超曲面上的几何曲率流及应用- Lecture 3

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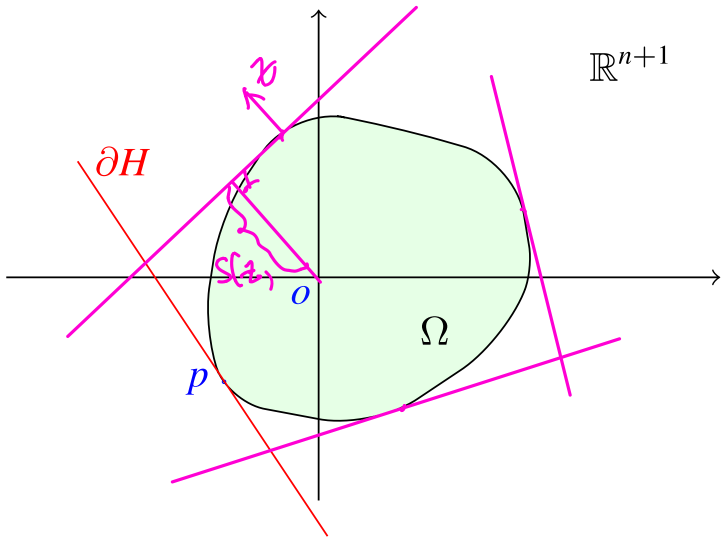
Outline of today's lecture

1. Convex geometry in Euclidean space.
2. Geometry of horospherical convex hypersurface in Hyperbolic space.
3. New quermassintegrals, curvature flows, geometric inequalities, some remarks.

Let Ω be a convex domain in $\mathbb{R}^{n+1}$. A closed Halfspace H is said to support Ω (at p) if $\Omega \subset H$ and $p \in \partial\Omega \cap \partial H$. The boundary of H is called a supporting hyperplane. The closure of Ω is the intersection of its supporting hyperplanes.



Support function of Ω is a function $s : \mathbb{S}^n \to \mathbb{R}$

$$s(z) = \sup_{x\in\Omega} \langle x, z\rangle, \qquad z \in \mathbb{S}^n,$$

which completely determines Ω

$$\Omega = \bigcap_{z\in\mathbb{S}^n} \left\{ y \in \mathbb{R}^{n+1} : \langle y, z\rangle < s(z) \right\}.$$

The boundary $\Sigma = \partial\Omega$ of a smooth bounded convex domain is a convex hypersurface in $\mathbb{R}^{n+1}$. Let

$$\nu : \Sigma \to \mathbb{S}^n$$

be the Gauss map of $\Sigma = \partial\Omega$. When Σ is strictly convex ($\kappa_i > 0$), the Weingarten operator $\mathcal{W} = d\nu$ is positive definite. Then $\nu : \Sigma \to \mathbb{S}^n$ is a diffeomorphism.

Lemma

$\kappa_i > 0$

Let $\Sigma = \partial\Omega$ be a strictly convex hypersurface in $\mathbb{R}^{n+1}$. Then $\nu^* g_{\mathbb{S}^n} = h^2$.

proof: $$(\nu^* g_{S^n})_{ij} = g_{S^n}\left(\frac{\partial \nu}{\partial x^i}, \frac{\partial \nu}{\partial x^j}\right)$$

$$= g_{S^n}\left(h_i^k \frac{\partial X}{\partial x^k}, h_j^l \frac{\partial X}{\partial x^l}\right)$$

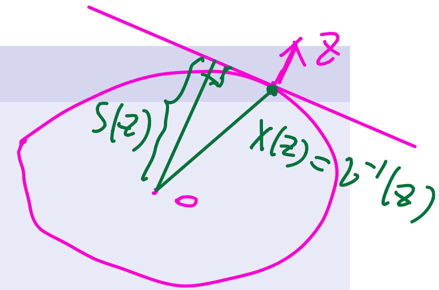
$$= h_i^k h_{jk} = (h^2)_{ij}$$

The support function $s : \mathbb{S}^n \to \mathbb{R}$ of a convex hypersurface Σ is defined by

$$\underline{s(z) = \langle \nu^{-1}(z), z \rangle,}$$

$\nu : \Sigma \to S^n$

where $z \in \mathbb{S}^n$ is the outer unit normal at the point $\nu^{-1}(z) \in \Sigma$.



Lemma $X : S^n \to \mathbb{R}^{n+1}, \quad \Sigma = X(S^n)$

Let $\Sigma = \partial\Omega$ be a strictly convex hypersurface in $\mathbb{R}^{n+1}$. Then

$$X(z) \quad \nu^{-1}(z) = \underline{s(z)z} + \underline{\bar{\nabla} s(z)}.$$

The second fundamental form h and support function are related by

$$\nu^* \left(\underline{\bar{\nabla}^2 s + s\bar{g}} \right) = h.$$

In other words, we can reparametrize a strictly convex Σ as an embedding

$$\begin{aligned} \bar{X} : \quad \mathbb{S}^n &\to \mathbb{R}^{n+1} \\ z &\mapsto s(z)z + \bar{\nabla} s(z) \end{aligned}$$

① $s(z) = \langle z, X(z) \rangle$

$\bar{\nabla}_i s(z) = \langle \bar{\nabla}_j z, X(z) \rangle + \cancel{\langle z, \partial_j X(z) \rangle}$

$\therefore \quad X(z)^T = \bar{\nabla} s(z)$

$\Longrightarrow \quad X(z) = s(z)\, z + \bar{\nabla} s(z)$

② : $\bar{\nabla}_i \bar{\nabla}_j s(z) = \underline{\langle \bar{\nabla}_i \bar{\nabla}_j z, X(z) \rangle} + \langle z, \bar{\nabla}_i \bar{\nabla}_j X(z) \rangle$

$+ \langle \bar{\nabla}_i z, \bar{\nabla}_j X(z) \rangle + \langle \bar{\nabla}_j z, \bar{\nabla}_i X(z) \rangle$

$= \underline{-\bar{g}_{ij} \langle z, X(z) \rangle} - \langle z, \bar{\nabla}_i \bar{\nabla}_j X(z) \rangle$

$= -\bar{g}_{ij}\, s(z) + h_{ij}$

$\Longleftrightarrow \quad \underline{h_{ij} = \bar{\nabla}_i \bar{\nabla}_j s(z) + s(z)\, \bar{g}_{ij}}$

$X : S^n \longrightarrow \mathbb{R}^{n+1}$

$X(z) = s(z)\, z + \bar{\nabla} s(z)$

$|r(z)|^2 = |X(z)|^2 = s^2(z) + |\bar{\nabla} s|^2$

$$\begin{cases} \bar{g}_{ij} = (h^2)_{ij} \\ \underbrace{\bar{\nabla}_i \bar{\nabla}_j s(z) + s(z)\, \bar{g}_{ij}}_{A[s]} = h_{ij} \end{cases}$$

$\Longrightarrow \quad \bar{g}^{ik} A[s]_{kj} = (w^{-1})_i^{\ j}$

eigenvalues $\{ r_1 \ldots\ r_n \} \longleftrightarrow \{ \frac{1}{\kappa_1}, \ldots \frac{1}{\kappa_n} \}$

principal curvatures $\qquad r_i = 1/\kappa_i \quad i = 1 \ldots n$

Denote $A[s] = \bar{\nabla}^2 s + s\bar{g}$. Then

$$\nu^* g_{\mathbb{S}^n} = h^2, \qquad \nu^* A[s] = h$$

It follows that the eigenvalues $r_1, \cdots, r_n$ of $A[s]$ with respect to $\bar{g} = g_{\mathbb{S}^n}$ are the same as the eigenvalues of h with respect to h^2, which are $1/\kappa_1, \cdots, 1/\kappa_n$. That is,

$$r_i = \frac{1}{\kappa_i}, \quad i = 1, \cdots, n$$

are the principal curvature radii.

$A[s] = w^{-1}$

$\frac{\partial}{\partial t} A[s]_{ij} = \bar{\nabla}_i \bar{\nabla}_j (\partial_t s) + \partial_t s \, \bar{g}_{ij}$

$= \bar{\nabla}_i \bar{\nabla}_j F + F \bar{g}_{ij}$

Lemma

Let $X : M \to \mathbb{R}^{n+1}$ be a strictly convex solution of

$$\frac{\partial}{\partial t} X = \underline{-F(\kappa)}\nu.$$

The flow is equivalent to the evolution of the support function of $\Sigma_t = X(M, t)$

$$\frac{\partial}{\partial t} s(z, t) = -F(A[s]^{-1}), \quad \text{on } \mathbb{S}^n \times [0, T).$$

" $\Rightarrow$ ". $\Sigma_t = X(M^n, t)$ strictly convex

$\nu_t : \Sigma_t \to S^n$ Gauss map

$\nu_t^{-1} : S^n \to \Sigma_t$ $\qquad X : \underbrace{M \to \Sigma_t \to S^n}_{\nu_t}$

$\nu_t^{-1} : S^n \to M$

support function : $s(z,t) = \langle z, X(\nu_t^{-1}(z)) \rangle$

$\frac{\partial}{\partial t} s(z,t) = \langle z, \frac{d}{dt}\left(X(\nu_t^{-1}(z), t)\right) \rangle$

$= \langle z, \frac{\partial}{\partial t} X + \underbrace{\frac{\partial X}{\partial x_i} \cdot \frac{\partial (\nu_t^{-1}(z))^i}{\partial t}}_{\text{tangential}} \rangle$

$= -F(\kappa)$

$= -F(A[s]^{-1})$ on $S^n \times [0, T)$

$A[s] = \bar\nabla^2 s + s\bar g$

" $\Leftarrow$ " suppose $s(z,t)$. $\underline{\partial_t s = -F(A[s]^{-1})}$. $A[s] > 0$

$s(z,t) \rightsquigarrow \Sigma_t$ strictly convex

for each t : $X(\cdot, t) = \nu_t^{-1} : S^n \to \Sigma_t \subset \mathbb{R}^{n+1}$

claim : $\exists$ diff. $\varphi(\cdot, t) : S^n \to S^n$. s.t

$\bar X(z,t) = X(\varphi(z,t), t)$ solves $\frac{\partial \bar X}{\partial t} = -F(\kappa)\vec{z}$

$\frac{\partial \bar X}{\partial t} = \frac{\partial X}{\partial t} + DX \cdot \frac{\partial \varphi}{\partial t}$

$= -F(\kappa) \cdot \varphi(z,t) + \underbrace{\left(\frac{\partial X}{\partial t}\right)^T + DX \cdot \frac{\partial \varphi}{\partial t}}_{=0}$

$\langle \frac{\partial X}{\partial t}, \partial_j \rangle + \underbrace{(D_i X \cdot \partial_j)}_{A[s] > 0} \frac{\partial \varphi^i}{\partial t} = 0$

We now describe an analogue geometry in hyperbolic space $\mathbb{H}^{n+1}$.

sectional curvature $\equiv 0$

- Supporting hyperplanes in Euclidean space are replaced by supporting horospheres in $\mathbb{H}^{n+1}$, which are hypersurfaces with constant principal curvatures

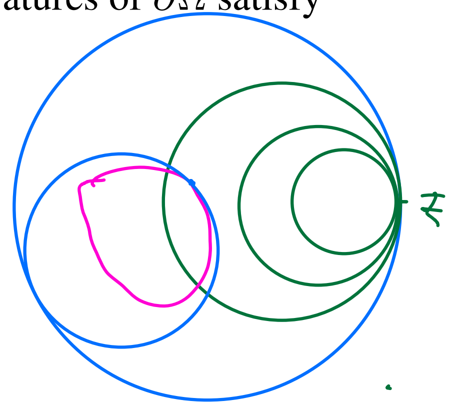
$$\kappa_i = 1, \qquad i = 1, \cdots, n$$

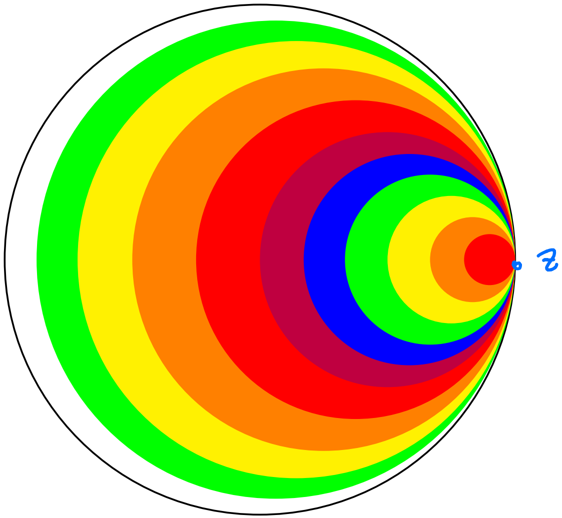
  everywhere.

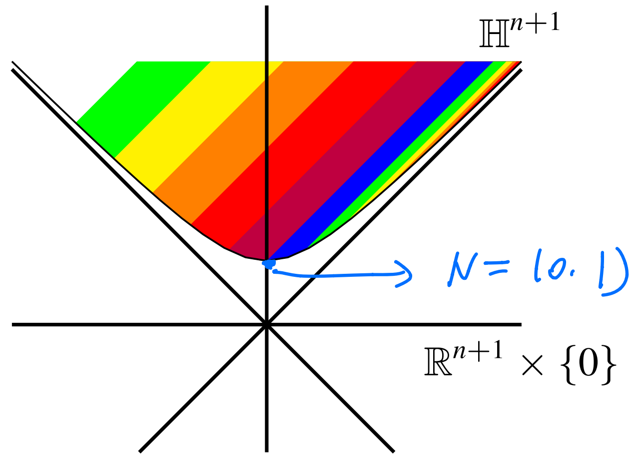
- A domain Ω in $\mathbb{H}^{n+1}$ is called "horospherically convex" (or "h-convex") for each $p \in \partial\Omega$, there is a horosphere enclosing Ω and touching at p. If $\partial\Omega$ is smooth, "h-convexity" is equivalent to that principal curvatures of $\partial\Omega$ satisfy $\kappa_i > 1$, $i = 1, \cdots, n$.

"strictly h-convex"

$k_i > 1$







In Ponincaré disc model, given each point $z \in S^n_\infty$ there exists a family of Horospheres touching at z and foliating the whole space.

In hyperboloid model, given a point $z \in S^n$, the horospheres touching at z are given by

$$H_z(s) = \{X : |X|^2 = -1, \langle X, (z, 1) \rangle = -e^s\}, \quad s \in \mathbb{R}$$

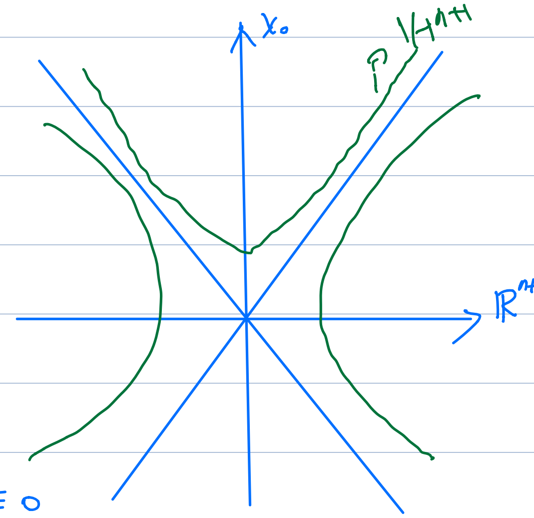
where s is the signed distance to the north pole $N = (0, 1)$. Denote $B_z(s)$ the horo-balls enclosed by horospheres $H_z(s)$.

$\mathbb{R}^{n+1,1}$. $\quad |X|^2 = -x_0^2 + \sum_{i=1}^{n+1} (x_i)^2$

$\mathbb{H}^{n+1}$: $\quad |X|^2 = -1, \quad x_0 > 0$

S_1^{n+1}: $\quad |X|^2 = 1$

$\mathbb{N}_+^{n+1}$: $\quad |X|^2 = 0.$



$\Sigma^n \subset \mathbb{H}^{n+1}$

$\vec{\nu}$: unit normal. $\quad |\vec{\nu}| = 1$

$\tilde{\nu} \cdot X = 0$

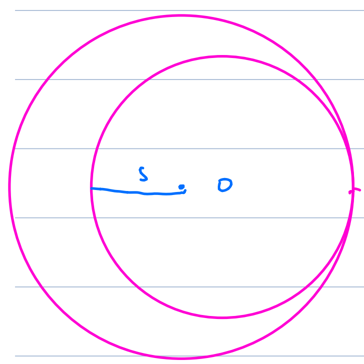
$\Rightarrow \quad X - \vec{\nu} \in \mathbb{N}_+^{n+1}$

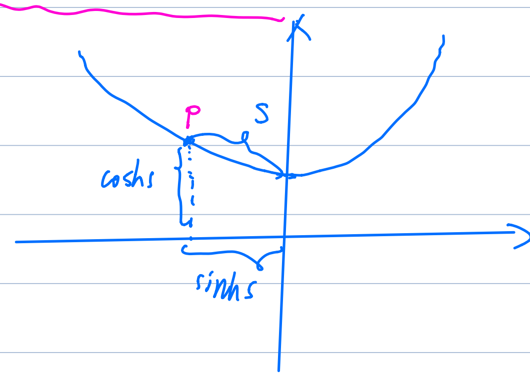
$\forall \ u \in T\Sigma, \quad D_u (X - \vec{\nu}) = (I - W)(u) \equiv 0.$

$\therefore \quad X - \vec{\nu}$ constant null vector

$\therefore \quad X - \vec{\nu} = \lambda (z, 1). \qquad z \in S^n$

$\Rightarrow \quad X \cdot (X - \vec{\nu}) = -1 \quad \Leftrightarrow \quad X \cdot (z, 1) = -\lambda^{-1}$





$$\left(\cosh s\,(0,1) - \sinh s\,(z, 0)\right) \cdot (z, 1) = -\frac{1}{\lambda}$$

$$\Leftrightarrow \quad \lambda = e^{-s}$$

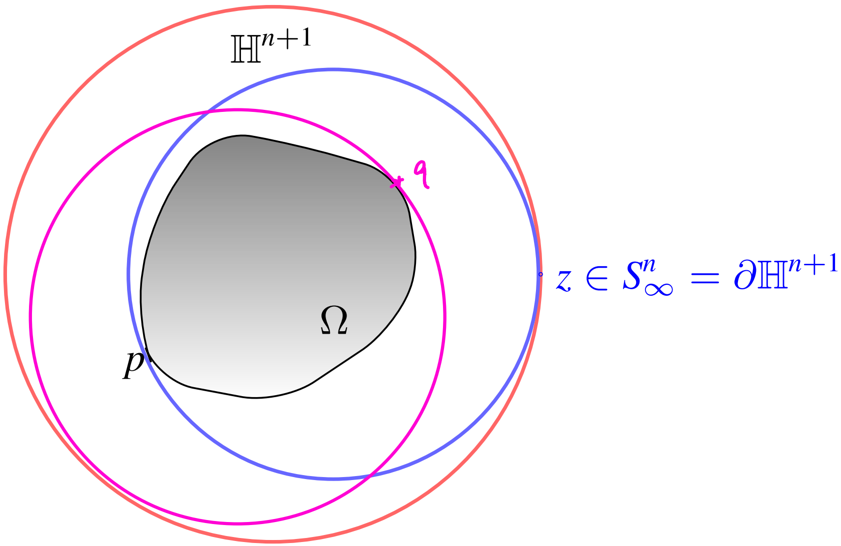
$\therefore$ Given $z \in S^n$, $\exists$ a family of horospheres

$$H_z(s) = \{ X \in \mathbb{H}^{n+1}. \quad X \cdot (z, 1) = -e^{s} \}$$

horo-balls: $B_z(s) = \{ X \in \mathbb{H}^{n+1}. \quad 0 > X \cdot (z, 1) > -e^{s} \}$

For each $z \in S^n$ we define the horospherical support function of an h-convex Ω by

$$u(z) := \inf\{\underbrace{s \in \mathbb{R}}\ :\ \underbrace{\Omega \subset B_z(s)}\} \ = \ \sup\{\log(-X \cdot (z,1)),\ X \in \partial\Omega\}.$$



The horospherical support function completely determines an h-convex domain Ω:

$$\Omega = \bigcap_{z \in S^n} B_z(u(z)).$$

Recovering the region from support function

The horospherical Gauss map $\mathbf{e}: \Sigma \to \mathbb{S}^n$: null vector $X - \nu \in \mathbb{N}^{n+1}_+$

$$\mathbf{e}(p) = \pi(\underline{X(p) - \nu(p)}) \in \mathbb{S}^n$$

where $\pi(x, y) = y/x$ is the radial projection from the light cone to $\mathbb{S}^n \times \{1\}$.

If $\kappa_i > 1$ everywhere,

$$\mathbf{D}\mathbf{e}(v) = D\pi\big|_{X-\nu}\left((\mathcal{W} - \mathrm{I})(v)\right), \quad v \in T\Sigma,$$

it follows that $\mathbf{e}$ is a diffeomorphism from Σ to $\mathbb{S}^n$, and we can reparametrize an h-convex hypersurface using $\mathbf{e}^{-1}$.

Proposition (Andrews-Chen-W., 2018)

We can reparametrize an h-convex $\Sigma = \partial\Omega$ using the horospherical support function u by $\bar{X} = X(\mathbf{e}^{-1}(\cdot)) : \mathbb{S}^n \to \mathbb{H}^{n+1}$,

$$\bar{X}(z) = \Big(\underbrace{-e^u\bar{\nabla}u + \big(\frac{1}{2}e^u|\bar{\nabla}u|^2 - \sinh u\big)z}_{\mathbb{R}^{n+1}},\ \underbrace{\frac{1}{2}e^u|\bar{\nabla}u|^2 + \cosh u}_{x^0}\Big).$$

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Q/A: $x - \nu \in \mathbb{N}_+^{n+1}$.

$|x-\nu|^2 = |x|^2 + |\nu|^2 - 2x\cdot\vec{\nu} = 0$

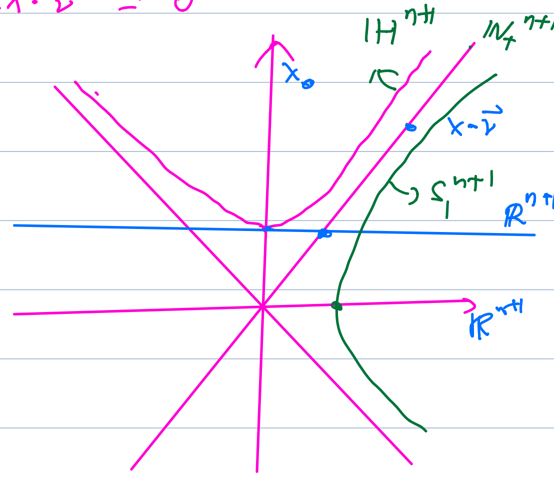
$e(p) = \pi(x - \vec{\nu}) \in S^n$

$e : \Sigma \longrightarrow S^n$

$x \in \mathbb{H}^{n+1}$

$\vec{\nu} \in S_1^{n+1}$

$x - \vec{\nu} \in \mathbb{N}_+^{n+1}$



Recovering the region from support function

If $\kappa_i > 1$ everywhere, for each $z \in \mathbb{S}^n$ there is a unique point of Σ touching $\partial(B_z(u(z)))$, which we label as $X(z)$.

Choose local coordinates $\{x^i\}$ for $\mathbb{S}^n$ near z. We write $\bar{X}(z)$ as a linear combination of the basis consisting of the two null elements $(z, 1)$ and $(-z, 1)$, together with $(z_j, 0)$, where $z_j = \frac{\partial z}{\partial x^j}$ for $j = 1, \cdots, n$:

$$\bar{X}(z) = \alpha(-z, 1) + \beta(z, 1) + \gamma^j(z_j, 0)$$

for some coefficients α, β, γ^j. The idea is to find the coefficients using the facts

- $\bar{X}(z) \in \mathbb{H}^{n+1}$. $\iff$ $|\bar{X}| = -1$ $\iff$ $\gamma^2 - 4\alpha\beta = -1$
- $\bar{X}(z) \cdot (z, 1) = -e^{u(z)}$ since $\bar{X}(z) \in \partial B_z(u(z))$. $\Rightarrow$ $\alpha = \frac{1}{2} e^{u(z)}$
- The normal to $\partial\Omega$ coincides with the normal to the horosphere $\partial B_z(u(z))$.

$$\nu = \bar{X}(z) - e^{-u(z)}(z, 1).$$

$$0 = \partial_j \bar{X} \cdot (\bar{X}(z) - \vec{\nu}(z))$$
$$= e^{-u(z)} \partial_j \bar{X} \cdot (z, 1) \qquad (\text{it } \lambda\ \bar{X})$$

$$\Rightarrow \quad \gamma_i = -e^{u(z)} u_i$$

A condition for h-convexity

$\bar{X} : S^n \longrightarrow \mathbb{H}^{n+1}$

Given a function $u \in C^\infty(\mathbb{S}^n)$, we can use the expression

$$\bar{X}(z) = \left(-e^u \bar{\nabla} u, 0\right) + \frac{1}{2}(e^u|\bar{\nabla} u|^2 + e^{-u})(z,1) + \frac{1}{2}e^u(-z,1)$$

to define a map to hyperbolic space. The unit normal is given by $\nu = \bar{X} - e^{-u}(z,1)$. Differentiation in z gives

$$(\mathcal{W} - \mathrm{I})(D_i\bar{X}) = u_i e^{-u}(z,1) - e^{-u}(\partial_i z, 0).$$

Taking the inner product with $-(\partial_j z, 0)$ gives

$$\boxed{(\mathcal{W} - \mathrm{I})_i^k A_{kj} = e^{-u}\bar{g}_{ij}}$$

where

$$A_{kj} = -\langle \partial_k \bar{X}, (\partial_j z, 0)\rangle = \boxed{\bar{\nabla}_k\bar{\nabla}_j e^u - \frac{1}{2}e^u|\bar{\nabla} u|^2\bar{g}_{kj} + \sinh u\,\bar{g}_{kj}}.$$

Proposition (Andrews-Chen-W., 2018)

The map $\bar{X} : \mathbb{S}^n \to \mathbb{H}^{n+1}$ defined in terms of a function $u \in C^\infty(\mathbb{S}^n)$ is an embedding of an h-convex hypersurface if and only if the matrix $A_{kj}[u]$ is positive definite.

$$\underbrace{e^u A_{ik}[u]\bar{g}^{kj}}=\underbrace{(W-I)}{}^{-1}{}_i^j$$

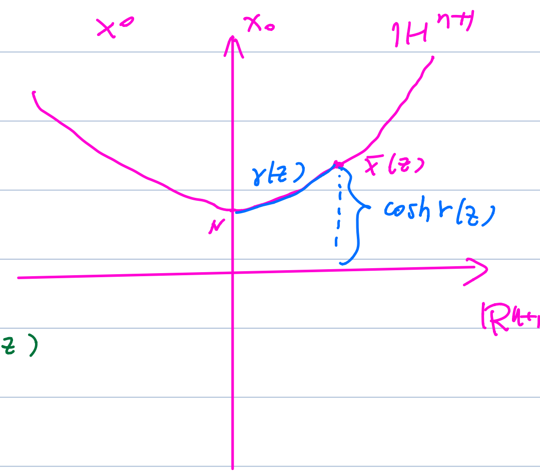
eigenvalues $\{\gamma_1 .. \gamma_n\}$ $\quad \left\{\frac{1}{k_1-1}, \ldots \frac{1}{k_n-1}\right\}$

$\gamma_i = \frac{1}{k_i-1}$ hyperbolic principal radii

$$\bar{X}(z) = \Big(\underbrace{-e^u\bar{\nabla}u + (\tfrac{1}{2}e^u|\bar{\nabla}u|^2 - \sinh u)z}_{\mathbb{R}^{n+1}}, \underbrace{\tfrac{1}{2}e^u|\bar{\nabla}u|^2 + \cosh u}_{x^0}\Big).$$

$\cosh r(z) = \frac{1}{2} e^u |\bar{\nabla}u|^2 + \cosh u$

where $r(z) = \mathrm{dist}_{\mathbb{H}^{n+1}}(\bar{X}(z), N)$



$\mathrm{Diff} \Rightarrow \quad \sinh r(z)\, \gamma_i(z) = \underbrace{A_{ik}[u]}_{>0} \cdot u_k(z)$

1. This motivates us to define the hyperbolic principal curvature radii as:

$$\boxed{r_i = \frac{1}{\kappa_i - 1}, \quad i = 1, \cdots, n.}$$

$\kappa_i > 1$

$\kappa_i \geqslant -1$

A similar development was presented by Espinar-Gálvez-Mira (2009), but in a slightly different context: In that paper the 'horospherically convex' regions are those which are intersections of complements of horo-balls (corresponding to principal curvatures greater than -1 everywhere, while we deal with regions which are intersections of horo-balls, corresponding to principal curvatures greater than 1. Our condition is more stringent but is more useful for the evolution equations we consider later.

2. Let $r(z) = \mathrm{dist}_{\mathbb{H}^{n+1}}(o, \bar{X}(z))$. Then

$$\boxed{\cosh r(z) = \frac{1}{2} e^u |\bar{\nabla} u|^2 + \cosh u}$$

Differentiation gives $\sinh r(z) r_i = A_{ij}[u] u_j$. It follows $\bar{\nabla} r = 0$ iff $\bar{\nabla} u = 0$, and $\max_{\mathbb{S}^n} r = \max_{\mathbb{S}^n} u$.

3. We can ask similar questions that have been studied in Euclidean convex geometry: for example, the Christoffel-Minkowski problem in hyperbolic space, prescribed curvature problem in hyperbolic space, etc.

4. We can study the geometric flows by smooth functions of the hyperbolic principal curvature radii (equivalently, of the shifted principal curvature $\tilde{\kappa}_i = \kappa_i - 1$), and study the geometric inequalities for h-convex hypersurfaces.

Lemma

$(\kappa_i > 1,\ i = 1 . \ n)$

Let $X : M \times [0, T) \to \mathbb{H}^{n+1}$ be a smooth h-convex solution to the flow

$$\frac{\partial}{\partial t} X = -F(\kappa - 1)\nu.$$

$\kappa = (\kappa_1, \cdot \quad \cdot \kappa_n)$
principal curvatures

in $\mathbb{H}^{n+1}$. Then it is equivalent to the following initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t} u = -F(e^{-u} A[u]^{-1}), \\ u(\cdot, 0) = u_0(\cdot) \end{cases}$$

on $S^n \times [0, T)$

on $\mathbb{S}^n \times [0, T)$, where u is the horospherical support function of $\Sigma_t = X(M, t)$.

New Quermassintegrals

We next discuss some results on curvature flows by shifted principal curvatures.

We define a new family of 'modified Quermassintegrals' in hyperbolic space, which are natural under the condition of h-convexity:

$$\widetilde{W}_k(\Omega) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} W_i(\Omega), \quad k = 0, \cdots, n.$$

$\kappa = (\kappa_1, \cdots \kappa_n)$

These are characterised by their variation equation: If $\frac{\partial X}{\partial t} = F\nu$, then

$$\frac{d}{dt}\widetilde{W}_k(\Omega_t) = \int_{\Sigma_t} F\tilde{E}_k \, d\mu_t,$$

$E_k(\kappa - 1) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} E_i(\kappa)$

where $\tilde{E}_k = E_k(\tilde{\kappa})$ is the kth elementary symmetric function of the 'shifted' principal curvatures $\tilde{\kappa}_i = \kappa_i - 1$. Denote $\tilde{f}_k(r) = \widetilde{W}_k(B(r))$. We proved

Theorem (Andrews-Chen-W. 2018)

Let $\Omega \subset \mathbb{H}^{n+1}$ be a smooth, bounded and h-convex domain. Then

$$\widetilde{W}_k(\Omega) \geq \tilde{f}_k \circ \tilde{f}_\ell^{-1}(\widetilde{W}_\ell(\Omega)), \quad 0 \leq \ell < k \leq n \tag{1}$$

with equality holding if and only if Ω is a geodesic ball.

$\tilde{f}_k(r) = \widetilde{W}_k(B(r))$

$k > \frac{n}{2}$, $\widetilde{W}_k(\Omega) \leq C$

$\therefore \quad \tilde{f}_k(r) \leq C$

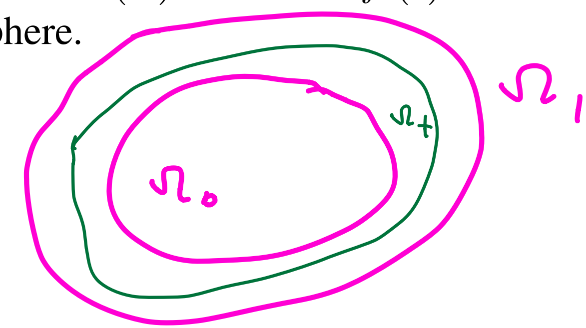
1. The inequalities (1) are new and can be viewed as an improvement of Wang-Xia's (2014) result.
2. The proof is by applying the following Quermassintegral preserving flow

$$\frac{\partial}{\partial t}X = \left(\phi(t) - \left(\frac{\tilde{E}_k}{\tilde{E}_\ell}\right)^{\frac{1}{k-\ell}}\right)\nu, \quad 0 \leq \ell < k \leq n$$

with $\phi(t)$ chosen to keep $\widetilde{W}_\ell(\Omega_t)$ constant.

3. Complication: If $k > n/2$, then $\lim_{r\to\infty} \tilde{f}_k(r) < \infty$. To make sense of the Theorem we first need to prove that $\tilde{W}_k(\Omega)$ is in the range of $\tilde{f}_k$.

We prove: $\tilde{W}_k$ is monotone with respect to inclusion for h-convex domains, i.e. if $\Sigma_i = \partial\Omega_i$ for $i = 1, 2$ are h-convex and $\Omega_1 \subset \Omega_2$, then $\tilde{W}_k(\Omega_1) \leq \tilde{W}_k(\Omega_2)$. The fact that $\tilde{W}_k(\Omega) < \lim_{r\to\infty} \tilde{f}_k(r)$ follows since we can always enclose Ω by a large sphere.



$\Omega_1 \subset \Omega_2$

$\Rightarrow \widetilde{W}_k(\Omega_1) \leq \widetilde{W}_k(\Omega_2)$

Idea: construct a family of h-convex Ω_t expanding from Ω_0 to Ω_1

Then $\frac{d}{dt} \widetilde{W}_k(\Omega_t) = \int_{\partial\Omega_t} \underbrace{F}_{\geq 0} \cdot \underbrace{\tilde{E}_k}_{\geq 0} \geq 0$

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let u_0, u_1 be horospherical support functions of Ω_0, Ω_1

$\therefore \quad u_0(z) \leq u_1(z), \quad \forall z \in S^n$

令 $u_t(z) = \ln\left((1-t)e^{u_0(z)} + t e^{u_1(z)}\right) \qquad z \in S^n$

$$A_{ij}[u_t] = \bar{\nabla}_k \bar{\nabla}_j e^u - \frac{1}{2} e^u |\bar{\nabla} u|^2 \bar{g}_{kj} + \sinh u \bar{g}_{kj}$$

令 $\varphi_t = e^{u_t}, \quad \varphi_0 = e^{u_0}, \quad \varphi_1 = e^{u_1}$

$\therefore \quad \varphi_t(z) = (1-t)\varphi_0 + t\varphi_1$

$$\Rightarrow A_{ij}[u_t] = \bar{\nabla}_i \bar{\nabla}_j \varphi_t - \frac{1}{2\varphi_t} |\bar{\nabla}\varphi_t|^2 \bar{g}_{ij} + \frac{1}{2}(\varphi_t - \varphi_t^{-1}) \bar{g}_{ij}$$

$$= (1-t) A_{ij}[u_0] + t A_{ij}[u_1] + t(1-t) \frac{|\varphi_0 \bar{\nabla}\varphi_1 - \varphi_1 \bar{\nabla}\varphi_0|^2 + |\varphi_1 - \varphi_0|^2}{2\varphi_0\varphi_1((1-t)\varphi_0 + t\varphi_1)} \bar{g}_{ij}$$

$$\geq (1-t) \underbrace{A_{ij}[u_0]}_{>0} + t \underbrace{A_{ij}[u_1]}_{>0}$$

$$> 0$$

$\Rightarrow \quad u_t$ defines an h-convex Ω_t

We also studied the following inverse mean curvature type flow (with Xianfeng Wang and Tailong Zhou (2019))

$\Sigma_t = \{ (\theta, \gamma_t(\theta)) \}$ $\theta \in S^n$

$$\frac{\partial}{\partial t} X = \frac{1}{H - n} \nu$$

$k_i > 1$

for h-convex hypersurfaces in hyperbolic space.

1. The solution Σ_t expands to infinity in finite time, and is asymptotically round in the sense that the oscillation decays to zero exponentially.
2. This is in contrast to the asymptotical behavior of IMCF

$$\frac{\partial}{\partial t} X = \frac{1}{H} \nu$$

in $\mathbb{H}^{n+1}$

in hyperbolic space, as André Neves (2010), and P.-K. Hung and M.-T. Wang (2015) constructed examples to show the limiting shape of IMCF in hyperbolic space is not necessarily round. $\mathbb{H}^{n+1}$

Asymptotically hyperbolic

With Haizhong Li and Yingxiang Hu (2020), we introduced a new locally constrained curvature flow

$$\frac{\partial}{\partial t}X = \Big(\underbrace{(\cosh r - \langle \sinh r\partial_r, \nu\rangle)\frac{\tilde{E}_{k-1}}{\tilde{E}_k} - \langle \sinh r\partial_r, \nu\rangle}_{F}\Big)\,\nu$$

C^0-estimate

$1 < \kappa_i \le C$

$\Rightarrow$ C^1-estimate

C^2-estimate

for hypersurfaces in hyperbolic space.

1. The convergence to a geodesic sphere can be proved for h-convex hypersurfaces in hyperbolic space.
2. The flow preserves $\tilde{W}_k$ and decreases $\tilde{W}_{k+1}$.
3. New optimal geometric inequalities

Q: star-shaped + $\tilde{E}_k > 0 \Rightarrow$ convergence?

$$\int_\Sigma uL_k \;\geq\; c_{n,k}|\Sigma|^{\frac{n+1-2k}{n}}$$

for h-convex hypersurfaces. The case $k = 1$ was proved by Brendle, Hung and Wang (2012) for mean convex and star-shaped hypersurfaces.

$$\frac{d}{dt}\tilde{W}_k(\Omega_t) = \int_{\partial\Omega_t} F\cdot \tilde{E}_k \, d\mu_t$$

$$= \int_{\partial\Omega_t}\Big((\cosh r - \langle \sinh r\partial_r, \vec{\nu}\rangle)\cdot \tilde{E}_{k-1} - \tilde{E}_k\langle \sinh r\partial_r, \vec{\nu}\rangle\Big)\,d\mu_t$$

$$= 0 \quad \left(\text{by } \int_{\partial\Omega_t} \cosh r \, E_{k-1} = \int_{\partial\Omega_t} \langle \sinh r \partial_r, \vec{\nu} \rangle \cdot E_k \right)$$

$$\frac{d}{dt} \widetilde{W}_{k+1}(\Omega_t) \leq 0$$

$$\int_\Sigma u L_k \geq c_{n,k} |\Sigma|^{\frac{n+1-2k}{n}}$$

$u = \langle \sinh r \partial_r, \vec{\nu} \rangle$

L_k : Gauss-Bonnet curvature

$$L_k(g) = \frac{1}{2^k} \delta^{i_1 i_2 \cdots i_{2k}}_{j_1 j_2 \cdots j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}$$

$L_1(g) =$ scalar curvature of g

$L_2(g) = |Rm|^2 - 4|Ric|^2 + R^2$

$\Sigma^n \subset \mathbb{H}^{n+1}$. induced metric g

$$L_k(g) = \binom{n}{2k}(2k)! \sum_{i=0}^{k} (-1)^i \binom{k}{i} E_{2k-2i} \qquad 2k < n$$

Ge-Wang-Wu 2013 :

$$\Sigma \subset \mathbb{H}^{n+1}, \ h\text{-convex} \Rightarrow \int_\Sigma L_k(g) \geq C_{n,k} |\Sigma|^{\frac{n-2k}{n}}$$

(PSC by Yingxiang Hu. H. Li 2018)

Brendle-Hung-Wang 2012

$$\int_\Sigma (\cosh r \cdot E_1 - u) \geq w_n^{\frac{1}{n}} |\Sigma|^{\frac{n-1}{n}}$$

$$\Leftrightarrow \quad \int_\Sigma u \underbrace{(E_2 - 1)}_{L_1(g)} d\mu_g \geq w_n^{\frac{2}{n}} |\Sigma|^{\frac{1}{n}}$$

our inequality $\int_\Sigma u L_k(g) \geq C_{n,k} |\Sigma|^{\frac{n+1-2k}{n}}$

Concluding remarks

In this course, we discussed the hypersurface curvature flows in Euclidean space and in hyperbolic space, with focus on their applications in geometric inequalities:

Isoperimetric inequailty, Alexandrov – Fenchel inequalities.

There are many other interesting topics on hypersurface flows:

1. Mean curvature flow and fully nonlinear contracting curvature flows: Singularity analysis and application in topology. J. Scheuer
2. S^{n+1}. warped product G. Wang
   Curvature flows in general ambient space, and for hypersurfaces with boundary.
   C. Xia
3. Application in convex geometry. in $\mathbb{R}^{n+1}$ ✓ in $\mathbb{H}^{n+1}$?
4. non compact

Other geometric flows: Ricci flow, Kähler Ricci flow, Harmonic map heat flow, Yang-Mills flow, G_2-Laplacian flow, etc. They are (degenerate) parabolic equations / systems of certain geometric structures on a Riemannian manifold. The main motivation is the application in geometric and topology of the underlying manifold.

Thank you!

- Q & A.

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