2020年7月24日 7:55

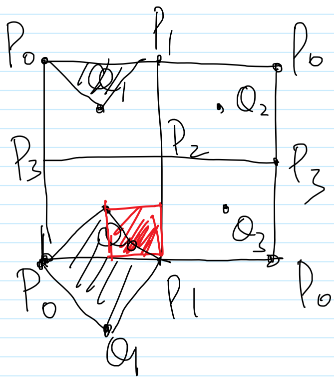
Recall. Choe – Soret. $\lambda_1(\xi_{m,k}) = 2$

Idea: (1). $\xi_{m,k}$ σ_a-inv. $a \in S^3 \subset \mathbb{R}^4$

$a \in \{a_j, b_e\}$ $\sigma_a(y) = y - 2\langle y, a\rangle a$

$= \begin{cases} y & y \perp a \\ -y & y \parallel a. \end{cases}$

(2). Courant Nodal thm $\lambda_1 \to f_1$. If $\lambda_1 < 2 \Rightarrow f_1$ has 3 nodal domain. Contra. $\Rightarrow \lambda_1 = 2$

(3) 

Furthmore:

Thm: $y_0 : \Sigma \to S^3$ one of $\xi_{m,k}$ Σ Re surface.

Assume $y : \Sigma \to S^n$ $n \geq 3$. conformal immersion. Then

$$W(y) = \int_\Sigma (1 + H^2)\, dM \geq W(y_0) = \text{Area}(y_0)$$

" = " $\Leftrightarrow$ y is confal congruent to y_0.

Pf: (1) Choe-Soret $\Rightarrow$ $E_{\lambda=2} = \{f \mid \Delta_\Sigma f + 2f = 0\}$

$\dim E_{\lambda=2} = 4$. $\text{span}_{\mathbb{R}}\{y_1, y_2, y_3, y_4\}$

Proof: Assume $f \perp_{L^2} \{y_j, j = 1 \sim 4\}$. $\dashrightarrow$ f has 3 nodal domain

(2). If y λ_1-minimal $\Rightarrow$ $y = B \cdot y_0$, $B \in SL(4, \mathbb{R})$

(2). If y ∂_1-minimal $\Rightarrow$ $\underline{y} = B \cdot y_0$ $\Rightarrow dy = B \cdot dy_0$ } $B \in SL(4, \mathbb{R})$

Montiel–Ros 1986 $\Rightarrow |dy|^2 = |dy_0|^2$ } $B \in O(4)$

$\Rightarrow$ y congruent to y_1

$\underline{\xi_{m,k}}$ is unique in the sense it is the only ∂_1-minimal immersion from Σ

(3). Li–Yau '82 $W(y) \geq A_c(\Sigma) = \text{Area}(\xi_{m,k})$

conformal area of Riemann surface Σ ; ∂_1-minimal

"=" $\Rightarrow$ ...... $\Rightarrow$ y is conformally congruent to y_0.

Rmk: ① Kusner – Willmore Conj. genus $\Sigma = g$ $W(\Sigma) \geq \text{Area}(\xi_{g,1})$

"=" $\Leftrightarrow$ $= \xi_{g,1}$

② Restriction of Riemann surface structure true!

③ $2\pi^2 \leq \text{Area}(\xi_{g,1}) < 8\pi$ $\rightarrow 8\pi$ $g \rightarrow +\infty$.

↓ Clifford torus.

§4. Index & nullity of $\underline{\xi_{g,1}}$

① Recall. in S^3. $\frac{d^2}{dt^2} \text{Area}(y_t)\Big|_{t=0} = -\int_\Sigma f \, J f \, dM$

$V = f \cdot \vec{n}$, $J = \Delta_\Sigma + 2 + |II|^2$.

↓ 2.1 → curv of S^3.

2. If $Jf + \mu f = 0$ μ - eigenvalue f eigenfunction.

$$\frac{d^2}{dt^2}(\ ) = \mu \int_\Sigma f^2 \qquad \mu<0 \quad \text{unstable}$$

3. Index(y) = $\#\{\mu<0\}$ 个数

Nullity(y) = $\#\{\mu=0\}$

4. J. Simons (70s). Variation + Simons Equality

① $\forall\, a \in \mathbb{R}^4$. $\vec{n}$ unit normal vector.

$$\Delta_\Sigma \vec{n} + |II|^2 \vec{n} = 0 \quad (\text{Ex: } \dots) \quad n_{z\bar{z}}$$

$$\Delta_\Sigma \langle n, \vec{a}\rangle + |II|^2 \langle \vec{n}, a\rangle = 0$$

$$(J-2)\langle \vec{n}\cdot\vec{a}\rangle = 0 \Rightarrow \mu = -2.$$

$\#\{\mu=-2\} \geq 1$. ("=1" $\Leftrightarrow$ totally geodesic)

⇕

$S^2 \hookrightarrow S^3$

$$1 = \mathrm{Conf}(S^3)/\mathrm{Iso}(S^3) - \mathrm{Conf}(S^2)/\mathrm{Iso}(S^2)$$
$$= 3 - 2 = 1$$

② Index $=1 \Rightarrow y = S^2$.

③. Prop: $y \neq S^2$ (oriented closed) $\#\{\mu=-2\} \geq 4$.

$y = M^n \to S^{n+1} \subset \mathbb{R}^{n+2}$ not totally geodesic $\#\{\mu=-n\} \geq n+2$.

5. Example 1. $S^2 \hookrightarrow S^3$ $|II|^2 = 0 \Rightarrow J = \Delta_\Sigma + 2$

"1st eigenvalue" $\quad \mathrm{Iso}(S^3)/\mathrm{Iso}(S^2)$

mult	1	3
2+Δ	0	2

multi	1	3	
λ of Δ	0	2	
μ of J	-2	0	>0

max nullity $\longleftrightarrow$ Killing field restricting on S^2

6. Example 2. $T^2 \hookrightarrow S^3$ Clifford torus. $K=0$

$\Rightarrow |\mathbb{I}|^2 = 2$ Gox.

$J = \Delta_\Sigma + 4$

"free" simple $\quad Or(S^3)/Iso(S^3) \cong \mathbb{H}^4$ $\quad\quad Iso(S^3)/Iso(T^2)$

$6 - 2 = 4.$

multi	1	4	(4)	
λ of Δ	0	2	4	
μ of J	-4	-2	0	>0

$\#(\mu=-2) \geq 4$ $\quad\quad$ $Ind(CT) = 5$, $Null = 4$

Remark $\quad Ind(\text{Clifford torus}) \leq Ind(y) \quad y \neq S^2(1).$

$\quad\quad \|$

$\quad\quad 5$

6. Thm: (Urbano thm '93). $y \neq S^2(1) \hookrightarrow S^3$ min

$Ind(y) \geq 5$ & "$=5$" $\Leftrightarrow$ y Clifford torus

Pf: (1) $Ind(y) \geq 1 + 4$

$\uparrow$ "free" $\quad\quad \updownarrow$ $Or(S^3)/Iso(S^3)$

(2) "=" $\quad \mu_1 < -2 \quad \left.\begin{array}{l}\mu_2 = -2 \\ \# = 4\end{array}\right\} \Rightarrow$

$\mu_1 \to \varphi_1 . \quad \varphi_1 > 0$ $\quad\quad \geq |\kappa^{2}|$

$M_1 \to \rho_1$, $\rho_1 > 0$

(Li-Yau '82) $\exists\, T \in \mathrm{Conf}(S^3)$ s.t. $\int_\Sigma \rho_1 \cdot (T y)\, dM = 0$

Hersh $\lambda_1(S^2)$
Yang-Yau $\lambda_1(\Sigma)$
Li-Yau $A_c(\Sigma)$

$\Rightarrow \tilde y = T\circ y = \begin{pmatrix} \tilde y_1 \\ \vdots \\ \tilde y_4 \end{pmatrix}$ test function

$$\mu_2 = \inf_{f \perp_{L^2} \rho_1} \frac{\int f J f}{\int f^2} = -2 \Rightarrow$$

$$\int J\tilde y_j\, \tilde y_j + 2\tilde y_j^2 \geq 0$$

$$\Rightarrow \int |\nabla \tilde y_j|^2 - |II|^2 \tilde y_j^2 \geq 0 \quad \Big\} \Rightarrow = 0.$$

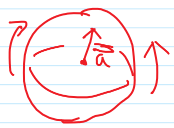
(≤ 0)

$$\sum_{j=1}^4 \int |\nabla \tilde y_j|^2 = 2\,\mathrm{Area}(\tilde y) \leq 2W(\tilde y) = 2W(y) = 2\,\mathrm{Area}(y)$$

$$\sum_{j=1}^4 \int_\Sigma |II|^2 \tilde y_j^2 = \int_\Sigma |II|^2 = 2\int_\Sigma (1-K) \geq 2\,\mathrm{Area}(\Sigma) \quad (H=0)$$

$\Rightarrow$ "=" $\Rightarrow K=0 \Rightarrow$ Clifford torus.

$$\varphi_a(x) = a + \frac{1-|a|^2}{1+2\langle a,x\rangle+|a|^2}(x+a) \qquad |a|<1,\ a \in \mathbb{R}^{n+1}$$



$\exists\, a$ $\varphi_a(y) \perp_{L^2} \rho_1$

$\approx \delta a - o(\delta) + \cdots$

7. Index of $\xi_{g,1}$ (Kapouleas – Wiygul)

① Thm: (1) Index $(\xi_{g,1}) = 2g+3$ ~~$= \text{Ind}(S^3)$~~

(2) Null. $(\xi_{g,1}) = 6$ $\quad = \quad g \geq 2. \quad (g=1 \;\; =4)$

⇓ / minimal

(3) Isolation ⇒ $\{\Sigma_t\} \to \xi_{g,1}$ $t \to +\infty$

Then t large enough ⇒ $\Sigma_t = \xi_{g,1}$

② Thm. (Kusner –)

$$\mu_1 < \mu_2 \leq \cdots \leq \mu_{2g-1} < \mu_{2g} = \cdots = \mu_{2g+3} < 0$$

$$\underbrace{\quad}_{2g-1} + \underbrace{\quad}_{4} = 2g+3$$

⇓

Thm: $\xi_{m,k}$ (1) W-stable in S^3 & S^n in the sense $S^3 \subset S^n$

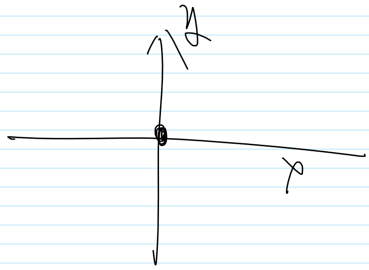
(2) local W-minimizer.

8. Idea of Montiel–Ros (Schrödinger operator ---)

$$(\Delta_\Sigma + q)(f)$$

Symmetry:

(1) $f: \mathbb{R} \to \mathbb{R}$ $\quad \sigma(x) = -x$



$$f = \tfrac{1}{2}(f + f\circ\sigma) + \tfrac{1}{2}(f - f\circ\sigma)$$

even $\qquad$ odd.

$$= f_{even} + f_{odd}$$

$f_{odd}|_{\partial\Omega} = 0 \leftarrow$ Dirichlet Condition

$\partial_n f_{even}|_{\partial\Omega} = 0$ Neumann Condition

(2) $a \in S^3 \subseteq \mathbb{R}^4$. $\sigma_a : S^3 \to S^3$

$f \in C^\infty(\Sigma)$. $\left. \underline{f - f\circ\sigma} \right|_{\langle a \rangle^\perp} = 0$ Dirichlet

$\left. \partial_n (f + f\circ\sigma) \right|_{\langle a\rangle^\perp} = 0$ Neumann.

(3) Σ $\underline{\sigma_a\text{-inv.}} \Rightarrow f$ μ-eigen-f, $f\circ\sigma$ ______ }

$\dim \underline{E_\mu} = \dim E_\mu^{odd} + \dim E_\mu^{even}$

$\Updownarrow$ Dirichlet. $\Updownarrow$ Neumann

8. CM-R (Schrödinger operator —) $\partial_D U \cap \partial_N U = \phi$

(1). $J = \Delta_\Sigma + 2 + |H|^2$. $\underline{U}$. $\partial U = (\partial_D U) \cup (\partial_N U)$ ↓ Diri ↓ Neum.

$\underline{\gamma \subset U}$ $U \setminus \gamma =$ n connected domain

$U = \bigcup_{i=1}^{n} U_i$

$\partial_D U_i = \partial U_i \cap \partial_D U$ $\partial_N U_i = \partial U_i \cap \partial_N U$

$\gamma_i = \partial U_i \cap \gamma$

$\Rightarrow \partial U_i = \gamma_i \cup \partial_D U_i \cup \partial_N U_i$.

(2) Prop. (i) $\#_{<\mu}(J, U; \underline{\partial_D U}, \underline{\partial_N U})$

$\geq \#_{<\mu}(J, U_1; \gamma_1 \cup \partial_D U_1, \partial_N U_1)$

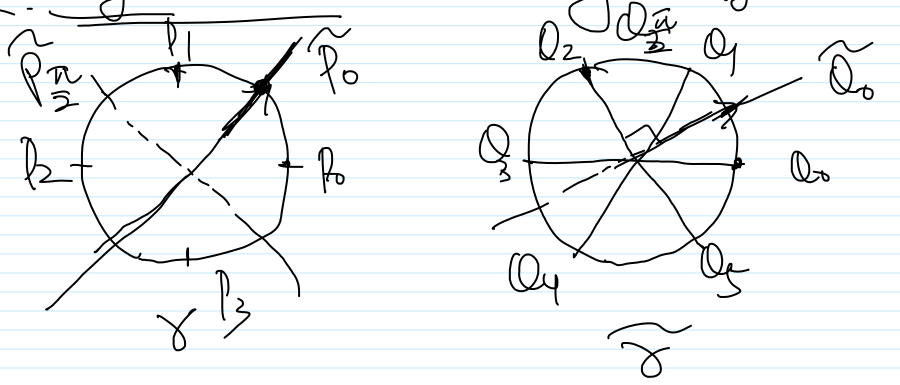
$$+ \sum_{i=2}^{n} \#_{\le \mu}\left(J, u_i; (\partial_D u_i) \cup \gamma_i, \partial_N u_i \right)$$

(ii) $\#_{\le \mu}\left(J, u; \partial_D u; \partial_N u \right)$

$$\le \#_{\le \mu}\left(J; u_1; \partial_D u_1; \gamma_1 \cup \partial_N u_1 \right)$$

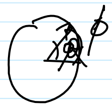
$$+ \sum_{i=2}^{n} \#_{< \mu}\left(J; u_i; \partial_D u_i; \gamma_i \cup \partial_N u_i \right).$$

9, Key idea 2 dividing $\xi_{g,1}$ with 4 two-spheres (o.n. to each other)



$k=1$. $m=2$.

$$\begin{cases} \Sigma^0 = \mathrm{Span}\{\tilde{P}^0, \tilde{\gamma}\} \cap S^3 \\ \Sigma^{\frac{\pi}{2}} = \mathrm{Span}\{\tilde{P}^{\frac{\pi}{2}}, \tilde{\gamma}\} \cap S^3 \\ \Sigma_0 = \mathrm{Span}\{\tilde{Q}_0, \tilde{\gamma}\} \cap S^3 \\ \Sigma_{\frac{\pi}{2}} = \mathrm{Span}\{\tilde{Q}_{\frac{\pi}{2}}, \tilde{\gamma}\} \cap S^3 \end{cases}$$



12, Killing fields of S^3.

(x_1, x_2, x_3, x_4)

① $R_\gamma^\phi = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\phi & \sin\phi \\ & & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ fixed Rotating ϕ-degree

$\begin{pmatrix} & & \vdots & \\ & & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \vdots \\ x_4 \end{pmatrix}$

② $K^\phi_\gamma(\phi) = \frac{d}{d\phi}\Big|_{\phi=0} \left(R^\phi_\gamma \cdot \phi \right) = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_4 \\ -x_3 \end{pmatrix}$

③ R^ϕ_γ restricting on $\xi_{m,k}(\Sigma)$ $\Longrightarrow$ $\qquad n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}$

Example: $K^\phi_\gamma(\phi) \cdot \vec{n} = \underbrace{n_3 x_4 - n_4 x_3}_{J_\gamma} \in E_{\mu=0}$.

④ Killing fields of S^3 $\Rightarrow$ eigenfunctions of $E_{\mu=0}$

Prop: $\dim \{ J_\gamma \mid \gamma \cdots \} = 6$. ($\Rightarrow$ null ≥ 6).

Prop: Symmetries of J_γ w.r.t. σ_{a_i}, σ_{b_j}

13.1 ① Def: $V^{\pm\pm} = \{ u \in C^\infty_{pw}(\Sigma) \mid u \circ R_{\Sigma,0} = \pm u,\ u \circ R_{\Sigma,\frac{\pi}{2}} = \pm u \}$

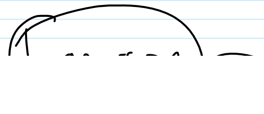
$R_{\Sigma,0}$, $R_{\Sigma,\frac{\pi}{2}}$ divides Σ into 4-parts.

V^{++}, V^{--}, V^{+-}, V^{-+}

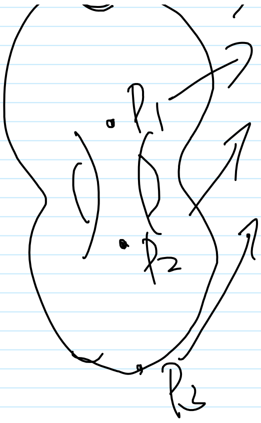
② Prop 6.3. $J_\gamma \in V^{--}$, $J_\gamma|_{(\ )} > 0$ $\Big\} \Rightarrow \begin{cases} \text{Index}(V^{--}) = 0 \\ \text{Null}(V^{--}) = 1 \end{cases}$

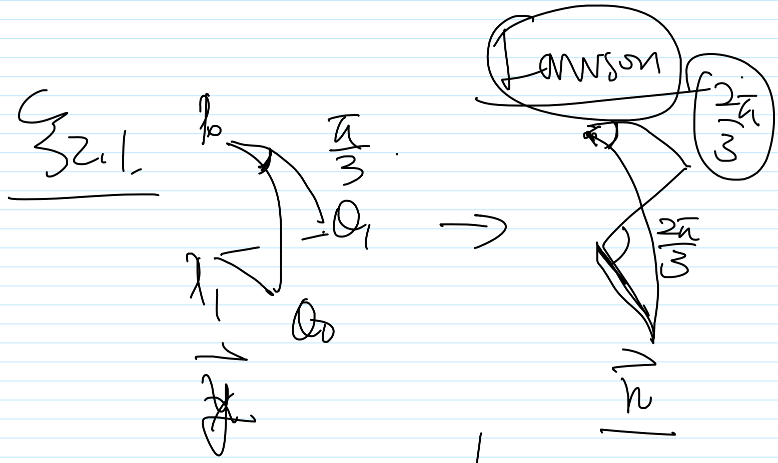
14. $\xi_{g,1}$ $\quad p_0$ ($\xi_{m,k}$?)

$\rightarrow$ umbilic points (Kapl. – Wiygul)



umbilic points





nodal domain of n_3 $= 4$

$= 2g.$

$N_\xi = -2$

$\gamma \geq 2g$

Index $\geq 2g - 1 + 4 = 2g + 3.$