Lawson's minimal surfaces in S^3

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survey

NOTES ON LAWSON MINIMAL SURFACES $\xi_{M,K}$ IN S^3

PENG WANG

Abstract.

Reference

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College of Mathematics and Informatics, FJKLMAA, Fujian Normal University, Fuzhou 350117, P. R. China
E-mail address: pengwang@fjnu.edu.cn, netwangpeng@hotmail.com.

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§1. Surface in S^n

1. $f: M \to S^n$ surface $\dim M = 2$

$\exists$ isothermal coordinate (Chern) (u,v). s.t.

$|f_u|^2 = |f_v|^2 = e^{2w}$. $\langle f_u, f_v\rangle = 0$.

$\Leftrightarrow |df|^2 = e^{2w}(du^2+dv^2) = e^{2w}|dz|^2$.

Set $z = u+iv$. $dz = du + i dv$.

z defines a Riemann surface str. on M.

f <u>conformal</u> immersion $\Leftrightarrow |df|^2 = e^{2w}|dz|^2$.

2. Structure equations.

① $z = u+iv$. $u = \frac{z+\bar z}{2}$. $v = \frac{z-\bar z}{2i}$

$f_z = f_u \cdot u_z + f_v \cdot v_z = \frac{1}{2}(f_u - i f_v)$ $\quad (\partial_z = \frac{1}{2}(\partial_u - i\partial_v))$

$\Rightarrow$ <u>$\langle f_z, f_z\rangle$</u> $= \frac{1}{4}\left(\underline{\langle f_u, f_u\rangle} - \underline{\langle f_v, f_v\rangle} - 2i\langle f_u, f_v\rangle\right) = 0$

$\Leftrightarrow |f_u|^2 = |f_v|^2$ & $\langle f_u, f_v\rangle = 0 \Leftrightarrow$ conformal.

(Rmk. <u>$\langle f_z, f_z\rangle dz^2$</u>)

② $|f_z|^2 = \frac{1}{2}e^{2w}$. $S^n \subset \mathbb{R}^{n+1}$

③ S^n; $|f| = 1$, $\{f, f_z, f_{\bar z}, Y_\alpha\}$ (normal frame; f: normal of S^n in $\mathbb{R}^{n+1}$)

$f_{zz} = ?$ $\quad \langle f, f\rangle = 1 \Rightarrow 2\langle f_z, f\rangle = 0$

$\Rightarrow \langle f_{zz}, f\rangle + \langle f_z, f_z\rangle = 0$ ($\langle f_z, f_z\rangle = 0$ conformal)

$\langle f_z, f_z\rangle = 0 \Rightarrow \langle f_{zz}, f_{\bar z}\rangle = 0$.

$$\boxed{f_{zz} = 2w_z f_z + \Omega} \qquad \Omega \in \Gamma(NM \otimes \mathbb{C})$$

(et. → $2w_z$)

④ $\boxed{\Omega dz^2}$, vector-valued Hopf differential $\Rightarrow$ <u>$\langle \Omega, \Omega\rangle dz^4$</u> 4-form.

(Rmk, S^3. $\Omega = \Theta \cdot \vec n \Rightarrow$ <u>Θdz^2</u> 2-form) (Hopf LNM 1000)

Pf: independent of choice of z w another complex coordinate

$\Rightarrow$ $N_z = 0$ $\Rightarrow$ $f_w = f_z z_w$

$\Rightarrow$ $f_{ww} = (z_w)_w f_z + f_{zz} \cdot (z_w)^2$

$= (\quad) f_w + \tilde{\Omega} \Rightarrow \tilde{\Omega} = \Omega (z_w)^2$

$\tilde{\Omega} dw^2 = \Omega (z_w)^2 dw^2 = \Omega (dz)^2$. global defined.

(6). $f_{z\bar{z}} = -\frac{1}{2} e^{2w} f + \frac{1}{2} e^{2w} \vec{H}$ $f_{z\bar{z}} \perp f_z, f_{\bar{z}}$.

$\vec{H}$: mean curv in S^n.

(Remark $= \frac{1}{2} e^{2w} \vec{H}_{\mathbb{R}^{n+1}}$)

$\vec{H}_{\mathbb{R}^{n+1}}$: mean curv. in $\mathbb{R}^{n+1}$.

$\vec{H}_{\mathbb{R}^{n+1}} = \vec{H} - f$

f: normal vector of S^n in $\mathbb{R}^{n+1}$.

$\{Y_\alpha\}$ o.n. basis of NM

$\text{II} = \left(\langle f_{zz}, Y_\alpha \rangle dz^2 + \langle f_{\bar{z}\bar{z}}, Y_\alpha \rangle d\bar{z}^2 + 2 \langle f_{z\bar{z}}, Y_\alpha \rangle |dz|^2 \right) Y_\alpha$.

$\Rightarrow \cdots \vec{H} = \frac{1}{2} \sum_\alpha (h^\alpha_{11} + h^\alpha_{22}) Y_\alpha$ $\Omega = \frac{1}{4} e^{2w} \sum_\alpha (h^\alpha_{11} - h^\alpha_{22} - 2i h^\alpha_{12}) Y_\alpha$ (?).

(7). Lemma: $\Delta_M = 4 e^{-2w} \partial_z \partial_{\bar{z}}$

Lemma: $\Delta_M f = -2f + 2H = 2H_{\mathbb{R}^{n+1}}$

$$(*) \begin{cases} f_{zz} = 2w_z f_z + \Omega \\ f_{z\bar{z}} = -\frac{1}{2} e^{2w} f + \frac{1}{2} e^{2w} H \\ Y_z = D_z Y - \langle Y, H \rangle f_z - 2e^{-2w} \langle Y, \Omega \rangle f_{\bar{z}} \end{cases}$$

eq.

$Y \in \Gamma(NM \otimes \mathbb{C})$; D_z: normal connection

$\begin{pmatrix} f_z \\ f_{\bar{z}} \\ Y \end{pmatrix}_z$ linear PDE

(8). Gauss. Codazzi. Ricci eqs ?

$(\Leftrightarrow \quad (\ )_{z\bar z} = (\ )_{\bar z z}) \qquad f_{z\bar z} = f_{\bar z z}$

(1). $H^2 - K + 1 = 4e^{-4w}|\Omega|^2 = |\overset{\circ}{\mathrm{II}}|^2$. Conf. inv.

H^2: Willmore energy; $|\overset{\circ}{\mathrm{II}}|^2$: square of norm of traceless part of II.

$K = -4e^{-2w} w_{z\bar z}$.

(2). Codazzi: $\frac{1}{2}e^{2w} D_{\bar z}\vec{H} = D_{\bar z}\Omega$.

$\Rightarrow$ Cor: $n=3$. $\vec{H} = h\vec{n}$. $\Omega = \theta \cdot \vec{n}$. Codazzi

$$\frac{1}{2}e^{2w} h_{\bar z} = \theta_{\bar z}.$$

$\Rightarrow h = \text{const} \Leftrightarrow \theta dz^2$ holomorphic.

Cor: $M = S^2$ $\theta dz^2 \equiv 0 \Rightarrow$ round S^2 (Hopf.)

(On S^2 a holomorphic differention form must vanish.)

Cor: f minimal $\vec{H} = 0 \Rightarrow \langle \Omega, \Omega \rangle dz^4$ holomorphic differential

(9). Calabi $f: S^2 \to S^n$, $\vec{H} = 0$, full. $\Rightarrow n = 2m$. &

$$\langle f_z^{(k)}, f_z^{(l)} \rangle \equiv 0.$$

(Projection of horizontal twistor curve in twistor bundle over S^{2m})

(first intro. of twistor bundle 1970 SDG.)

Proof: $\langle f_{zz}, f_{zz} \rangle$

$= \langle \Omega, \Omega \rangle dz^4$ holo.

$+ S^2 \Rightarrow \langle \Omega, \Omega \rangle \equiv 0 \Rightarrow \langle f_{zz}, f_{zz} \rangle = 0$.

Induction: $\langle f_{zzz}, f_{zzz} \rangle dz^6$ holo.

$\Rightarrow$ $\overset{||}{0}$ ... $f \in S^{2m}$

$f \oplus \{f_z, f_{zz}, \cdots, f^{(m)}\} \oplus ?$ ... $\subset \mathbb{R}^{2m+1}$.

$f \Leftrightarrow \{f_z, f_{zz}, \cdots f^{(m)}_{z\cdots z}\} \oplus \{\qquad\qquad\} \subset \mathbb{R}^{2m+1}$

f totally isotropic.

(10). Ricci: $D_{\bar z} D_z \psi - D_z D_{\bar z} \psi = 2e^{-2\omega}(\langle \Omega, \psi\rangle \bar\Omega - \langle \bar\Omega, \psi\rangle \Omega)$.

3. Minimal surfaces in S^n $\quad \vec{H} \equiv 0$.

$f_{z\bar z} = -\frac{1}{2}e^{2\omega} f \Rightarrow \underbrace{4e^{-2\omega} f_{z\bar z}}_{\Delta f} = -2f \Leftrightarrow \Delta f = -2f$.

Takahashi: $\Delta_M f = -2f \Leftrightarrow f$ minimal in S^n.

$\Delta_M f = -mf \Leftrightarrow f$ —— S^n. $\quad m = \dim M$.

$\Rightarrow f = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix} \in S^n \subset \mathbb{R}^{n+1} \Rightarrow \Delta_M f_j + 2 f_j = 0 \quad j = 0 \cdots n$

$\Rightarrow \{f_j\}$ are eigenfunctions of Δ_M w.r.t. eigenvalue $\lambda = 2$.

*. (M, g). Δ_M $\quad \boxed{\Delta_M h + \lambda h = 0}$. $\lambda \in \mathbb{R}$ fixed number. $h \in C^\infty(M)$ $h \not\equiv 0$.

$\exists h$. λ

$\underbrace{\lambda \in \mathrm{Spec}(\Delta_M)}$ h. eigenfunction

λ eigenvalue

*. $h = 1$ $\lambda = 0$ $\quad 0 < \underline{\lambda_1} \le \lambda_2 \le \cdots < +\infty$

4. Yau. Conj. $f: M^2 \to S^3$ co-dim 1, $H = 0$

if embedding + oriented + closed. Then $\lambda_1 = 2$.

$\boxed{\Delta_M f_j = -2 f_j}$

$2 \in \mathrm{Spec}(\Delta_M) \Rightarrow \lambda_1 \le 2$

$\boxed{\lambda_1 = n-1 \quad S^n. \text{ hypersurface}}$

minimal $\Rightarrow \lambda_1 \le 2$ $\quad (\lambda_1 \le n-1)$

5. Choe-Soret 2009. Lawson minimal surface $\xi_{m,k}$ $\lambda_1 = 2$.

5. Choe - Soret 2009. Lawson minimal surface $\Sigma_{m,k}$ $\lambda_1 = 2$.

6. Example 1. $S^2 \hookrightarrow S^2 \subset \mathbb{R}^3$. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

Spec.

λ_j	0	2	6 (?)	...
φ_j	1	x, y, z	$x^2+y^2-2z^2$, xy, yz, zx, x^2-y^2	...

Veronese in S^4

$n(n+1)$?

$\frac{2}{n(n+1)}$

② Example 2. Clifford torus. $f = \frac{1}{\sqrt{2}}(\cos\sqrt{2}u, \sin\sqrt{2}u, \cos\sqrt{2}v, \sin\sqrt{2}v)$

(i) $|df|^2 = du^2 + dv^2$. $u, v \in [0, \frac{2\pi}{\sqrt{2}}]$

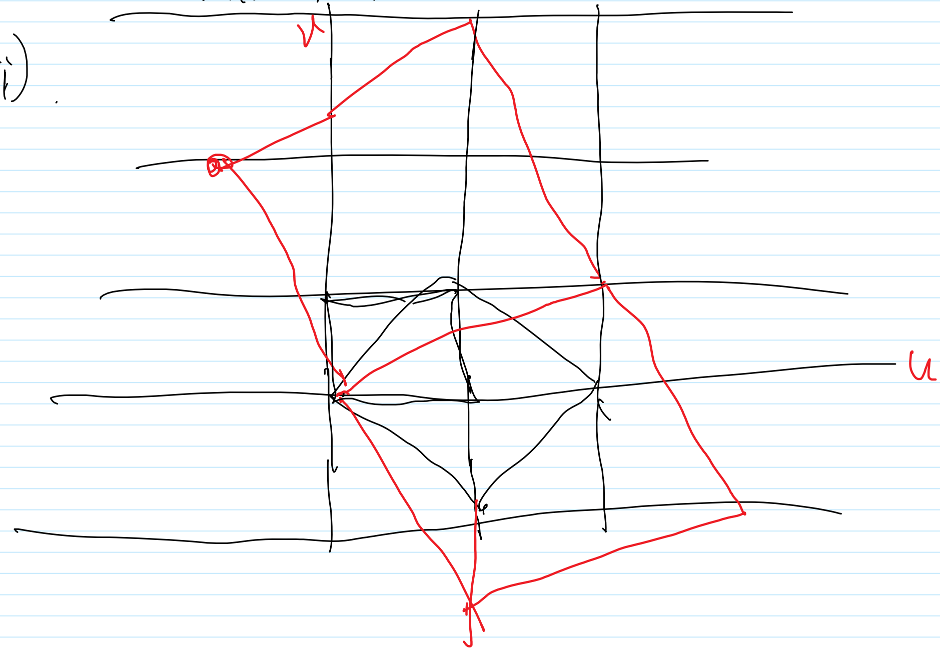
$\Delta_M = \partial_u^2 + \partial_v^2$

(ii) $\{\cos\sqrt{2}(pu+qv), \sin\sqrt{2}(pu+qv), p \in \mathbb{Z}$, $q \in \mathbb{Z}^+ \cup \{0\}$, if $q=0$ $p \geq 0\}$

$\lambda(p,q) = 2(p^2+q^2)$.

λ	0	2	4		...
φ	1	$\cos\sqrt{2}u, \cos\sqrt{2}v$, $\sin\sqrt{2}u, \sin\sqrt{2}v$	$\varphi(u \pm v)$		

(iii)



$(p, q) = (2, \pm 1)$

$(\pm 1, 2)$

$y_t = \frac{1}{\sqrt{2}}\left(t\cos(2u+v), \sin(2u+v), \cos(u-2v), \sin(u-2v)\right.$

$$y_t = \frac{1}{\sqrt{2}}\Big(t\cos(2u+v),\ t\sin(2u+v),\ \cos(u-2v),\ \sin(u-2v),$$
$$\sqrt{1-t^2}\cos(2u-v),\ \sqrt{1-t^2}\sin(2u-v),\ \sqrt{1-t^2}\cos(u+2v),$$
$$\sqrt{1-t^2}\sin(u+2v)\Big).$$

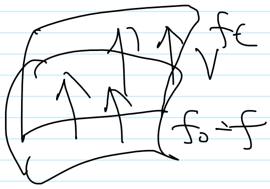
Family of minimal T^2 in S^7.

4. Stability.

(1) Thm f minimal $\Leftrightarrow$ Critical surface w.r.t area.

$\{f_t\}$ $\quad \left.\frac{dF_t}{dt}\right|_{t=0} = 0 \Leftrightarrow \vec{H} = 0.$

(2) $V = (f_t)_*|_{t=0}\left(\frac{\partial}{\partial t}\right)$. $\quad f_0 \quad \vec{H} = 0$



$$\left.\frac{d^2F_t}{dt^2}\right|_{t=0} = -\int_M \langle V, (\Delta^{\perp} + 2 + \tilde{A})(V)\rangle dM$$

$J = \Delta^{\perp} + 2 + \tilde{A}$, $\quad \langle \tilde{A}(V), W\rangle = \langle A(V), A(W)\rangle$

$\Delta^{\perp}$ ↓ normal Δ. $\quad 2$ ↓ $2 \cdot 1$ ↓ curv. of S^n. $\quad$ (A : Shape operator).

(J. Simons. Minimal Variety).

$*$. $n=3$. $\quad J = \Delta + 2 + |II|^2 = \Delta + 2 + S$.

$*$. Spec J. $= \{\sigma_j\}$. $\quad J(V) + \sigma_j(V) = 0$.

(i) $n=3$. J acts on function.

$\sigma_1 < \sigma_2 \leq \cdots \to +\infty$ $\quad$ example Clifford.

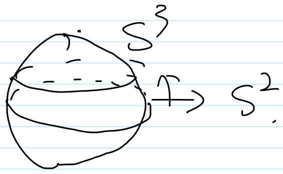
(ii) $n=4$. J acts on sections

$\sigma_1 \leq \sigma_2 \leq \cdots \to +\infty$ $\quad$ example Veronese

$\sigma_1 = -2$

dim ≥ 10

(iii). All minl surface in S^n unstable!



J. Simons. $\mathrm{Ind}(S^2) = 1$. nullity $= 3$.

(iv). $\mathrm{Ind}(f) = \#\{\sigma_j \mid \sigma_j < 0\}$ {记重数}

= dim of space spanned by eigenfunctions / sections w.r.t. to negative σ_j

unstability.

(v). Nullity $\mathrm{Null}(f)$ = Multiplicity of 0 as eigenvalue of J.

$\geq \dim SO(n+1) - \dim \mathrm{Iso}(M^2)$

$S^2 \subset S^3$. $\frac{4\times 3}{2} - \frac{2\times 3}{2} = 3$

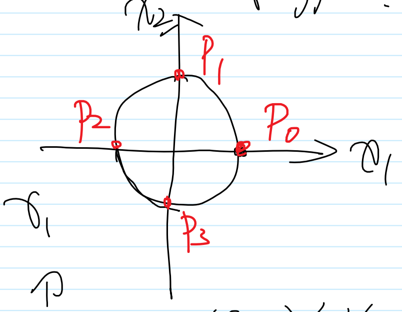
$\mathrm{Null}(f) \overset{?}{=} \dim SO(n+1) - \dim \mathrm{Iso}(M^2)$.

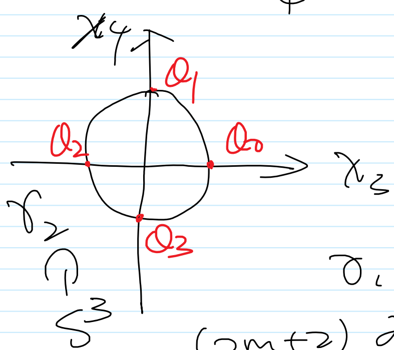
⇓ Rotation

§2. Lawson minimal surface $\xi_{m,k}$ in S^3

1. Lawson polygon on $S^3 \subset \mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ (x_1, x_2, x_3, x_4)

$\underset{\psi}{\mathbb{R}^2} \oplus \underset{\psi\, S^1}{\mathbb{R}^2}$





$\gamma_1 \perp \gamma_2$ $\gamma_1 \cap \gamma_2 = \emptyset$.

$(2m+2)$ 等分

$\gamma_1 = (\cos t, \sin t, 0, 0)$ $\gamma_2 = (0, 0, \cos t, \sin t)$

$P_j = \left(\cos \frac{2j\pi}{2(k+1)}, \sin \frac{j\pi}{k+1}, 0, 0\right)$ $Q_l = \left(0, 0, \cos \frac{l\pi}{m+1}, \sin \frac{l\pi}{m+1}\right)$

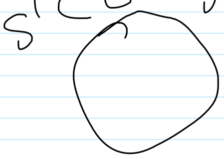
$P_j = \left(\cos\frac{2j\pi}{2(k+1)}, \sin\frac{j\pi}{k+1}, 0, 0\right)$, $j \in \mathbb{Z}_{2(k+1)}$

$Q_l = \left(0, 0, \cos\frac{l\pi}{m+1}, \sin\frac{l\pi}{m+1}\right)$.

考虑 球面四边形 $P_0Q_0P_1Q_1$ $(P_jQ_lP_{j+1}Q_{l+1})$

2. Reflections w.r.t. geodesic. l

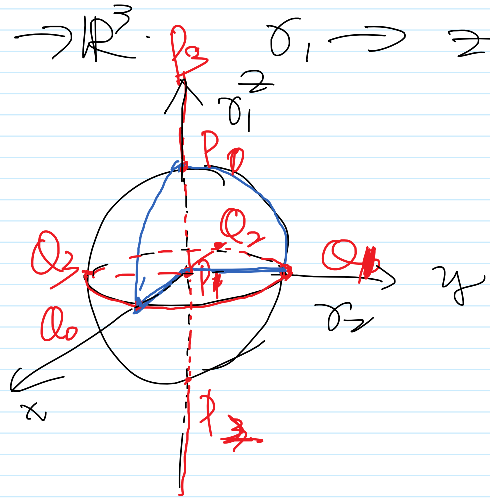
$S^1 \subset S^3$ geodesic

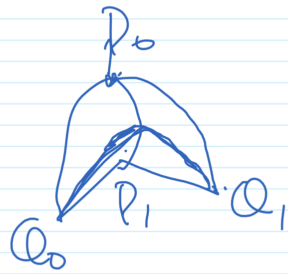


例 $\gamma_1 = (\cos t, \sin t, 0, 0)$

$R_{\gamma_1}(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, -x_4)$

3. $S^3 \to \mathbb{R}^3$. $\gamma_1 \to$ z轴 $\gamma_2 \to$ xy平面上单位圆.





4. Lawson $\exists\ \Sigma_{0,1}$. $\partial\Sigma_{0,1} = P_0Q_0P_1Q_1$ (Area$(\Sigma_{0,1})$ minimizer.)

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