \nabla_s^2 h_{22} + 2\left(\nabla_s h_{22} \nabla_s h_{11} - (\nabla_s h_{12})^2 \right)$$

$$+ 4 \sum_{p>2} \Gamma_{1p} \kappa_2 \nabla_s h_{1p} + 4 \sum_{p>2} \Gamma_{2p} \kappa_1 \nabla_s h_{2p}$$

$$+ 2 \sum_{p>2} \Gamma_{1p}^2 \kappa_2 (\kappa_p - \kappa_1) + 2 \sum_{p>2} \Gamma_{2p}^2 \kappa_1 (\kappa_p - \kappa_2). \quad (2)$$

Since G has a minimum at $(x_0, t_0, \mathbb{O}_0)$, the right-hand side of (2) is non-negative for any choice of Γ. Minimizing over Γ gives

$$0 \leq \kappa_2 \nabla_s^2 h_{11} + \kappa_1 \nabla_s^2 h_{22} + 2 \left(\nabla_s h_{22} \nabla_s h_{11} - (\nabla_s h_{12})^2 \right)$$

$$- 2 \sum_{p>2} \frac{\kappa_2}{\kappa_p - \kappa_1} (\nabla_s h_{1p})^2 - 2 \sum_{p>2} \frac{\kappa_1}{\kappa_p - \kappa_2} (\nabla_s h_{2p})^2.$$

By a direct and lengthy calculation, and using the properties for $F = E_k^{1/k}$, we obtain that the spatial minimum of G satisfies

$$\frac{\partial}{\partial t} G \Big|_{(x_0, t_0, \mathbb{O}_0)} \geq \boxed{\text{nonnegative gradient terms}} - CG.$$

Parabolic maximum principle implies that $G > 0$ is preserved.

Thank you!

- Q & A.