Motivation and a classical introduction: Complex analytic theory
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Abel (1802-29), Jacobi (1804-1851), Weiestrass(1815-1897), Kronecker (1823-1891), Eisenstein(1823-1852).....: elliptic functions—periodic meromorphic functions on $\mathbb{C}$ with period $\Lambda = \mathbb{Z} + \mathbb{Z} \tau$, $\tau \in \mathbb{C} \setminus \mathbb{R}$.

Klein (1849-1925), Fricke(1861-1930): study the Riemann surface $\Gamma \backslash \mathcal{H}$

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Gelfand(1913 - 2009 ), Godement (1921-2016), Harish-Chandra (1923-1983 ): automorphic representations....


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Π-module structure on the space of modular forms $\overline{M(\mathbb{Q})}$.

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Example with fixed level: the graded \( \mathbb{C} \)-algebra

\[ M(\mathbb{SL}_2(\mathbb{Z})) \]

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WANG Shanwen

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Example with fixed level: the graded $\mathbb{C}$-algebra $M(\text{SL}_2(\mathbb{Z}))$, $\Pi$-module structure on the space of modular forms $M(\overline{\mathbb{Q}})$

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Example with fixed level: the graded \( \mathbb{C} \)-algebra \( M(\text{SL}_2(\mathbb{Z})) \) with \( \mathbb{Q} \)-module structure on the space of modular forms \( M(\mathbb{Q}) \).

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WANG Shanwen
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Monster’s moonshine

- In the classification of finite simple group, except from some uniform families, there are 26 isolated finite simple groups. Among these 26 groups, the biggest one is called the monster group, which is discovered via the representation theory. The first few dimensions of the irreducible representations of the monster group:
  \[
  f_1 = 1, \quad f_2 = 196883, \quad f_3 = 21296876, \quad f_4 = 842609326, \\
  f_5 = 18538750076, \quad f_6 = 19360062527, \ldots
  \]

- J. McKay remarked in 1977 that 196883 is related to the Fourier coefficients of the modular function \( j(\tau) \): if we write
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  j(\tau) = \frac{1}{q} + 744 + \sum_{n \geq 1} c_n q^n \text{ with } q = e^{2i\pi \tau},
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  then
  \[
  c_1 = f_2 + f_1, \quad c_2 = f_3 + f_2 + f_1, \quad c_3 = f_4 + f_3 + 2f_2 + 2f_1 \ldots
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  This mysterious relation is well-known with the name ”monsters moonshine”.

- This has been proved by R. Borcherds (Fields medal 1998) in 1992, using the objects in mathematical physics.
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Algebraic number theory: study the galois group of $\mathbb{Q}$.

1967 Langlands to Weil: conjecture that there exists a correspondance between the automorphic representations of $GL_n$ and the Galois representations of $Gal_{\mathbb{Q}}$ dimension $n$.

- $n = 1$: Class field theory.
- $n = 2$: some evidences from modular forms (works of Deligne, Eichler-Shimura.......)

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Example with fixed level: the graded $\mathbb{C}$-algebra $M(\text{SL}_2(\mathbb{Z}))$ has $\prod_{\mathbb{Q}}$-module structure on the space of modular forms $\bar{M}(\mathbb{Q})$.

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- The theory of modular forms starts from the complex analytic theory. We will introduce the significant results in the theory and point out how they allows us to pass from the complex analytic geometry to the algebraic geometry over $\mathbb{Q}$.
- We study the tower of modular curves (thus the space of modular forms of all the level). A uniform way to describe them is to use the adelic language. The latter allows us to pass to the theory of representations of $\text{GL}_2(\mathbb{A}_f)$.
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The modular form theory gives a concrete example of the theory of motives.

- Motive: Roughly speaking, a motive (pure motive) defined over a field $K$ is a piece of a 'universal' cohomology of an algebraic variety (resp, smooth projective variety) defined over $K$.

- Instead of studying the motive itself, we try to understand it by its various realizations and the relations between them (i.e. comparison theorems). In the following, to simplify the life, we set $K = \mathbb{Q}$. Let $M$ be a motive defined over $\mathbb{Q}$, we can associate to $M$ two $\mathbb{Q}$-vector spaces of the same dimension.

  - Betti Realization: a $\mathbb{Q}$-vector space $M_B$ equipped with an action of $\text{Gal}(\mathbb{C}/\mathbb{R})$. Thus a decomposition $M_B = M_B^+ \oplus M_B^-$. 
  - de Rham realization: a $\mathbb{Q}$-vector space $M_{\text{dR}}$ equipped with a decreasing filtration $(M_{\text{dR}}^i)_{i \in \mathbb{Z}}$ consisting of sub-$\mathbb{Q}$-vector spaces.
  - A 'complex period' isomorphism $\mathbb{R} \otimes M_{\text{dR}} \cong (\mathbb{C} \otimes M_{\text{dR}})^{\text{Gal}(\mathbb{C}/\mathbb{R})}$.
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\[ \mathcal{H}^+ = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \] the Poincaré upper half plane.

- \( \text{GL}_2(\mathbb{R})_+ = \{ \gamma \in \text{GL}_2(\mathbb{R}) : \det \gamma > 0 \} \).

- Linear fractional action of \( \text{GL}_2(\mathbb{R})_+ \) on \( \mathcal{H} \): \( \gamma \tau = \frac{a \tau + b}{c \tau + d} \), for \( \tau \in \mathcal{H} \) and \( \gamma = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \text{GL}_2(\mathbb{R})_+ \). In other words, there is a group homomorphism

\[
\text{GL}_2(\mathbb{R})_+ \to \text{Aut}(\mathcal{H}^+).
\]

- If \( k \in \mathbb{N}, j \in \mathbb{Z} \), define a weight \((k, j)\) action of \( \text{GL}_2(\mathbb{R})_+ \) on the space of functions \( C^\infty(\mathcal{H}^+, \mathbb{C}) \):

\[
(f|_{(k,j)} \gamma)(\tau) = \frac{(\det \gamma)^{k-j}}{(c \tau + d)^k} f\left(\frac{a \tau + b}{c \tau + d}\right), \text{ if } \gamma = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \text{GL}_2(\mathbb{R})_+. \quad (1)
\]
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Example with fixed level: the graded \( \mathbb{C} \)-algebra \( \mathcal{M}(\text{SL}_2(\mathbb{Z})) \) \( \mathbb{Q} \)-module structure on the space of modular forms \( \overline{\mathcal{M}(\mathbb{Q})} \).

Definition

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Complex analytic definition

- \( \mathcal{H}^+ = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) the Poincaré upper half plane.
- \( \text{GL}_2(\mathbb{R})_+ = \{ \gamma \in \text{GL}_2(\mathbb{R}) : \det \gamma > 0 \} \).
- Linear fractional action of \( \text{GL}_2(\mathbb{R})_+ \) on \( \mathcal{H} \): \( \gamma \tau = \frac{a\tau + b}{c\tau + d} \), for \( \tau \in \mathcal{H} \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{R})_+ \). In other words, there is a group homomorphism \( \text{GL}_2(\mathbb{R})_+ \to \text{Aut}(\mathcal{H}^+) \).
- If \( k \in \mathbb{N}, j \in \mathbb{Z} \), define a weight \( (k, j) \) action of \( \text{GL}_2(\mathbb{R})_+ \) on the space of functions \( \mathcal{C}^\infty(\mathcal{H}^+, \mathbb{C}) \):

\[
(f|_{(k,j)} \gamma)(\tau) = \frac{(\det \gamma)^{k-j}}{(c\tau + d)^k} f\left(\frac{a\tau + b}{c\tau + d}\right), \quad \text{if} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{R})_+. \quad (1)
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Example with fixed level: the graded $\mathbb{C}$-algebra $M(\SL_2(\mathbb{Z}))$ $\Pi_0$-module structure on the space of modular forms $\tilde{M}(\mathbb{Q})$

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Classical definition

Definition
Let $\Gamma < \text{SL}_2(\mathbb{Z})$ be a subgroup of finite index. Let $\chi$ be a finite order character of $\Gamma$ (i.e. $\chi(\Gamma) \subset \mu_N$). A holomorphic function $f : \mathcal{H}^+ \to \mathbb{C}$ is a modular form of weight $k$ and of character $\chi$ for $\Gamma$, if

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- We denote by $M_k(\Gamma, \chi)$ the $\mathbb{C}$-vector space of modular forms of weight $k$, level $\Gamma$ and of character $\chi$ for $\Gamma$.
- In particular, the $\mathbb{C}$-v.s. $M_k(\Gamma)$ of modular forms of weight $k$, level $\Gamma$ is $M_k(\Gamma, \chi)$ with $\chi$ the trivial character for $\Gamma$.
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Example with fixed level: the graded $\mathbb{C}$-algebra $M(\text{SL}_2(\mathbb{Z}))$ $\Pi_Q$-module structure on the space of modular forms $\tilde{M}(0)$

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Example with fixed level: the graded \( C \)-algebra \( M(\text{SL}_2(\mathbb{Z})) \)
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q-expansion

- If $f \in C^\infty(\mathcal{H}^+, \mathbb{C})$ is fixed by a finite index subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$. Then $f$ is a periodic function of period $N$ for some $N \geq 1$. Thus we have the $q$-expansion of $f$ as following:

$$f(\tau) = \sum_{n \in \mathbb{Z}[\frac{1}{N}]} a_n(f) e^{2i\pi n \tau} = \sum_{n \in \mathbb{Z}[\frac{1}{N}]} a_n(f) q^n, \text{ where } q = e^{2i\pi \tau}.$$

- This gives a decomposition of $\mathbb{C}$-v.s.:

$$M_k(\Gamma) = S_k(\Gamma) \oplus \text{Eis},$$

where $S_k(\Gamma) = \{f \in M_k(\Gamma) : a_0(f|_{k\gamma}) = 0, \forall \gamma \in \text{GL}_2(\mathbb{Q})_+\}$ and $\text{Eis} = \{f \in M_k(\Gamma) : \exists \gamma \in \text{GL}_2(\mathbb{Q}), a_0(f|_{k\gamma}) \neq 0\}$.

- The q-expansion leads us to consider the following subspace of modular forms: Let $A$ be a subring of $\mathbb{C}$ and we have the $A$-module $M_k(\Gamma, A) := \{f \in M_k(\Gamma) : a_n(f) \in A, \forall n\}$. 

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- If \( f \in \mathcal{C}^{\infty}(\mathcal{H}^+, \mathbb{C}) \) is fixed by a finite index subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \). Then \( f \) is a periodic function of period \( N \) for some \( N \geq 1 \). Thus we have the \( q \)-expansion of \( f \) as following:

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Definition

Let $A$ be a ring, define $A[\Gamma \backslash G / \Gamma]$ to be the set of $\phi : G \to A$ satisfying the following two conditions:

1. $\phi(\gamma x) = \phi(x \gamma) = \phi(x)$, for all $x \in G$, $\gamma \in \Gamma$.
2. There exists a finite set $I$ such that $\phi = \sum_{i \in I} \lambda_i 1_{\Gamma x_i}$.
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- If \( \phi = \sum_{i \in I} \lambda_i 1_{\Gamma x_i} \) and \( \phi' = \sum_{j \in J} \mu_j 1_{\Gamma x_j} \) in \( A[\Gamma \backslash G / \Gamma] \), then for \( i \in I, j \in J \), we have
  \[
  \phi * \phi' = \sum_{(i,j) \in I \times J} \lambda_i \mu_j 1_{\Gamma x_i y_j} \in A[\Gamma \backslash G / \Gamma],
  \]
  which doesn’t depend on the choice.

- \( (A[\Gamma \backslash G / \Gamma], +, *) \) is an associative \( A \)-algebra with \( 1_{\Gamma} \) as a unit.

- If \( M \) is a right \( G \)-module with \( G \) action \( m \mapsto m * g \), and \( \phi = \sum_{i \in I} \lambda_i 1_{\Gamma x_i} \in A[\Gamma \backslash G / \Gamma] \), then for any \( m \in M^r \), we have
  \[
  m * \phi = \sum_{i \in I} \lambda_i m * x_i
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  \[
  m * (\phi_1 \ast \phi_2) = (m * \phi_1) \ast \phi_2, m * (\phi_1 + \phi_2) = (m * \phi_1) + (m * \phi_2).
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Remark: \( A[\Gamma \backslash G / \Gamma] \) is commutative if and only if \( G \) is commutative.
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Remark: \( A[\Gamma \setminus G / \Gamma] \) is commutative if and only if \( G \) is commutative.
Apply to modular forms: $G = \text{GL}_2(\mathbb{Q})_+, \Gamma = \text{SL}_2(\mathbb{Z})$

**Lemma**

Let $g \in G \cap M_2(\mathbb{Z})$, then there exists a unique pair $(a, d) \in \mathbb{N} - \{0\}$ and $b \in \mathbb{Z}$ which is unique mod $d\mathbb{Z}$ such that $\Gamma g = \Gamma \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right)$.

**Definition**

$$R_n = 1_{\Gamma \left( \begin{array}{cc} n & 0 \\ 0 & n \end{array} \right)} \in \mathbb{Z}[\Gamma \backslash G / \Gamma], \quad T_n = 1_{\{g \in M_2(\mathbb{Z}), \det g = n\}} \in \mathbb{Z}[\Gamma \backslash G / \Gamma].$$

**Remark:** If $p$ is a prime, then $T_p = 1_{\Gamma \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right)} \Gamma$.

**Theorem**

If $p$ and $l$ are coprime, then $T_p T_l = T_l T_p = T_{pl}$.

If $p$ is prime and $r \geq 1$, then $T_p^r = T_p + \frac{r-1}{p} R_p T_p^{r-1}$.
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Apply to modular forms: \( G = \text{GL}_2(\mathbb{Q})_+, \Gamma = \text{SL}_2(\mathbb{Z}) \)

Lemma

Let \( g \in G \cap M_2(\mathbb{Z}) \), then there exists a unique pair \((a, d) \in \mathbb{N} - \{0\}\) and \( b \in \mathbb{Z} \) which is unique \( \mod d\mathbb{Z} \) such that \( \Gamma g = \Gamma \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \).

Definition

\[ R_n = 1_{\Gamma \left( \begin{array}{cc} n & 0 \\ 0 & n \end{array} \right)} \in \mathbb{Z}[\Gamma \backslash G / \Gamma], \quad T_n = 1_{\{g \in M_2(\mathbb{Z}), \det g = n\}} \in \mathbb{Z}[\Gamma \backslash G / \Gamma]. \]

Remark: If \( p \) is a prime, then \( T_p = 1_{\Gamma \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right)} \Gamma \).

Theorem

- \( \forall n \geq 1 \) and \( l \geq 1 \), then \( R_nR_l = R_{nl} = R_lR_n \) and \( R_nT_l = T_lR_n \).
- If \((l, n) = 1\), then \( T_lT_n = T_{ln} = T_nT_l \).
- If \( p \) is prime and \( r \geq 1 \), then \( T_p^r = T_p^{r+1} + pR_pT_{p^r} \).
- Let \( T_n \) be the subalgebra of \( \mathbb{Z}[\Gamma \backslash G / \Gamma] \) generated by \( R_n \) and \( T_n \) for \( n \geq 1 \).

Then it is a commutative algebra.
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**Lemma**

Let $g \in G \cap M_2(\mathbb{Z})$, then there exists a unique pair $(a, d) \in \mathbb{N} - \{0\}$ and $b \in \mathbb{Z}$ which is unique mod $d\mathbb{Z}$ such that $\Gamma g = \Gamma \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right)$.

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- If $p$ is prime and $r \geq 1$, then $T_p T_p = T_{pr + 1} + pR_p T_{pr - 1}$.
- Let $T_2$ be the subalgebra of $\mathbb{Z}[\Gamma \backslash G / \Gamma]$ generated by $R_n$ and $T_n$, for $n \geq 1$. Then it is a commutative algebra.
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- $M_k(\mathbb{C})$ and $S_k(\mathbb{C})$ are stable under the action of $\text{GL}_2(\mathbb{Q})_+$.
  - For $f \in M_k(H, \mathbb{C})$ and $\gamma \in H$, we have
    \[ f|_{(k,j)} \alpha = (f|_{(k,j)} \alpha)|_{(k,j)}(\alpha^{-1} \gamma \alpha). \]

  Thus $f$ is invariant under $\alpha^{-1} H \alpha \cap \text{SL}_2(\mathbb{Z})$ which is a finite index subgroup.
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- Explicit formulae:
  - For $f \in M_k(H, \mathbb{C})$, and $T_n = \alpha^{-n} T \alpha^n$,
    \[ f|_{(k,j)} T_n = n^{-k} f|_{(k,j)} \tau + \sum_{a \geq 1, b \mod d} \frac{d}{d^k} f|_{(a \tau + b d, d)}. \]
    In terms of polynomials $g(t) = \sum_{j=0}^{\infty} c_j t^j$, we have
    \[ f|_{(k,j)} T_n = n^{-k} f|_{(k,j)} \tau + \sum_{a \geq 1, b \mod d} \frac{d}{d^k} g(t)|_{(a \tau + b d, d)}. \]
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  - In terms of q-expansion: $S_m(f|_{(k,j)} T_n) = n^{k-2j} \sum_{a \geq 1, a \mid m} a^{k-1} q^m f$. As a result, we have
    $$a_0(f|_{(k,j)} T_n) = n^{k-2j} a_{k-1} q^m f$$
    and
    $$a_1(f|_{(k,j)} T_n) = n^{k-2j} a_n f.$$
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  - In terms of q-expansion: $a_m(f|_{(k,j)} T_n) = n^{1-j} \sum_{a \geq 1, a | (m,n)} a^{k-1} a^m n^{\frac{a}{a^2}} (f)$. As a result, we have
    \[ a_0(f|_{(k,j)} T_n) = n^{1-j} \sigma_{k-1} a_0(f) \] and $a_1(f|_{(k,j)} T_n) = n^{1-j} a_n(f)$.
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    \[
    f \big|_{(k,j)} \alpha = (f \big|_{(k,j)} \alpha)(\alpha^{-1}\gamma\alpha).
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  - In terms of \( q \)-expansion: \( a_m(f_{(k,j)}T_n) = n^1-j \sum_{a \geq 1, a \mid (m,n)} a^{k-1}a_{\frac{mn}{a^2}}(f) \). As a result, we have
    \[
    a_0(f_{(k,j)}T_n) = n^{1-j}\sigma_{k-1}a_0(f) \quad \text{and} \quad a_1(f_{(k,j)}T_n) = n^{1-j}a_1(f).
    \]
**Definition**

If \((\tau, z) \in \mathcal{H}^+ \times \mathbb{C}\), let \(q = e^{2i\pi \tau}\) and \(q_z = e^{2i\pi z}\). We set \(e(a) = e^{2i\pi a}\). If \(k \in \mathbb{N}\), \(\tau \in \mathcal{H}^+\), and \(z, u \in \mathbb{C}\), the Eisenstein-Kronecker series is

\[
H_k(s, \tau, z, u) = \frac{\Gamma(s)}{(-2i\pi)^k} \left( \frac{\tau - \bar{\tau}}{2i\pi} \right)^{s-k} \sum_{\omega \in \mathbb{Z}+\mathbb{Z}\tau}^{'} \frac{\omega + z^k}{|\omega + z|^{2s}} e\left( \frac{\omega \bar{u} - u\bar{\omega}}{\tau - \bar{\tau}} \right),
\]

which converges for \(\Re(s) > 1 + \frac{k}{2}\), and has a meromorphic continuation to \(\mathbb{C}\) with simple poles at \(s = 1\) (if \(k = 0\) and \(u \in \mathbb{Z} + \mathbb{Z}\tau\)) and at \(s = 0\) (if \(k = 0\) and \(z \in \mathbb{Z} + \mathbb{Z}\tau\)). In the above formula, \(\sum_{\omega}^{'}\) means (if \(z \in \mathbb{Z} + \mathbb{Z}\tau\)) that we get rid of the term corresponding to \(\omega = -z\). Moreover, it satisfies the functional equation:

\[
H_k(s, \tau, z, u) = e\left( \frac{z\bar{u} - u\bar{z}}{\tau - \bar{\tau}} \right) H_k(k + 1 - s, \tau, u, z).
\]
Elliptic functions

- If $k \geq 1$, we define the following elliptic functions:

$$E_k(\tau, z) = \text{H}_k(k, \tau, z, 0)(= \frac{\Gamma(k)}{(-2i\pi)^k} \sum'_{\omega \in \mathbb{Z} + \mathbb{Z}_\tau} \frac{1}{(\omega + z)^k} \text{if} \ k \geq 3), \quad F_k(\tau, z) = \text{H}_k(k, \tau, 0, z)(= \frac{\Gamma(k)}{(-2i\pi)^k} \sum'_{\omega \in \mathbb{Z} + \mathbb{Z}_\tau} \frac{1}{(\omega)^k} e^{\left(\frac{\omega \bar{z} - z \bar{\omega}}{\tau - \bar{\tau}}\right)} \text{if} \ k \geq 3).$$

- The functions $E_k(\tau, z)$ and $F_k(\tau, z)$ are periodic function in variable $z$ with period $\mathbb{Z}_\tau + \mathbb{Z}$. Moreover, we have:

$$E_{k+1}(\tau, z) = \partial_z E_k(\tau, z), \text{ if } k \in \mathbb{N} \text{ and } E_0(\tau, z) = \log |\theta(\tau, z)| \text{ if } z \notin \mathbb{Z} + \mathbb{Z}_\tau,$$

where $\theta(\tau, z)$ is given by the infinite product:

$$\theta(\tau, z) = q^{1/12}(q_z^{1/2} - q_z^{-1/2}) \prod_{n \geq 1}((1 - q^n q_z)(1 - q^n q_z^{-1})).$$
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Eisenstein series as evaluation of elliptic functions at torsion points

- If \((\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2\) and \((a, b) \in \mathbb{Q}^2\) whose image in dans \((\mathbb{Q}/\mathbb{Z})^2\) is \((\alpha, \beta)\).
- If \(k = 2\) and \((\alpha, \beta) \neq (0, 0)\), or if \(k \geq 1\) and \(k \neq 2\), we define:
  
  \[ E^{(k)}_{\alpha, \beta} = E_k(\tau, a\tau + b) \text{ and } F^{(k)}_{\alpha, \beta} = F_k(\tau, a\tau + b). \]
- If \(k = 2\) and \((\alpha, \beta) = (0, 0)\), we define \(^1E_{0,0}^{(2)} = F_{0,0}^{(2)} := \lim_{s \to 2} H_2(s, \tau, 0, 0).\)

Theorem

1. \(E_{0,0}^{(2)} = F_{0,0}^{(2)} = -\frac{1}{24} E_2^*\), where \(E_2^* = \frac{6}{i\pi(\tau - \bar{\tau})} + 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n) q^n\) is the usual non-holomorphic Eisenstein serie of weight 2.
2. If \(N\alpha = N\beta = 0\), then \(H_2(s, \tau, 0, 0)\) converge pour \(\Re(s) > 2\) and it doesn't converge for \(s = 2\).
**Eisenstein series as evaluation of elliptic functions at torsion points**

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E^{(k)}_{\alpha, \beta} = E_k(\tau, a\tau + b) \quad \text{and} \quad F^{(k)}_{\alpha, \beta} = F_k(\tau, a\tau + b).
\]

- If \(k = 2\) and \((\alpha, \beta) = (0, 0)\), we define\(^1\)

\[
E^{(2)}_{0,0} = F^{(2)}_{0,0} := \lim_{s \to 2} H_2(s, \tau, 0, 0).
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**Theorem**

1. \(E^{(2)}_{0,0} = F^{(2)}_{0,0} = -\frac{1}{24} E^*_2\), where \(E^*_2 = \frac{6}{\pi(\tau - \bar{\tau})} + 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n) q^n\) is the usual non-holomorphic Eisenstein series of weight 2.
2. If \(N\alpha = N\beta = 0\), then

\[
E^{(k)}_{\alpha, \beta} \in M_k(\Gamma(N), \mathbb{Q}(\mu_N)) \quad \text{if} \quad k \geq 1 \quad \text{and} \quad k \neq 2.
\]

\[
F^{(k)}_{\alpha, \beta} \in M_k(\Gamma(N), \mathbb{Q}(\mu_N)) \quad \text{if} \quad k \geq 1 \quad \text{and} \quad k \neq 2 \quad \text{or} \quad k = 2, \quad (\alpha, \beta) \neq (0, 0).
\]

\(^1\) \(H_2(s, \tau, 0, 0)\) converge pour \(\Re(s) > 2\) and it doesn't converge for \(s < 2\).
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1. $E^{(2)}_{0,0} = F^{(2)}_{0,0} = -\frac{1}{24} E^*_2$, where $E^*_2 = \frac{6}{i\pi (\tau - \overline{\tau})} + 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n)q^n$ is the usual non-holomorphic Eisenstein serie of weight 2.
2. If $N\alpha = N\beta = 0$, then
   
   (i) $E^{(3)}_{\alpha, \beta} = E^{(3)}_{\alpha, \beta} - E^{(3)}_{0,0} \in M_3(\Gamma(N), \mathbb{Q}(\mu_N))$ and $E^{(k)}_{\alpha, \beta} \in M_k(\Gamma(N), \mathbb{Q}(\mu_N))$ if $k \geq 1$ and $k \neq 2$.
   
   (ii) $F^{(k)}_{\alpha, \beta} \in M_k(\Gamma(N), \mathbb{Q}(\mu_N))$ if $k \geq 1$, $k \neq 2$ or if $k = 2$, $(\alpha, \beta) \neq (0, 0)$.

---

$1 H_2(s, \tau, 0, 0)$ converge pour $\Re(s) > 2$ and it doesn't converge for $s = 2$. 

WANG Shanwen
Introduction to modular forms
If \((\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2\) and \((a, b) \in \mathbb{Q}^2\) whose image in dans \((\mathbb{Q}/\mathbb{Z})^2\) is \((\alpha, \beta)\).

If \(k = 2\) and \((\alpha, \beta) \neq (0, 0)\), or if \(k \geq 1\) and \(k \neq 2\), we define:

\[ E_{\alpha, \beta}^{(k)} = E_k(\tau, a\tau + b) \text{ and } F_{\alpha, \beta}^{(k)} = F_k(\tau, a\tau + b). \]

If \(k = 2\) and \((\alpha, \beta) = (0, 0)\), we define

\[ E_{0, 0}^{(2)} = F_{0, 0}^{(2)} = -\frac{1}{24} E^*_2, \]

where \(E^*_2 = \frac{6}{i\pi(\tau - \bar{\tau})} + 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n)q^n\) is the usual non-holomorphic Eisenstein serie of weight 2.

(2) If \(N\alpha = N\beta = 0\), then

(i) \( \tilde{E}_{\alpha, \beta}^{(2)} = E_{\alpha, \beta}^{(2)} - E_{0, 0}^{(2)} \in M_2(\Gamma(N), \mathbb{Q}(\mu_N)) \) and \( E_{\alpha, \beta}^{(k)} \in M_k(\Gamma(N), \mathbb{Q}(\mu_N)) \) if \(k \geq 1\) and \(k \neq 2\).

(ii) \( F_{\alpha, \beta}^{(k)} \in M_k(\Gamma(N), \mathbb{Q}(\mu_N)) \) if \(k \geq 1\), \(k \neq 2\) or if \(k = 2\), \((\alpha, \beta) \neq (0, 0)\).
Eisenstein series as evaluation of elliptic functions at torsion points

- If \((\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2\) and \((a, b) \in \mathbb{Q}^2\) whose image in \((\mathbb{Q}/\mathbb{Z})^2\) is \((\alpha, \beta)\).
- If \(k = 2\) and \((\alpha, \beta) \neq (0, 0)\), or if \(k \geq 1\) and \(k \neq 2\), we define:
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- If \(k = 2\) and \((\alpha, \beta) = (0, 0)\), we define\(^1\) \(E_{0,0}^{(2)} = F_{0,0}^{(2)} := \lim_{s \to 2} H_2(s, \tau, 0, 0)\).

**Theorem**

1. \(E_{0,0}^{(2)} = F_{0,0}^{(2)} = -\frac{1}{24} E_2^* = \frac{6}{\pi(\tau - \bar{\tau})} + 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n)q^n\) is the usual non-holomorphic Eisenstein serie of weight 2.
2. If \(N\alpha = N\beta = 0\), then
   - If \(k \geq 1\) and \(k \neq 2\):
     \[
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     \]
   - If \(k \geq 1, k \neq 2\) or if \(k = 2, (\alpha, \beta) \neq (0, 0)\).

\(^1\)\(H_2(s, \tau, 0, 0)\) converge pour \(\Re(s) > 2\) and it doesn’t converge for \(s = 2\).
Eisenstein series as evaluation of elliptic functions at torsion points

- If \((\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2\) and \((a, b) \in \mathbb{Q}^2\) whose image in \(\mathbb{Q}/\mathbb{Z}\) is \((\alpha, \beta)\).
- If \(k = 2\) and \((\alpha, \beta) \neq (0, 0)\), or if \(k \geq 1\) and \(k \neq 2\), we define:
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  E^{(k)}_{\alpha, \beta} = E_k(\tau, a\tau + b) \quad \text{and} \quad F^{(k)}_{\alpha, \beta} = F_k(\tau, a\tau + b).
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Theorem

1. \(E^{(2)}_{0,0} = F^{(2)}_{0,0} = -\frac{1}{24} E^*_2\), where \(E^*_2 = \frac{6}{\pi(\tau - \bar{\tau})} + 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n)q^n\) is the usual non-holomorphic Eisenstein serie of weight 2.
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Outline

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   - History
   - Two amazing examples
   - Plan of this min-course

2 What is a modular form?
   - Definition
   - $q$-expansion
   - Hecke operators
   - Examples: existence of modular form

3 Example with fixed level: the graded $\mathbb{C}$-algebra $M(\text{SL}_2(\mathbb{Z}))$
   - Setting
   - $k_{12}$-formula
   - Rational and integral structure of $M(\text{SL}_2(\mathbb{Z}))$
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4 $\Pi'_Q$-module structure on the space of modular forms $M(\overline{\mathbb{Q}})$
   - Setting
   - The group $\Pi'_Q$
The fundamental domain

We will study the space $M(SL_2(\mathbb{Z})) = \bigoplus_k M_k(SL_2(\mathbb{Z}), \mathbb{C})$ of modular forms with fixed level $SL_2(\mathbb{Z})$. By the definition of modular form, we reduce to study the holomorphic functions on the quotient space $SL_2(\mathbb{Z}) \backslash \mathcal{H}^+$.  

**Definition**

Let $G$ be a group and let $X$ be a topological space equipped with a left $G$-action. Then a closed subset $D$ of $X$ is called a fundamental domain of $G$ in $X$ if $X$ is the union of conjugates of $D$, i.e., $X = \bigcup_{g \in G} gD$ and the intersection of any two conjugates has no interior.

Let

$$D \subset \{ \tau = x + iy \in \mathbb{C} : -\frac{1}{2} < x \leq \frac{1}{2} \}$$

be the fundamental domain of $SL_2(\mathbb{Z})$ in $\mathcal{H}^+$.  

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Introduction to modular forms
For any $z \in D$, we denote by $v_z(f)$ the order of zeros of $f$ at point $z$. The following theorem is the famous $\frac{k}{12}$-formula.

**Theorem**

Let $f \in M_k(SL_2(\mathbb{Z})) - \{0\}$, then we have

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum_{z \in D \setminus \{i, \rho\}} v_z(f) = \frac{k}{12}.$$ 

The $\frac{k}{12}$-formula can be used to compute the dimension of the space of modular forms.

- $\dim M_0(SL_2(\mathbb{Z})) = 1$.
- $\dim M_k(SL_2(\mathbb{Z})) = 0$, if $2 \nmid k$.
- The map $f \mapsto (a_n(f))_{0 \leq n \leq \frac{k}{12}}$ is an injection of vector spaces from $M_k(SL_2(\mathbb{Z}))$ to $\mathbb{C}^{1+\lceil \frac{k}{12} \rceil}$. Thus, $\dim M_k(SL_2(\mathbb{Z})) < \infty$. 

**Setting**

- $\frac{k}{12}$-formula
- Rational and integral structure of $M(SL_2(\mathbb{Z}))$
- As inner product space

**Motivation**

- What is a modular form?
- Example with fixed level: the graded $\mathbb{C}$-algebra $M(SL_2(\mathbb{Z}))$
- $\Pi'_\mathbb{Q}$-module structure on the space of modular forms $M(\overline{\mathbb{Q}})$

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Introduction to modular forms
The $\frac{k}{12}$-formula

For any $z \in D$, we denote by $v_z(f)$ the order of zeros of $f$ at point $z$. The following theorem is the famous $\frac{k}{12}$-formula.

**Theorem**

Let $f \in M_k(SL_2(\mathbb{Z})) \setminus \{0\}$, then we have

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Introduction to modular forms
The $\frac{k}{12}$-formula

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Motivation

What is a modular form?

Example with fixed level: the graded \( \mathbb{C} \)-algebra \( M(\text{SL}_2(\mathbb{Z})) \)

\( \mathbb{Q} \)-module structure on the space of modular forms \( \hat{M}(\mathbb{Q}) \)

Setting

\( k \)-formula

Rational and integral structure of \( M(\text{SL}_2(\mathbb{Z})) \)

As inner product space

An application

Lemma

*The modular function \( j(\tau) \) takes all the complex value strictly once (i.e. \( j : \text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}^+ \cong \mathbb{C} \)).*

Proof.

- \( j \) function is a meromorphic function of level \( \text{SL}_2(\mathbb{Z}) \) and \( v_{\infty}(f) = -1 \).
- For any \( a \in \mathbb{C} \), \( f(\tau) = j(\tau) - a \) is still a modular function for \( \text{SL}_2(\mathbb{Z}) \).
- Applying the \( k \)-formula, we have

\[
-1 + \frac{1}{2} v_{\infty}(f) + \frac{1}{3} v_{\rho}(f) + \sum_{z \in \mathcal{D}\{i, \rho\}} v_z(f) = 0.
\]

Thus \( f \) has zero at \( i \) (if \( a = 0 \)) with multiplicity 2 or at \( \rho \) with multiplicity 3 (if \( a = 1 \)) or at \( \tau \in \mathcal{H}^+ \) with \( \tau \neq i, \rho \) with multiplicity 1.
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Example with fixed level: the graded \( C \)-algebra \( M(\text{SL}_2(\mathbb{Z})) \)
\( \Pi_0 \)-module structure on the space of modular forms \( M(\mathbb{Q}) \)

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Motivation

What is a modular form?

Example with fixed level: the graded $\mathbb{C}$-algebra $M(\text{SL}_2(\mathbb{Z}))$

$\mathbb{Q}$-module structure on the space of modular forms $M(\mathbb{Q})$

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**Motivation**

What is a modular form?

Example with fixed level: the graded \( \mathbb{C} \)-algebra \( M(\text{SL}_2(\mathbb{Z})) \) \( \mathbb{Q} \)-module structure on the space of modular forms \( \overline{M(\mathbb{Q})} \)

**Setting**

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As inner product space

**An application**

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*The modular function* \( j(\tau) \) *takes all the complex value strictly once (i.e. \( j : \text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}^+ \cong \mathbb{C} \)).*

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**Lemma**

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Motivation

What is a modular form?

Example with fixed level: the graded \( \mathbb{C} \)-algebra \( M(\text{SL}_2(\mathbb{Z})) \)

\( \mathbb{Q} \)-module structure on the space of modular forms \( M(\mathbb{Q}) \)

Setting

\( k \)-formula

Rational and integral structure of \( M(\text{SL}_2(\mathbb{Z})) \)

As inner product space

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1. **Multiplication** \( \cdot : M_k(\Gamma) \times M_j(\Gamma) \to M_{k+j}(\Gamma) \). Thus \( (M(\Gamma, A) = \bigoplus_k M_k(\Gamma, A), +, \ast) \) is an \( A \)-algebra.

2. Let \( \sigma_k(n) = \sum_{d|n, d \geq 1} d^k \), \( G_4 = 1 + \sum_{n \geq 1} \sigma_3(n)q^n \) and \( G_6 = \frac{-1}{504} + \sum_{n \geq 1} \sigma_3(n)q^n \).

---

**Theorem**

Remainder: The proof uses the discriminant form \( \Delta = \frac{1}{1728} ((240G_4)^3 - (506G_6)^2) \), which is a cusp form of weight 12. One can use the \( k \)-formula to prove that it has no zero on \( D \) and has no zero on \( \mathcal{H}^+ \). Then we can prove the multiplication by \( \Delta \) defines an isomorphism between \( M_{k-12}(\text{SL}_2(\mathbb{Z})) \) and \( S_k(\text{SL}_2(\mathbb{Z})) \).
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What is a modular form?

Example with fixed level: the graded $\mathbb{C}$-algebra $M(\text{SL}_2(\mathbb{Z}))$

$\mathbb{Q}$-module structure on the space of modular forms $M(\mathbb{Q})$

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Rational and integral structure of $M(\text{SL}_2(\mathbb{Z}))$

As inner product space

1. Multiplication $\cdot : M_k(\Gamma) \times M_j(\Gamma) \to M_{k+j}(\Gamma)$. Thus $(M(\Gamma, A) = \bigoplus_k M_k(\Gamma, A), +, *)$ is an $A$-algebra.

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Theorem

- We have an isomorphism of graded $\mathbb{Q}$-algebras
  
  $M(\text{SL}_2(\mathbb{Z}), \mathbb{Q}) \cong \mathbb{Q}[X, Y],$

  where $X = G_4$, $Y = G_6$.

- $M(\text{SL}_2(\mathbb{Z}), \mathbb{C}) = \mathbb{C} \otimes M(\text{SL}_2(\mathbb{Z}), \mathbb{Q})$.

Remark: The proof uses the discriminant form

$\Delta = \frac{1}{1728}((240G_4)^3 - (506G_6)^2)$, which is a cusp form of weight 12. One can use the $k_{\frac{12}{12}}$-formula to prove that it has no zero on $\mathbb{H}$ and has no zero on $\mathbb{H}^+$. Then we can prove the multiplication by $\Delta$ defines an isomorphism between $M_{k-12}(\text{SL}_2(\mathbb{Z}))$ and $S_k(\text{SL}_2(\mathbb{Z}))$. 

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Introduction to modular forms
Multiplication $\cdot : M_k(\Gamma) \times M_j(\Gamma) \to M_{k+j}(\Gamma)$. Thus $(M(\Gamma, A) = \bigoplus_k M_k(\Gamma, A), +, \ast)$ is an $A$-algebra.

Let $\sigma_k(n) = \sum_{d|n, d \geq 1} d^k$, $G_4 = 1 + \sum_{n \geq 1} \sigma_3(n) q^n$ and $G_6 = \frac{-1}{504} + \sum_{n \geq 1} \sigma_3(n) q^n$.

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Example with fixed level: the graded $\mathbb{C}$-algebra $M(\text{SL}_2(\mathbb{Z}))$

$\mathbb{Q}$-module structure on the space of modular forms $M(\mathbb{Q})$

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Rational and integral structure of $M(\text{SL}_2(\mathbb{Z}))$

As inner product space

\begin{enumerate}
  \item Multiplication $\cdot : M_k(\Gamma) \times M_j(\Gamma) \to M_{k+j}(\Gamma)$. Thus $(M(\Gamma, A) = \bigoplus_k M_k(\Gamma, A), +, \ast)$ is an $A$-algebra.
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### Multiplication

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\]

### Let

\[
\sigma_k(n) = \sum_{d|n, d \geq 1} d^k, \quad G_4 = 1 + \sum_{n \geq 1} \sigma_3(n)q^n \quad \text{and} \quad G_6 = \frac{-1}{504} + \sum_{n \geq 1} \sigma_3(n)q^n .
\]

### Theorem

1. We have an isomorphism of graded \( \mathbb{Q} \)-algebras

\[
M(\text{SL}_2(\mathbb{Z}), \mathbb{Q}) \cong \mathbb{Q}[X, Y],
\]

where \( X = G_4, Y = G_6 \).

2. \( M(\text{SL}_2(\mathbb{Z}), \mathbb{C}) = \mathbb{C} \otimes M(\text{SL}_2(\mathbb{Z}), \mathbb{Q}) \).

### Remark

The proof uses the discriminant form

\[
\Delta = \frac{1}{1728} ((240G_4)^3 - (506G_6)^2), \quad \text{which is a cusp form of weight } 12. \quad \text{One can use the } k_{12} \text{-formula to prove that it has no zero on } D \text{ and has no zero on } \mathcal{H}^+. \quad \text{Then we can prove the multiplication by } \Delta \text{ defines an isomorphism between } M_{k-12}(\text{SL}_2(\mathbb{Z})) \text{ and } S_k(\text{SL}_2(\mathbb{Z})).
\]
integral structure and \mod p modular forms

1. By the rational structure theorem and the explicit formula of $G_4$ and $G_6$, one can deduce that $M(\text{SL}_2(\mathbb{Z}), \mathbb{Q}) = M(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) \otimes \mathbb{Q}$.

2. Ramanujan’s congruence. Let the $q$-expansion of the weight 12 cusp form $\Delta$ be

$$
\sum_{n \geq 1} \tau(n) q^n = q \prod_{m \geq 1} (1 - q^m)^{24}.
$$

Then Ramanujan’s congruence says: $\tau(n) \equiv \sigma_{11}(n) \mod 691$.

- $\tau(1) = 1$, $\tau(2) = -24$, $\tau(3) = 252$, $\tau(4) = -1472$, $\tau(nm) = \tau(n)\tau(m) \forall (n, m) = 1$.

- (Ramanujan-Deligne) $|\tau(p)| < 2p^{1/2}$ for all prime $p$.

Develop the following identity:

$$(4 \cdot 504)_2 = (1 - q(1 + \sum_{n=1}^{\infty} \sigma_{11}(n) q^n) q_{252} \Delta) \in \mathbb{Z}[[q]].$$

Hence

$$1 - 1008 q + 65520 \alpha q^2 \equiv 1 + 65520 \sigma_{11}(n) q \mod q^2.$$

Thus

$$\alpha = -1008 = \frac{65520}{691} \in \mathbb{Q},$$

which implies $\alpha \equiv 65520 \mod 691$.

On the other hand,

$$\tau(3) = 252 \equiv 65520 \sigma_{11}(3) \mod 691.$$
By the rational structure theorem and the explicit formula of $G_4$ and $G_6$, one can deduce that $M(\operatorname{SL}_2(\mathbb{Z}), \mathbb{Q}) = M(\operatorname{SL}_2(\mathbb{Z}), \mathbb{Z}) \otimes \mathbb{Q}$.

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- $(504G_6)^2 = (1 - 504 \sum_{n \geq 1} \sigma_5(n)q^n)^2 \in \mathbb{Z}[[q]]$.
- Develop $(504G_6)^2$ in terms of the basis $\{\Delta, E_{12} = 1 + \frac{65520}{691} \sum_{n \geq 1} \sigma_{11}(n)q^n\}$ of the 2-dimensional vector space $M_{12}(\operatorname{SL}_2(\mathbb{Z}))$: $(504G_6)^2 = E_{12} + \alpha \Delta$. Thus

$$1 - 1008q = 1 + \frac{65520}{691} + \alpha q \pmod{q^2}.$$ 

- $\alpha = -1008 - \frac{65520}{691} = -\frac{a}{691} \in \mathbb{Q}$, which implies $a \equiv 65520 \pmod{691}$.
- On the other hand, $\frac{65520}{691} \sigma_{11}(n) + \frac{a}{691} \tau(n) \in \mathbb{Z}$.
By the rational structure theorem and the explicit formula of $G_4$ and $G_6$, one can deduce that $M(SL_2(\mathbb{Z}), \mathbb{Q}) = M(SL_2(\mathbb{Z}), \mathbb{Z}) \otimes \mathbb{Q}$.

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$$1 - 1008q \equiv 1 + \frac{65520}{691} + \alpha q \mod q^2.$$

- $\alpha = -1008 - \frac{65520}{691} = \frac{2}{691} \in \mathbb{Q}$, which implies $\alpha \equiv 65520 \mod 691$.
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Introduction to modular forms
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**Motivation**

Example with fixed level: the graded $\mathbb{C}$-algebra $M(\text{SL}_2(\mathbb{Z}))$

$\mathbb{Q}$-module structure on the space of modular forms $M(\mathbb{Q})$

**Setting**

$k$-formula

Rational and integral structure of $M(\text{SL}_2(\mathbb{Z}))$

As inner product space
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- \( (504G_6)^2 = (1 - 504 \sum_{n \geq 1} \sigma_5(n)q^n)^2 \in \mathbb{Z}[[q]]. \)

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Introduction to modular forms
The differential \( \frac{dx dy}{y^2} = \frac{i}{y^2} d\tau \wedge d\bar{\tau} \) is invariant under the action of \( SL_2(\mathbb{R}) \), and it defines a hyperbolic measure on \( D \).

Let \( f \in M_k(SL_2(\mathbb{Z})) \) and \( g \in S_k(SL_2(\mathbb{Z})) \). We define the Petersson inner product of \( f \) and \( g \) by the formula:

\[
\langle f, g \rangle = \int_{SL_2(\mathbb{Z}) \setminus \mathcal{H}^+} \overline{f(\tau)} g(\tau) y^k \frac{dx dy}{y^2}.
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Remark: In general, for arbitrary level \( \Gamma \), we define the normalized Petersson inner product by

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\langle f, g \rangle = \frac{1}{[SL_2(\mathbb{Z}) : \Gamma]} \int_{\Gamma \setminus \mathcal{H}^+} \overline{f(\tau)} g(\tau) y^k \frac{dx dy}{y^2},
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which is independent of choice of \( \Gamma \).
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Introduction to modular forms
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What is a modular form?

Example with fixed level: the graded \( \mathbb{C} \)-algebra \( M(\text{SL}_2(\mathbb{Z})) \)

\( \mathbb{Q} \)-module structure on the space of modular forms \( M(\overline{\mathbb{Q}}) \)

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Normality of Hecke operator

Lemma

- If \( f \in M_k(\Gamma(N)) \), \( g \in S_k(\Gamma(N)) \), \( \alpha \in M_2(\mathbb{Z}) \) with determinant \( m > 0 \).

  Then

  \[
  \langle f|_k \alpha, g \rangle = \langle f, g|_k \alpha' \rangle,
  \]

  where \( \alpha' = m \alpha^{-1} \).

- For \( \Gamma = \text{SL}_2(\mathbb{Z}) \), Hecke operators are normal operators.
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Introduction to modular forms
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   Then
   \[
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   where \( \alpha' = m\alpha^{-1} \).

2. For \( \Gamma = \text{SL}_2(\mathbb{Z}) \), Hecke operators are normal operators.
A finite index subgroup $\Gamma$ of $SL_2(\mathbb{Z})$ is called a congruence subgroup if it contains the principal congruence subgroup

$$\Gamma(N) = \{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \},$$

for some $N$.

- In general, for the congruence subgroup of level $N$, the Hecke operators $T_n$ with $(n, N)$ are normal and $U_p$ with $p \mid N$ usually is not normal.
- Note that there are plenty of non-congruence subgroup. Already in the late 19th century, Fricke and Klein showed the existence of non-congruence subgroup. Indeed, since the free group of rank 2 is the principal congruence subgroup $\Gamma(2)$, any 2-generated finite group is a quotient of this group. It turns out that the finite, simple groups which can occur as quotients by congruence subgroups are the groups $PSL_2(\mathbb{F}_p)$ for primes $p$. But there are many finite, simple 2-generated groups (for example $A_n (n > 5)$, $PSL_3(\mathbb{F}_q)$ for odd prime $q$) which are not isomorphic to $PSL_2(\mathbb{F}_p)$. So, the corresponding kernel cannot be a congruence subgroup of $SL_2(\mathbb{Z})$. 

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Introduction to modular forms
Definition

- A modular form is called eigenform if it is the eigenvector for all the Hecke operators.
- newform: an eigenform of level \( \Gamma \) is new if it doesn’t come from lower level.
- primitive form: normalized newform. (Atkin-Lehner theory says if a newform is eigenform, then it is uniquely (up to multiplication by a constant) determined by the eigenvalue for the Hecke operator not dividing the level).
Outline

1 Motivation
   - History
   - Two amazing examples
   - Plan of this min-course

2 What is a modular form?
   - Definition
   - q-expansion
   - Hecke operators
   - Examples: existence of modular form

3 Example with fixed level: the graded $\mathbb{C}$-algebra $M(\text{SL}_2(\mathbb{Z}))$
   - Setting
   - $k_{12}$-formula
   - Rational and integral structure of $M(\text{SL}_2(\mathbb{Z}))$
   - As inner product space

4 $\Pi'_Q$-module structure on the space of modular forms $M(\overline{\mathbb{Q}})$
   - Setting
   - The group $\Pi'_Q$
Modular forms for congruence subgroups

Let $K$ be a subfield of $\mathbb{C}$. Now we consider

- the space of all the modular forms with Fourier coefficients in $K$.
  \[ M(K) = \bigcup \Gamma M(\Gamma, K), \]
  where $\Gamma$ runs through all the finite order subgroup of $\text{SL}_2(\mathbb{Z})$.

- the space of all the modular forms for congruence subgroups with Fourier coefficients in $K$
  \[ M^\text{cong}(K) = \bigcup \Gamma M(\Gamma, K), \]
  where $\Gamma$ runs through all the congruence subgroup of $\text{SL}_2(\mathbb{Z})$.

- We denote $\Pi_K$ the group of automorphisms of $M(\bar{K})$ over $M(\text{SL}_2(\mathbb{Z}), K)$, which is a profinite group.
  
  If $K$ is algebraic closed, then $\Pi_K$ is the profinite completion of $\text{SL}_2(\mathbb{Z})$.
  In the general case, we have the exact sequence
  \[ 1 \rightarrow \Pi_Q = \Pi_K ightarrow \text{Gal}(\bar{K}/K) \rightarrow 1. \]
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  - If $K$ is algebraic closed, then $\Pi_K$ is the profinite completion of $SL_2(\mathbb{Z})$.
  - In the general case, we have an exact sequence

$$1 \to \Pi_{\bar{K}} \to \Pi_K \to \text{Gal}(\bar{K}/K) \to 1.$$
Let $K$ be a subfield of $\mathbb{C}$. Now we consider

- the space of all the modular forms with Fourier coefficients in $K$.
  
  $$M(K) = \bigcup_\Gamma M(\Gamma, K),$$

  where $\Gamma$ runs through all the finite order subgroup of $SL_2(\mathbb{Z})$.

- the space of all the modular forms for congruence subgroups with Fourier coefficients in $K$
  
  $$M^{\text{cong}}(K) = \bigcup_\Gamma M(\Gamma, K),$$

  where $\Gamma$ runs through all the congruence subgroup of $SL_2(\mathbb{Z})$.

- We denote $\Pi_K$ the group of automorphisms of $M(\bar{K})$ over $M(SL_2(\mathbb{Z}), K)$, which is a profinite group.
  
  - If $K$ is algebraic closed, then $\Pi_K$ is the profinite completion of $SL_2(\mathbb{Z})$.
  - In the general case, we have an exact sequence

    $$1 \to \Pi_{\bar{K}} \to \Pi_K \to \text{Gal}(\bar{K}/K) \to 1.$$
We denote by $\Pi'_Q$ the sub-group of $\text{Aut}(M(\overline{Q}))$ generated by $\Pi_Q$ and $\text{GL}_2(\mathbb{Q})_+$. 

The sub-algebra $M^{\text{cong}}$ is stable under the action of $\Pi$ and $\Pi'$ which act via $\text{GL}_2(\mathbb{Z})$ and $G(\mathbb{A}_f)$ respectively. Here, $\mathbb{A}_f$ is the finite adeles of $\mathbb{Q}$.

We have a commutative diagram of groups:

$$
\begin{array}{ccccccc}
1 & \rightarrow & \Pi & \rightarrow & \Pi_K & \rightarrow & G_K & \rightarrow & 1 \\
& \downarrow & & \downarrow \rho_{\text{cong}} & & \\
1 & \rightarrow & \text{SL}_2(\hat{\mathbb{Z}}) & \rightarrow & \text{GL}_2(\hat{\mathbb{Z}}) & \rightarrow & \hat{\mathbb{Z}}^* & \rightarrow & 1.
\end{array}
$$

The kernel $H$ of the natural map $\Pi'_Q \rightarrow \text{GL}_2(\mathbb{A}_f)$ is contained in $\Pi_Q$ since it fixes $M(\text{SL}_2(\hat{\mathbb{Z}}), \mathbb{Q})$. We have a commutative diagram of groups:

$$
\begin{array}{ccccccc}
1 & \rightarrow & H & \rightarrow & \Pi_Q & \overset{\rho_{\text{cong}}}{\rightarrow} & \text{GL}_2(\hat{\mathbb{Z}}) & \rightarrow & 1 \\
& \| & & \downarrow & & \\
1 & \rightarrow & H & \rightarrow & \Pi'_Q & \overset{\rho_{\text{cong}}}{\rightarrow} & \text{GL}_2(\mathbb{A}_f) & \rightarrow & 1.
\end{array}
$$
We denote by $\Pi_Q'$ the sub-group of $\text{Aut}(M(\overline{Q}))$ generated by $\Pi_Q$ and $\text{GL}_2(Q)_+$. 

The sub-algebra $M^{\text{cong}}$ is stable under the action of $\Pi$ and $\Pi'$ which act via $\text{GL}_2(\mathbb{Z})$ and $G(\mathbb{A}_f)$ respectively. Here, $\mathbb{A}_f$ is the finite adeles of $\mathbb{Q}$.

We have a commutative diagram of groups:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi & \longrightarrow & \Pi_K & \longrightarrow & G_K & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \rho_{\text{cong}} & & \\
1 & \longrightarrow & \text{SL}_2(\hat{\mathbb{Z}}) & \longrightarrow & \text{GL}_2(\hat{\mathbb{Z}}) & \longrightarrow & \hat{\mathbb{Z}}^* & \longrightarrow & 1.
\end{array}
$$

The kernel $H$ of the natural map $\Pi_Q' \rightarrow \text{GL}_2(\mathbb{A}_f)$ is contained in $\Pi_Q$ since it fixes $M(\text{SL}_2(\hat{\mathbb{Z}}), \mathbb{Q})$. We have a commutative diagram of groups:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & H & \longrightarrow & \Pi_Q & \longrightarrow & \text{GL}_2(\hat{\mathbb{Z}}) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \rho_{\text{cong}} & & \\
1 & \longrightarrow & H & \longrightarrow & \Pi_Q' & \longrightarrow & \text{GL}_2(\mathbb{A}_f) & \longrightarrow & 1.
\end{array}
$$
We denote by $\Pi'_Q$ the sub-group of $\text{Aut}(M(\bar{Q}))$ generated by $\Pi_Q$ and $\text{GL}_2(\mathbb{Q})_+$. 

The sub-algebra $M^{\text{cong}}$ is stable under the action of $\Pi$ and $\Pi'$ which act via $\text{GL}_2(\mathbb{Z})$ and $G(\mathbb{A}_f)$ respectively. Here, $\mathbb{A}_f$ is the finite adeles of $\mathbb{Q}$.

We have a commutative diagram of groups:

$$
\begin{CD}
1 @>>> \Pi @>>> \Pi_K @>>> G_K @>>> 1 \\
@. @VVV @VV\rho_{\text{cong}}V @. \\
1 @>>> \text{SL}_2(\hat{\mathbb{Z}}) @>>> \text{GL}_2(\hat{\mathbb{Z}}) @>>> \hat{\mathbb{Z}}^* @>>> 1.
\end{CD}
$$

The kernel $H$ of the natural map $\Pi'_Q \rightarrow \text{GL}_2(\mathbb{A}_f)$ is contained in $\Pi_Q$ since it fixes $M(\text{SL}_2(\hat{\mathbb{Z}}), \mathbb{Q})$. We have a commutative diagram of groups

$$
\begin{CD}
1 @>>> H @>>> \Pi_Q @>>> \text{GL}_2(\hat{\mathbb{Z}}) @>>> 1 \\
@. @. @VV\rho_{\text{cong}}V @. \\
1 @>>> H @>>> \Pi'_Q @>>> \text{GL}_2(\mathbb{A}_f) @>>> 1.
\end{CD}
$$
We denote by $\Pi'_Q$ the sub-group of $\text{Aut}(M(\bar{Q}))$ generated by $\Pi_Q$ and $\text{GL}_2(\mathbb{Q})_+$. The sub-algebra $M^{\text{cong}}$ is stable under the action of $\Pi$ and $\Pi'$ which act via $\text{GL}_2(\mathbb{Z})$ and $G(\mathbb{A}_f)$ respectively. Here, $\mathbb{A}_f$ is the finite adeles of $\mathbb{Q}$.

We have a commutative diagram of groups:

$$
\begin{array}{cccccc}
1 & \longrightarrow & \Pi & \longrightarrow & \Pi_K & \longrightarrow & G_K & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \rho_{\text{cong}} & & \\
1 & \longrightarrow & \text{SL}_2(\hat{\mathbb{Z}}) & \longrightarrow & \text{GL}_2(\hat{\mathbb{Z}}) & \overset{\text{det}}{\longrightarrow} & \hat{\mathbb{Z}}^* & \longrightarrow & 1.
\end{array}
$$

The kernel $H$ of the natural map $\Pi'_Q \rightarrow \text{GL}_2(\mathbb{A}_f)$ is contained in $\Pi_Q$ since it fixes $M(\text{SL}_2(\hat{\mathbb{Z}}), \mathbb{Q})$. We have a commutative diagram of groups:

$$
\begin{array}{cccccc}
1 & \longrightarrow & H & \longrightarrow & \Pi_Q & \overset{\rho_{\text{cong}}}{\longrightarrow} & \text{GL}_2(\hat{\mathbb{Z}}) & \longrightarrow & 1 \\
\vert & & \vert & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H & \longrightarrow & \Pi'_Q & \overset{\rho_{\text{cong}}}{\longrightarrow} & \text{GL}_2(\mathbb{A}_f) & \longrightarrow & 1.
\end{array}
$$
Part II

Geometric modular forms
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Deligne-Rapoport\textsuperscript{2} : For $N \geq 3$, the functor

$$S \mapsto \text{the set of isomorphism classes of triples } (E, e_1, e_2)$$

where $E$ is an elliptic curve over $S$ and $(e_1, e_2)$ is a pair of sections of $E$ over $S$ which forms a $\mathbb{Z}/N\mathbb{Z}$-basis of $\text{Ker}(N : E \to E)$, is represented by a smooth connected irreducible affine curve $Y(N)$ over $\mathbb{Q}$, called the open modular curve over $\mathbb{Q}$ of level $N$.

2. $Y(N)$ is not geometrically connected and the constant field of $\mathcal{O}(Y(N))$ is $\mathbb{Q}(\mu_N)$.

3. There is a bijection between the set of connected components of $Y(N)_{\mathbb{Q}(\mu_N)}$ and the set $\mu_N^*$ of the primitive $N$-th roots of unity.

4. $X(K)^*$: the compactified curve $X(K)$ of $Y(K)$ equipped with the logarithmic structure at the points.

\textsuperscript{2}Les schemas de courbes elliptiques, Lecture Notes in Math. 349 (1973), SpringerVerlag, p.143-316
Modular curves
Local systems on modular curves
Geometric modular forms
Relation with classical modular forms

Complex uniformization

Let $S = \mathbb{C}$.

1. $Y(N)(\mathbb{C})$ is not connected and we have a complex analytic isomorphism:
$$\Gamma(N) \backslash \mathcal{H}^+ \times (\mathbb{Z}/N\mathbb{Z})^* \rightarrow Y(N)(\mathbb{C}); (\tau, a) \mapsto (\mathbb{C}/\Lambda_{\tau}, a\tau/N, 1/N).$$

2. The inverse map is given by the Weil pairing:
$$(\mathbb{C}/\Lambda_{\tau}, a\tau/N, 1/N) \mapsto (\tau, \langle a\tau/N, 1/N \rangle).$$

3. $X(K)(\mathbb{C}) = \Gamma(N) \backslash (\mathcal{H}^+ \cup \mathbb{P}^1(\mathbb{Q})) \times (\mathbb{Z}/N\mathbb{Z})^*.$

4. Diagram:

\[\begin{array}{cccc}
\mathcal{H}^+ & \times & \mathbb{C} & \rightarrow (E^{univ}(\mathbb{C}), \text{level structure}) \\
& \downarrow & & \downarrow \\
\mathcal{H} & \rightarrow & Y(N)(\mathbb{C}) \\
& \downarrow & & \downarrow \\
\tau & \rightarrow & (\mathbb{C}/\Lambda_{\tau}, \frac{\tau}{N}, \frac{1}{N}) \\
\end{array}\]

where $\mathcal{P}$ is the Weierstrass $\mathcal{P}$-function.

WANG Shanwen
Introduction to modular forms
The Group $G(\mathbb{R})$ and Poincaré upper half plane

1. Set $A_f = \mathbb{Q} \otimes \mathbb{Z}$ and $A = \mathbb{R} \times A_f$ the ring of finite adèles of $\mathbb{Q}$ and of adèles of $\mathbb{Q}$ respectively.

2. Let $G$ be the algebraic group $GL_2$ defined over $\mathbb{Q}$. $G(A) = G(\mathbb{R}) \times G(A_f)$.

- the group $^3G(\mathbb{R})$:
  - The map $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \mapsto (\begin{smallmatrix} ai+b \\ ci+d \end{smallmatrix}) = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})(\begin{smallmatrix} i \\ 1 \end{smallmatrix})$ sends $G(\mathbb{R}) \hookrightarrow \mathbb{C}^2$.
  - The map $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \mapsto a + ib$ gives an identification $\mathbb{C}^* = \{(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}) \in G(\mathbb{R}) : a + ib \neq 0\}$.
  - The image of $G(\mathbb{R})$ under the map $G(\mathbb{R}) \hookrightarrow \mathbb{C}^2 \twoheadrightarrow \mathbb{C}^2/\mathbb{C}^* = \mathbb{P}^1(\mathbb{C})$, where the map is given by $(z_1/z_2) \mapsto z_1/z_2$, is $\mathcal{H} := \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) = \mathcal{H}^+ \cup \mathcal{H}^-$.
  - The map $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \mapsto \frac{a+ib}{c+id}$ induces an isomorphism $G(\mathbb{R})/\mathbb{C}^* \cong \mathcal{H} := \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) = \mathcal{H}^+ \cup \mathcal{H}^-$.  
  - The normalizer $\mathbb{C}^* \cup \mathbb{C}^* (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})$ of $\mathbb{C}^*$ acts on the right (i.e. right multiplication on $GL_2(\mathbb{R})$) on $\mathcal{H}$, and $\mathbb{C}^* (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})$ sends $\tau \in \mathcal{H}$ to $\bar{\tau} \in \mathcal{H}$. But be careful to distinguish this action and the linear fractional action of $GL_2(\mathbb{Q})$ on $\mathcal{H}$ which is the multiplication on the left.

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$^3$This part can be found in Deligne’s paper: Formes modulaires et representation de $GL(2)$, Modular Functions of One Variable, Lect. Notes in Math 349 (1973), 55-106.
Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. The $\mathbb{C}$-points of the level $K$ modular curve $Y(K)$ is:

$$Y(K)(\mathbb{C}) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / \mathbb{C}^* K = G(\mathbb{Q}) \backslash \mathcal{H} \times G(\mathbb{A}_f) / K,$$

which is a non-compact Riemann surface whose set of connected component is $\hat{\mathbb{Z}}^* / \det K$. The last map is given by

$$G(\mathbb{R}) / \mathbb{C}^* \to \mathcal{H}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{ai + b}{ci + d}$$

with the inverse $\tau = x + yi \mapsto \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$.

1. $X(K)(\mathbb{C})$ the smooth compactification of $Y(K)(\mathbb{C})$ by adding the points to $Y(K)(\mathbb{C})$.
2. $X(K)(\mathbb{C})^*$: the curve $X(K)(\mathbb{C})$ equipped with the logarithmic structure at the points.
3. $X(K)(\mathbb{C})$ is an algebraic curve which has a connected model $X(K)/\mathbb{Q}$ defined over $\mathbb{Q}$ (not geometrically connected).
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Let $W_{k,j}^*$ be the $\mathbb{Q}$-representation $\text{Sym}^k \otimes \det^{-j}$ of $G(\mathbb{Q})$, where $\text{Sym}^k$ is the $k$-th symmetric power of the standard representation $\text{Sym}^1 = \mathbb{Q}e_1^* \oplus \mathbb{Q}e_2^*$ defined by the formulae:

$$(\begin{array}{cc} a & b \\ c & d \end{array}) \ast e_1^* = ae_1^* + ce_2^*, (\begin{array}{cc} a & b \\ c & d \end{array}) \ast e_2^* = be_1^* + de_2^*.$$ 

$$(\begin{array}{cc} a & b \\ c & d \end{array}) \ast (e_1^* \wedge e_2^*) = (ad - bc)(e_1^* \wedge e_2^*).$$

Let $e_1, e_2$ be the dual basis of $(\text{Sym}^1)^*$ of $e_1^*, e_2^*$ and $W_{k,j} = (W_{k,j}^*)^*$. A simple computation gives

$$(\begin{array}{cc} a & b \\ c & d \end{array}) \ast \frac{(e_1 - \tau e_2)^k}{(e_1 \wedge e_2)^j} = \frac{(ad - bc)^j}{(-c\tau + a)^k} \frac{(e_1 - \tau e_2)^k}{(e_1 \wedge e_2)^j}.$$
Definition

Let $X$ be a topological space. There are three things which one might mean by a local system:

1. (Betti definition) a vector bundle $\pi : M \to X$ with parallel transport, i.e. for each homotopy class of paths in $X$, a vector space isomorphism between fibers, respecting composition.

2. (deRham definition) if $X$ is a differentiable manifold, a vector bundle $\pi : M \to X$ with a flat connection (i.e. a connection $\nabla : M \to M \otimes \Omega^1_X$ such that $\nabla \circ \nabla = 0$).

3. (sheaf definition) a locally constant sheaf of vector spaces on $X$. 
Fix a prime $p$. Set $F^B = \mathbb{Q}$, $F^{dR} = \mathcal{O}$ and $F^{\text{ét}} = \mathbb{Q}_p$, where $B$, $dR$, $\text{ét}$ stand for Betti, de Rham and étale respectively.

The cohomology $\mathcal{H}^1_{\text{truc}}$, with $\text{truc} \in \{dR, B, \text{ét}\}$, of the universal elliptic curve over $Y(K)$ gives $F^\text{truc}$-local systems on $Y(K)$.

For $k \in \mathbb{N}, j \in \mathbb{Z}$, we set $W^\text{truc}_{k,j} = \text{Sym}^k \mathcal{H}^1_{\text{truc}} \otimes (\mathcal{H}^2_{\text{truc}})^{-j}$.

$W^\text{truc}_{\text{tot}} = \bigoplus_{k,j} W^\text{truc}_{k,j} \otimes_{\mathbb{Q}} W^*_k,j$.

Comparaison of $F^\text{truc}$-local systems: if $\text{machin} \in \{\text{tot}, (k,j)\}$, then we have

$W^\text{ét}_{\text{machin}} = \mathbb{Q}_p \otimes W^B_{\text{machin}}, W^{dR}_{\text{machin}} = \mathcal{O} \otimes W^B_{\text{machin}}$. 
The rank 2 vector bundle $\mathcal{H}^1_{dR}$ has a decreasing filtration:

$$\text{Fil}^i \mathcal{H}^1_{dR} = \begin{cases} 
\mathcal{H}^1_{dR}, & \text{if } i \leq 0, \\
\omega, & \text{if } i = 1, \\
0, & \text{if } i \geq 2,
\end{cases}$$

where $\omega = \pi_* \Omega^1_{E/Y}$ is the rank 1 sheaf of holomorphic differential forms on the universal elliptic curve.

Since $\mathcal{H}^2_{dR} = \wedge^2 \mathcal{H}^1_{dR}$, the vector bundle $\mathcal{H}^2_{dR}$ is equipped with the filtration

$$\text{Fil}^i \mathcal{H}^2_{dR} = \begin{cases} 
\mathcal{H}^2_{dR}, & \text{if } i \leq 1, \\
0, & \text{if } i \geq 2.
\end{cases}$$

The vector bundle $W_{k,j}^{dR}$ is equipped with the filtration obtained by the tensor product:

$$\text{Fil}^{k-j-r} W_{k,j}^{dR} = \text{Fil}^{k-r} W_{k,0}^{dR} \text{ and } \text{Fil}^{k-r} W_{k,0}^{dR} = \omega^{k-r} \otimes \text{Sym}^r \mathcal{H}^1_{dR}.$$
Denote $\pi : E \to Y$. Let $\mathcal{H}^1_B = R^1\pi_*\mathbb{Z}$ the $\mathbb{Q}$-local system of rank 2 on the modular curve $Y$ and $\mathcal{H}^1_B = \text{Hom}(\mathcal{H}^1, \mathbb{Z})$.

$\mathcal{H}^2_B = \wedge \mathcal{H}^1_B$.

Let $\nu : \mathcal{H}^+ \to Y$ be the complex uniformization.

We denote by $\mathcal{H}^{1,\text{an}}_B = \mathcal{H}^1_B \otimes \mathbb{C}$ the analytification of $\mathcal{H}^1_B$.

Identify $\mathcal{H}^{1,\text{an}}_B$ with $(\text{Sym}^1)^*$ via $\int_{e_1^*} dq/z = 2i\pi$ and $\int_{e_2^*} dq/z = -2i\pi\tau$.

This induces an identification of $\mathcal{H}^1_B$ and Sym$^1$.

We identify $W_{k,j}^*$ with $W_{k,j}^* = (\text{Sym}^k \mathcal{H}^1) \otimes \mathcal{H}^2_B$ via the identification of Sym$^1$ and $\mathcal{H}^1_B$.

By the comparison, we have $\nu^* \mathcal{H}^1_{dR} \cong \mathcal{H}^{1,\text{an}}_B \otimes O(\mathcal{H}^+)$.

$$\frac{dq/z}{q/z} = 2i\pi(e_1 - e_2\tau).$$
• Convention: the Hodge-Tate weight of the Tate module of an Elliptic curve is 0, 1.

• $H^1_{dR}$ is the dual of the Tate module and the action of $GL_2(\mathbb{Q})$ on $H^2_{dR}$ is via $\det^{-1}$.

• $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ast (d\tau \otimes (H^2_{dR})^j) = \frac{(ad-bc)^{1-j}}{(-c\tau+a)^2} (d\tau \otimes (H^2_{dR})^j)$. As a consequence, $d\tau$ transfer under the $\ast$ action of $G(\mathbb{Q})$ as $\frac{(e_1-\tau e_2)^2}{(e_1 \wedge e_2)}$. 
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Holomorphic modular forms

1. Let $H^0(\omega^k) := \lim_{\rightarrow} K H^0(X(K)^*, \omega^k) = \lim_{N \geq 1} H^0(X(N)^*, \omega^k)$, where $K \subset G(\mathbb{A}_f)$ compact open subgroup. This is an infinite dimensional $\mathbb{Q}$-vector space equipped with an action of $G(\hat{\mathbb{Z}})$.

2. K-S iso. : $\omega \otimes 2 \cong \Omega^1_{X(N)/\mathbb{Q}}(\log(\text{cusps})).$

3. By GAGA, we have $H^0(\omega^k) \otimes _\mathbb{Q} \mathbb{C} = \lim_{\rightarrow} H^0(X(N)^*, \text{an}(\mathbb{C}), \omega_{\text{an}, k})$.

4. Using the complex uniformization, for any $F \in H^0(X(N)^*, \omega^k) \otimes _\mathbb{Q} \mathbb{C}$, the pull-back of $F$ to $\mathcal{H}^+ \times \{1\}$ can be written in the form

$$F = f(\tau) \otimes \left( \frac{dq_z}{q_z} \right)^k = f(\tau)(2i\pi dz)^k,$$

where $f(\tau)$ is holomorphic on $\mathcal{H}^+$ satisfying $f\left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau)$ for all $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(N)$.

Remark: This analytic description only describes the form on one connected component. To give the full information, we use the adelic description of the complex modular curves.
Nearly holomorphic modular forms: classical definition

Definition

A complex valued $C^\infty$-function $f$ on $\mathcal{H}^+$ is a nearly holomorphic modular form of weight $(k, j)$ and of depth $r$ for a congruence group $\Gamma \subset SL_2(\mathbb{Z})$ if $f$ satisfies the following properties:

- $f|_{(k, j)} \gamma = f, \forall \gamma \in \Gamma$,
- there are holomorphic functions $f_0, \cdots, f_r$ on $\mathcal{H}^+$ such that $f(\tau) = \sum_{i=0}^{r} \frac{1}{y^i} f_i(\tau)$.
- $f$ has a finite limit at the cusps.

We denote the space of nearly holomorphic modular forms of weight $(k, j)$ depth $r$ for $\Gamma$ by $\mathcal{N}_{k,j}^r(\Gamma, \mathbb{C})$.

Shimura’ $\epsilon$ operator $\epsilon : \mathcal{N}_{k,j}^r(\Gamma, \mathbb{C}) \to \mathcal{N}_{k-2,j}^{r-1}(\Gamma, \mathbb{C})$ defined by the formula $(\epsilon f)(\tau) = 8i\pi y^2 \frac{\partial f}{\partial \bar{\tau}}(\tau)$.

Maass-Shimura operator $\delta_k : \bigoplus_r \mathcal{N}_{k,j}^r(\Gamma, \mathbb{C}) \to \bigoplus_r \mathcal{N}_{k+2,j}^{r+1}(\Gamma, \mathbb{C})$ is defined by

$$\delta_k(f) = \frac{1}{2i\pi} y^{-k} \frac{\partial}{\partial \tau} (y^k f) = \frac{1}{2i\pi} \left( \frac{\partial f}{\partial \tau} + \frac{k}{2iy} f \right).$$

Denote by $\delta_k^s = \delta_{k+2s-2} \circ \cdots \circ \delta_k$. 

Lemma

Let \( f \in \mathcal{N}_{k,j}^r(\Gamma, \mathbb{C}) \). Assume that \( k > 2r \). Then there exist \( g_0, \cdots, g_r \in M_{k-2i}(\Gamma, \mathbb{C}) \) such that

\[
f = g_0 + \delta_{k-2} g_1 + \cdots + \delta_{k-2r} g_r.
\]

- Holomorphic projection (due to Shimura) \( H(f) := g_0 \).
- Lemma is wrong if \( k > 2r \) is not satisfied. For example, if \( k = 2 = 1 \), then the weight 2 Eisenstein serie of the full level is of level 1 and it can’t be decomposed in the form in the lemma.
- The Hodge filtration on $\mathcal{H}^1_{dR}$ induces an exact sequence:

$$0 \to \omega \to \mathcal{H}^1_{dR} \to \omega^\vee \to 0.$$ 

And in the $C^\infty$-topos, it splits (i.e $\mathcal{H}^1_{dR} = \omega \oplus \omega^\vee$).

- Using the complex uniformization, the pull-back of $\mathcal{H}^1_{dR}$ to $\mathcal{H}^+$ with $C^\infty$-topos is

$$\mathcal{H}^1_{dR} \otimes C^\infty(\mathcal{H}^+) = C^\infty(\mathcal{H}^+)dz \oplus C^\infty(\mathcal{H}^+)d\bar{z}.$$ 

- $M_{k,0,r}(\Gamma) = H^0(X(\Gamma)^\times, \omega^{k-r} \otimes \text{Sym}^r_{dR}) = H^0(X(\Gamma)^\times, \text{Fil}^{k-r}\text{Sym}^k_{dR}).$
Lemma

The Hodge decomposition induces a canonical isomorphism

\[ M^{nh}_{k,0,r}(\Gamma) \otimes \mathbb{C} \cong \mathcal{M}_k^r(\Gamma, \mathbb{C}). \]

Proof.

1. \( \{(dz)^{(k-l)}e_1^l\}_{0 \leq l \leq r} \) is a basis of \( \text{Fil}^{k-r}\text{Sym}^k \).
2. Let \( \eta \in M^{nh}_{k,0,r}(\Gamma) \otimes \mathbb{C} \). Then \( \nu^* \eta = \sum_{l=0}^{r} f_l(\tau)(dz)^{k-l}e_1^l \) with \( f_l(\tau) \in \mathcal{O}(\mathcal{H}^+) \).
3. Since \( e_1 = \frac{1}{2iy}(dz - d\bar{z}) \), we get

\[
\nu^* \eta(\tau) = \sum_{l=0}^{r} \frac{f_l(\tau)}{(2iy)^l} \sum_{i=0}^{l} (-1)^i \binom{l}{i} (dz)^{k-i} (d\bar{z})^i.
\]

4. The projection of \( \nu^* \eta(\tau) \) to the \( (k,0) \)-component gives \( f(\tau) = \sum_{l=0}^{r} \frac{f_l(\tau)}{(2iy)^l} \).

5. Use the fact that the projection to \( (k,0) \)-component is injective.
Via Poincaré duality, the quotient by the first step of the de Rham filtration of $\text{Fil}^{k-r} \mathcal{W}^k_{dR}$ induces

$$0 \to \omega^k \to \text{Fil}^{k-r} \mathcal{W}^k_{dR} \to \text{Fil}^{k-r-1} \mathcal{W}^{k-2}_{dR} \to 0.$$ 

$\forall k \geq 0$, the Gauss-Manin connection $\nabla : \text{Sym}^k_{dR} \to \text{Sym}^k_{dR} \otimes \Omega^1$ satisfies the Griffiths transversality (i.e. $\nabla(\text{Fil}^i \text{Sym}^k_{dR}) \subset \text{Fil}^{i-1} \text{Sym}^k_{dR}$).

Using the Kodaira-Spencer isomorphism $\kappa : \Omega^1 \cong \omega^2$, we can view $\nabla$ as a connection: $\text{Sym}^k_{dR} \to \text{Sym}^k_{dR} \otimes \omega^2$.

**Lemma**

*Passing to the global section, we get a decomposition:*

$$M^{nh}_{k+2,k}(\Gamma) = \nabla \cdot M^{nh}_{k,k}(\Gamma) \oplus M_{k+2}(\Gamma).$$
Let $\mathcal{A}$ be the ring of functions $\phi : G(\mathbb{A}) \rightarrow \mathbb{C}$ satisfying the following properties:

- $\phi$ factors through $\mathcal{H} \times G(\mathbb{A}_f)$,
- if $g_f \in G(\mathbb{A}_f)$, then the function $\tau \mapsto \phi(\tau, g_f)$ is slowly increasing harmonic function,
- there exists a open subgroup $K_\phi \subset G(\mathbb{A}_f)$, such that, for all $\kappa \in K_\phi$, we have $\phi(g_\infty, g_f \kappa) = \phi(g_\infty, g_f)$.

2 Let $\mathcal{A}^+(\text{resp. } \mathcal{A}^-)$ be the subspace of $\mathcal{A}$ consisting of the holomorphic (resp. antiholomorphic) functions at $\tau \in \mathbb{H}$. Then we have a short exact sequence of $\mathbb{C}$-vector spaces:

$$0 \rightarrow \text{LC}(G(\mathbb{A}), \mathbb{C}) \rightarrow \mathcal{A}^+ \oplus \mathcal{A}^- \rightarrow \mathcal{A} \rightarrow 0.$$

3 Let $\mathcal{A}_c$ be the ideal of $\mathcal{A}$ consisting of the rapidly decreasing functions at $\infty$. Let $\mathcal{A}_c^+ = \mathcal{A}_c \cap \mathcal{A}^+$ and $\mathcal{A}_c^- = \mathcal{A}_c \cap \mathcal{A}^-$. Then we have $\mathcal{A}_c = \mathcal{A}_c^+ \oplus \mathcal{A}_c^-$. 
Actions of $G(\mathbb{Q})$ and $G(\mathbb{A}_f)$ on $\mathcal{A}$ and $\mathcal{A}_c$:

$$(\gamma * \phi)(x) = \phi(\gamma^{-1}x), \quad g * \phi = \phi(xg),$$

for $\gamma \in G(\mathbb{Q})$ and $g \in G(\mathbb{A}_f)$. These two actions commute with each other.

1. The action of $G(\mathbb{A}_f)$ is smooth.
2. We can extend the action of $G(\mathbb{A}_f)$ to an action of $G(\mathbb{A})$ commuting with $G(\mathbb{Q})$ by setting

$$(g_\infty * \phi)(\tau, x_f) = \begin{cases} \phi(\tau, x_f), & \text{if } \text{sign}(g_\infty) = 1 \\ \phi(\bar{\tau}, x_f), & \text{if } \text{sign}(g_\infty) = -1 \end{cases}.$$
Recall: Let $W^*_{k,j}$ be the $\mathbb{Q}$-representation $\text{Sym}^k \otimes \text{det}^{-j}$ of $G(\mathbb{Q})$, where $\text{Sym}^k$ is the $k$-th symmetric power of the standard representation $\text{Sym}^1 = \mathbb{Q}e_1^* \oplus \mathbb{Q}e_2^*$ defined by the formulae:

$$(\begin{array}{cc} a & b \\ c & d \end{array}) \ast e_1^* = ae_1^* + ce_2^*, \quad (\begin{array}{cc} a & b \\ c & d \end{array}) \ast e_2^* = be_1^* + de_2^*.$$  

$$(\begin{array}{cc} a & b \\ c & d \end{array}) \ast (e_1^* \wedge e_2^*) = (ad - bc)(e_1^* \wedge e_2^*).$$

Let $e_1, e_2$ be the dual basis of $(\text{Sym}^1)^*$ of $e_1^*, e_2^*$ and $W_{k,j} = (W^*_{k,j})^*$. $d\tau$ transfer under the $\ast$ action of $G(\mathbb{Q})$ as $\frac{(e_1 - \tau e_2)^2}{(e_1 \wedge e_2)}$.

The $\mathcal{A}$-module $\mathcal{A} \otimes W_{k,j}$ is equipped with a diagonal action of $G(\mathbb{Q})$ and an action of $G(\mathbb{A}_{\mathbb{Q}})$ commuting with the action of $G(\mathbb{Q})$. 

If $r \leq k$, the sub-$\mathcal{A}^+$-module $\mathcal{A}^+ \otimes W_{k,j}$ is equipped with a $G(\mathbb{Q}) \times G(\mathbb{A}_f)$-stable decreasing filtration:

$$\text{Fil}^{k-r-j}(\mathcal{A}^+ \otimes W_{k,j}) = \bigoplus_{l=0}^{r} (\mathcal{A}^+ \otimes \frac{(\tau e_2 - e_1)^{k-l} e_1}{(e_1 \wedge e_2)^j}).$$

For $k \in \mathbb{N}, r \leq k, j \in \mathbb{Z}$, we define $\mathbb{C}$-vector space of the nearly holomorphic adelic modular form of weight $(k, j)$ and depth $r$ by

$$M_{k,j,r}^{nh}(\mathbb{C}) = H^0(G(\mathbb{Q}), \text{Fil}^{k-r}(\mathcal{A}^+ \otimes W_{k,j})).$$

For $k \in \mathbb{N}, j \in \mathbb{Z}$, let $M_{k,j}(\mathbb{C}) = M_{k,j,0}^{nh}(\mathbb{C})$ and $M_{k,j}^{nh}(\mathbb{C}) = M_{k,j,k}^{nh}(\mathbb{C})$. Both of them are equipped with a smooth action of $G(\mathbb{A}_f)$. 
Outline

5 Modular curves
- Modular curves as moduli spaces of elliptic curves
- Complex uniformization
- Complete modular curves: adelic description

6 Local systems on modular curves
- Generalities
- The filtration on de Rham cohomology
- Betti local system
- Complements on the weight

7 Geometric modular forms
- Definition and analytic description
- Space of adelic functions
- Nearly holomorphic adelic modular forms

8 Relation with classical modular forms
- Central character and weight
- Rational structure of space of adelic modular forms
- Cuspidal forms: $q$-expansion and Kirillov model
Let $w: \mathbb{A}^* \to \mathbb{C}^*$ be a continuous character.

1. If $v$ is a place of $\mathbb{Q}$, we denote by $w_v$ be the restriction of $w$ to $\mathbb{Q}_v^*$.
2. There exists an integer $N \geq 1$ and a character $\tilde{w}: (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ such that $w|_{\hat{\mathbb{Z}}^*}$ is the composition of $\tilde{w}$ and the projection $\hat{\mathbb{Z}}^* \to (\mathbb{Z}/N\mathbb{Z})^*$.
3. If $w$ factors through $\mathbb{A}^*/\mathbb{Q}^*$ and $p \nmid N$, then $w_p(p) = \tilde{w}^{-1}(p)w_\infty(p)^{-1}$.

Examples of multiplicative characters:

1. The character of norm $|\cdot|_\mathbb{A}: |x|_\mathbb{A} = \prod_v |x_v|_v$. By product formula, it factors through $\mathbb{A}^*/\mathbb{Q}^*$.
2. The character $\delta : x \mapsto \delta(x) = x_\infty^{-1}|x|_\mathbb{A}$.
3. Algebraic character: a continuous character $\chi : \mathbb{A}^*/\mathbb{Q}^*$ is called algebraic of weight $k \in \mathbb{Z}$, if $\chi(x_\infty) = x_\infty^k, \forall x_\infty \in \mathbb{R}^*$. 

Let $w_1, w_2 : \mathbb{A}^* \to \mathbb{C}^*$ be two continuous characters. For any $x_\infty \in \mathbb{R}^*$, we have $w_1(x_\infty)w_2(x_\infty) = w_1(x_\infty)w_2(x_\infty)$.
Lemma

Let $0 \neq \phi \in \mathcal{M}_{k,j}(\mathbb{C})$.

- If \( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \circ \phi = w(a)\phi, \forall a \in \mathbb{A}_f \), then \( w \) is the restriction of a character of \( \mathbb{A}^*/\mathbb{Q}^* \to \mathbb{C}^* \) to \( \mathbb{A}_f^* \), whose restriction to \( \mathbb{R}^* \) is \( x_\infty \mapsto x_\infty^{k-2j} \). Thus, it is algebraic of weight \( k-2j \).

- If \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \phi = w(d)\phi, \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \hat{\Gamma}_0(N) \), and if \( \phi(\tau, 1) = f(\tau) \frac{(\tau e_2 - e_1)^k}{(e_1 \wedge e_2)^j} \), then \( f \in \mathcal{M}_k(N, \tilde{w}^{-1}) \).

- We have a natural map:

\[ \mathcal{M}_{k,j}(\mathbb{C}) \to \mathcal{M}_k^{\text{class}}(\mathbb{C}); \phi \otimes \frac{(\tau e_2 - e_1)^k}{(e_1 \wedge e_2)^j} \mapsto f_\phi = \phi(\tau, 1). \]

The Koraida-Spencer iso. \( \omega^2 \cong \Omega^1(\log - \text{cusp}) \) is not \( G(\mathbb{Q}) \)-equivariant. The \( G(\mathbb{Q}) \)-equivariant one is \( \omega^2 \cong \Omega^1(\log - \text{cusp}) \otimes \mathbb{H}_{dR}^2 \).
The map $M_{k,j}(\mathbb{C}) \to M_{k}^{\text{class}}(\mathbb{C})$; $\phi \mapsto f_{\phi}$ defined by

$$\phi = \phi_0 \otimes \left( \frac{\tau e_2 - e_1}{e_1 \wedge e_2} \right)^k$$

and $f_{\phi}(\tau) = \phi_0(\tau, 1_f)$.

gives that, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Q})_+$, we have

$$\gamma_f \star \phi \mapsto f_{\gamma_f \star \phi} = \frac{(ad - bc)^{-j}}{(-c_\tau + a)^k} f_{\phi} \left( \frac{d_\tau - b}{-c_\tau + a} \right).$$

From the above formula, we get a left action of weight $(k, j)$ of $\gamma \in G(\mathbb{Q})_+$ which relates to the classical formula by

$$\gamma \ast_{(k,j)} f = f | _{k,j} \gamma^{-1}.$$

This map is surjective and for any $j \in \mathbb{Z}$ and $f \in M_{k}^{\text{class}}(\mathbb{C})$, there exists a unique $\phi \in M_{k,j}(\mathbb{C})$ such that $\phi(\tau, (u \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}))$ for all $u \in \hat{\mathbb{Z}}^*$. Moreover, if $\chi : (\mathbb{Z}/N)^* \to \mathbb{C}^*$ is a Dirichlet character, and if $f | _{k,j}(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) = \chi(d)f$, for all $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \in \Gamma_0(N)$, then $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \star \phi_{f,j} = \chi(d)^{-1} \phi_{f,j}$, for $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \in \hat{\Gamma}_0(N)$. 

WANG Shanwen
1. If we consider all the connected components, we have a map $M_{k,j}(\mathbb{C}) \to \text{LC}(\hat{\mathbb{Z}}^*, M_k^{\text{class}}(\mathbb{C})); \phi \mapsto (u \mapsto \phi_0(\tau, (u \ 0 \ \ 0 \ \ 1)))$. This is an isomorphism.

2. Since the field of definition of general level modular curve is $\mathbb{Q}^{cycl}$, we can naturally descent the above isomorphism to an iso.

$$M_{k,j}(\mathbb{Q}^{cycl}) \cong \text{LC}(\hat{\mathbb{Z}}^*, M_k^{\text{class}}(\mathbb{Q}^{cycl})).$$

3. Define the $q$-expansion of $\phi \in M_{k,j}(\mathbb{C})$ by composing the isomorphism and the $q$-expansion of the classical modular forms, thus we get

$$M_{k,j}(\mathbb{C}) \to \text{LC}(\hat{\mathbb{Z}}^*, \mathbb{C}[[q^{\mathbb{Q}^+}]]),$$

where $\mathbb{C}[[q^{\mathbb{Q}^+}]]$ is the ring of series $\sum_{n \in \mathbb{Q}^+} a_n q^n$ such that there exists a $M \in \mathbb{N}$ verifying $a_n = 0$ if $n \notin \frac{1}{M} \mathbb{Z}$ and $q = e^{2i\pi \tau}$ as usual.
Recall that we have an identification

$$H^0(\omega^k \otimes (H^2_{dR})^{-j}) \otimes \mathbb{Q}^{cycl} = M_{k,j}(\mathbb{Q}^{cycl}).$$

To descent to $\mathbb{Q}$, one need to take the Galois invariant on both side.

Define the $q$-expansion of $\phi \in M_{k,j}(\mathbb{C})$ by composing the isomorphism and the $q$-expansion of the classical modular forms for congruence subgroups, thus we get

$$M_{k,j}(\mathbb{C}) \to \text{LC}(\hat{\mathbb{Z}}^*, \mathbb{C}[\lbrack \lbrack q^{\mathbb{Q}^+} \rbrack \rbrack]),$$

where $\mathbb{C}[\lbrack \lbrack q^{\mathbb{Q}^+} \rbrack \rbrack]$ is the ring of series $\sum_{n \in \mathbb{Q}^+} a_n q^n$ such that there exists a $M \in \mathbb{N}$ verifying $a_n = 0$ if $n \notin \frac{1}{M} \mathbb{Z}$ and $q = e^{2i\pi \tau}$ as usual.

The Galois action on $M_{k,j}(\mathbb{Q}^{cycl})$ is described via its $q$-expansion. In fact, $\text{Gal}_\mathbb{Q}$ acts on the geometry part (i.e. permutes the connected components) and on the coefficients of the $q$-expansion.
The cyclotomic character $\chi_{\text{cycl}} : G_Q \to \hat{\mathbb{Z}}^*$ induces an identification $\chi_{\text{cycl}} : G_Q^{ab} \cong \hat{\mathbb{Z}}^*$. For $u \in \hat{\mathbb{Z}}^*$, we will denote by $\sigma_u$ the inverse image of $u$ in $G_Q^{ab}$.

For $a \in (\mathbb{Z}/N\mathbb{Z})^*$, we denote $\infty_a \in (\Gamma(N) \backslash \mathcal{H}^+ \cup \mathbb{P}^1(\mathbb{Q})) \times \{a\}$ the $\infty$ point in the connected component corresponding to $a$. Then $\sigma_u(\infty_a) = \infty_{ua}$.

**Theorem**

*We have an isomorphism*

$$H^0(\omega^k) \cong \{\phi \in \text{LC}(\hat{\mathbb{Z}}^*, M_k^{\text{class}}(Q_{\text{cycl}})) : \sigma_u(\phi(a)) = \phi(ua)\}. $$
Additive character

- The character of additive group $e_{\infty} : \mathbb{C} \to \mathbb{C}^*$ is defined by
  
  $$e_{\infty}(\tau) = e^{-2i\pi \tau}.$$ 

- The character of additive group $e_{\mathbb{A}} : \mathbb{A} \to \mathbb{C}^*$ factors through $\mathbb{A}/\mathbb{Q}$:
  - thus its restriction to $\mathbb{R} \subset \mathbb{A}$ is $e_{\infty}$.
  - If $\ell$ is a prime number, we denote by $e_{\ell}$ the restriction of $e_{\mathbb{A}}$ to $\mathbb{Q}_\ell \subset \mathbb{A}$. Then $e_{\ell}$ factors through $\mathbb{Q}_\ell/\mathbb{Z}_\ell = \mathbb{Z}[\frac{1}{\ell}]/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$ and we have
    
    $$e_{\ell}(x_{\ell}) = e^{2i\pi x_{\ell}},$$

    where $x_{\ell}$ is the image of $x_{\ell}$ in $\mathbb{R}/\mathbb{Z}$ by the above inclusion.

- $e_{\mathbb{A}}((x_v)_{v}) = \prod_v e_v(x_v)$. 
Suppose $\phi \in M_{k,j}(\mathbb{C})$ is cuspidal, i.e. there exists functions $u \mapsto a(n, u)$, for $n \in \mathbb{Q}$, $n > 0$, such that we have

$$\phi_0(\tau, (\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix})) = \sum_{n > 0} a(n, u)e_{\infty}(n\tau).$$

The condition that $\phi$ is invariant by $\left( \begin{smallmatrix} n^{-1} & 0 \\ 0 & 1 \end{smallmatrix} \right) \in G(\mathbb{Q})$ can be translated into the functional equation

$$\phi_0(n\tau, (\begin{smallmatrix} nu & 0 \\ 0 & 1 \end{smallmatrix})) = n^{j-k}\phi_0(\tau, (\begin{smallmatrix} n^{-1} & 0 \\ 0 & 1 \end{smallmatrix})).$$

**Definition**

Let $\kappa(\phi, u) = a(1, u)$ be the Kirillov function. Then $a(n, u) = n^{k-j}\kappa(\phi, nu)$.

As a consequence, the $q$-expansion of $\phi$ can be written in terms of Kirillov function:

$$\phi(\tau, (\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix})) = \left( \sum_{n \in \mathbb{Q}_+^*} n^{k-j}\kappa(\phi, nu) e_{\infty}(n\tau) \right) \otimes \frac{(\tau e_2 - e_1)^k}{(e_1 \wedge e_2)^j}.$$
Lemma

Let $e^{\infty}[x_f] = \prod_v e_v(x_v)$. Then for $\gamma_f = (a_f b_f \atop 0 1) \in M(\mathbb{A}_f)$, we have

$$\kappa((a_f b_f \atop 0 1) \ast \phi, u) = e^{\infty}([bfu] \kappa(\phi, afu)).$$

Proof.

- If $n \in \mathbb{Q}$, we have $(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}) \ast ((a_f b_f \atop 0 1) \ast \phi_0) = (a_f b_f \atop 0 1) \ast \phi_0$.

$$((a_f b_f \atop 0 1) \ast \phi_0)((\tau \begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix})) = \phi_0(\tau - n, (\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix})(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix})(a_f b_f \atop 0 1))$$

$$= \phi_0(\tau - n, (a_f u b_f u - n) \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}).$$

- If we choose $n$ close to $bfu$, then

$$\phi_0(\tau - n, (a_f u b_f u - n)) = \phi_0(\tau - n, (a_f u \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix})), \quad e^{\infty}(-n) = e^{\infty}([n]) = e^{\infty}([bfu]).$$

- Comparing the Fourier expansion of the two side, we get the result.
We have the natural identifications

\[ \text{LC}((\mathbb{A}_f)^*, \mathbb{C}) = \text{LC}((\mathbb{A}_f)^* \times \hat{\mathbb{Z}}^*, \mathbb{C}) \hat{\mathbb{Z}}^* = (\text{LC}((\mathbb{A}_f)^*, \mathbb{C}) \otimes \mathbb{Q}^{\text{cycl}}) \hat{\mathbb{Z}}^* , \]

where

- \( \hat{\mathbb{Z}}^* \) acts on \( \text{LC}((\mathbb{A}_f)^* \times \hat{\mathbb{Z}}^*, \mathbb{C}) \) by \( (a \cdot \phi)(x, u) = \phi(a^{-1}x, au) \) and on \( \text{LC}((\mathbb{A}_f)^*, \mathbb{C}) \otimes \mathbb{Q}^{\text{cycl}} \) by \( (a \cdot (\phi \otimes \alpha))(x) = \phi(a^{-1}x) \otimes \sigma_a(\alpha) \),
- the first map \( \phi \mapsto \tilde{\phi} \) is given by \( \tilde{\phi}(x, u) = \phi(ux) \) and the inverse of the second one \( \phi \otimes \alpha \mapsto \tilde{\phi} \) is given by \( \tilde{\phi}(x, u) = \sigma_u(\alpha)\phi(x) \).

**Definition**

Let \( \Lambda \) be a subring of \( \mathbb{C} \). We say \( \phi \) is defined over \( \Lambda \), if we have

\[ (u \mapsto \kappa(\phi, u)) \in \text{LC}((\mathbb{A}_f)^*, \Lambda \otimes \mathbb{Z}^{\text{cycl}}) \hat{\mathbb{Z}}^* . \]

We denote by \( M_{k,j}(\Lambda) \) the subring of \( M_{k,j}(\mathbb{C}) \) of the forms defined over \( \Lambda \).
Part III

Eichler-Shimura maps
9 Eichler-Shimura maps
- Description via group cohomology
- Description via coherent cohomology
- Recollection of all the weight

10 Cohomological representations
- Multiplicities
- Description of $\Pi$
Locally constant functions

- Let $\Lambda$ be a ring. Let $\text{LC}(G(\mathbb{A}), \Lambda)$ be the space of locally constant functions $\phi : G(\mathbb{A}) \to \Lambda$, such that there exists an open subgroup $K_\phi$ of $G(\mathbb{A}_f)$ such that $\phi(g\kappa) = \phi(g)$, $\forall \kappa \in K_\phi$.

- If $\Lambda$ is totally disconnected (for example, a finite extension $L$ of $\mathbb{Q}_p$), then such a function is constant on the class modulo $G(\mathbb{R})_+$ since this group is connected.

- If $\Lambda \to \Lambda'$ is a ring homomorphism, then

  $\text{LC}(G(\mathbb{A}), \Lambda') = \Lambda' \otimes_\Lambda \text{LC}(G(\mathbb{A}), \Lambda)$.

- The space $\text{LC}(G(\mathbb{A}), L)$ is equipped with the action of $G(\mathbb{Q})$ and of $G(\mathbb{A})$ given by:

  $$(\gamma \ast \phi)(x) = \phi(\gamma^{-1}x), \text{ for } \gamma \in G(\mathbb{Q}), \quad (g \ast \phi)(x) = \phi(xg), \text{ for } g \in G(\mathbb{A}).$$

  These two actions commute. Thus define an action of $G(\mathbb{Q}) \times G(\mathbb{A})$.

- The action of $G(\mathbb{A})$ is smooth.
Formalism

Let $\tau = x + iy$ be the natural parameter of $\mathcal{H}$ and $d\tau$ the natural base of $\Omega^1(\mathcal{H})$ over $\mathcal{O}(\mathcal{H})$.

1. If $k \in \mathbb{N}$ and $j \in \mathbb{Z}$, we have the exact sequences:

$$0 \rightarrow LC(G(\mathbb{A})/\mathbb{C}^*) \otimes W_{k,j} \rightarrow \mathcal{A}^+ \otimes W_{k,j} \rightarrow \mathcal{A}^+ d\tau \otimes W_{k,j} \rightarrow 0,$$

$$0 \rightarrow LC(G(\mathbb{A})/\mathbb{C}^*) \otimes W_{k,j} \rightarrow \mathcal{A} \otimes W_{k,j} \rightarrow (\mathcal{A}^+ d\tau \oplus \mathcal{A}^- d\bar{\tau}) \otimes W_{k,j} \rightarrow 0.$$

2. $LC(G(\mathbb{A})/\mathbb{C}^*, \mathbb{C}) = LC(G(\mathbb{A}), \mathbb{C})$.

3. Eichler-Shimura maps:

$$\iota_{ES} : H^0(G(\mathbb{Q}), \mathcal{A}^+ d\tau \otimes W_{k,j}) \rightarrow H^1(G(\mathbb{Q}), LC(G(\mathbb{A}), \mathbb{C}) \otimes W_{k,j}),$$

$$\iota_{ES} : H^0(G(\mathbb{Q}), (\mathcal{A}^+ d\tau \oplus \mathcal{A}^- d\bar{\tau}) \otimes W_{k,j}) \rightarrow H^1(G(\mathbb{Q}), LC(G(\mathbb{A}), \mathbb{C}) \otimes W_{k,j}).$$

4. Since $M_{k+2,j+1}(\mathbb{C}) \cong M_{k+2,j+1,k}^{nh}(\mathbb{C})/dM_{k,j}^{nh}(\mathbb{C})$, we have a $G(\mathbb{A}_f)$-equivariant Eichler-Shimura injection:

$$\iota_{ES} : M_{k+2,j+1}(\mathbb{C}) \rightarrow H^1(G(\mathbb{Q}), LC(G(\mathbb{A}), \mathbb{C}) \otimes W_{k,j}).$$
By holomorphic Poincaré lemma, we have an exact sequence

$$0 \to \mathbb{C} \to \mathcal{O}(Y(N)) \to \Omega^1(Y(N))(\log - \text{cusp}) \to 0.$$  

Tensoring with $W_{k,j}$ over $\mathbb{Q}$, we get

$$0 \to W_{k,j} \otimes_{\mathbb{Q}} \mathbb{C} \to \mathcal{O}(Y(N)) \otimes_{\mathbb{Q}} W_{k,j} \to \nabla W_{k,j}^{dR} \otimes \mathcal{O}(Y(N)) \Omega^1(Y(N)) \to 0.$$  

Taking the cohomology of $Y(N)$, we get

$$H^0(Y(N), W_{k,j}^{dR}) \to \nabla H^0(Y(N), W_{k,j}^{dR} \otimes \Omega^1(Y(N))) \to H^1(Y(N), W_{k,j} \otimes_{\mathbb{Q}} \mathbb{C}).$$

$\iota_{ES}$:

$$\frac{H^0(Y(N), W_{k,j}^{dR} \otimes \Omega^1(Y(N)))}{\nabla H^0(Y(N), W_{k,j}^{dR})} \hookrightarrow H^1(Y(N), W_{k,j} \otimes_{\mathbb{Q}} \mathbb{C}).$$
Let $L$ be a field of characteristic 0. We denote by $W_{k,j}^*(L)$ the algebraic representation of $G(L)$

$$W_{k,j}^*(L) = L \otimes_{\mathbb{Q}} W_{k,j}^*.$$ 

$W_{\text{tot}} = \bigoplus_{k,j} W_{k,j} \otimes W_{k,j}^*(L)$

Let $\text{Alg}(G, L)$ be the $L$-space of algebraic functions on $G(L)$ with values in $L$, equipped with the action of $G(\mathbb{Q}) \times G(L)$ by

$$(h_1, h_2) \phi(g) = \phi(h_1^{-1}g h_2).$$

$W_{\text{tot}} \cong \text{Alg}(G, L); \tilde{\nu} \otimes \nu \mapsto \phi_{\tilde{\nu}, \nu}(g) = \langle g \tilde{\nu}, \nu \rangle$ as $G(\mathbb{Q}) \times G(L)$-representation.

We define the space consisting of locally algebraic functions on $G(\mathbb{A})$

$$\text{LP}(G(\mathbb{A}), L) = \text{LC}(G(\mathbb{A}), L) \otimes_L \text{Alg}(G, L).$$

It is equipped with a diagonal action of $G(\mathbb{Q})$ and an action of $G(\mathbb{A}) \times G(L)$.

$$\bigoplus_{k,j} M_{k+2,j+1}(\mathbb{C}) \otimes W_{k,j}^* \hookrightarrow H^1(G(\mathbb{Q}), \text{LP}(G(\mathbb{A}), \mathbb{C})).$$
In general, if $X(G(\mathbb{A}))$ is a space of functions on $G(\mathbb{A})$, stable under the right translation of $G(\mathbb{A})$ and the left translation of $G(\mathbb{Q})$, and with trivial action of $G(\mathbb{R})_+ \subset G(\mathbb{A})$, we denote $H^1(G(\mathbb{Q}), X(G(\mathbb{A})))^\pm$ the subspace of $H^1(G(\mathbb{Q}), X(G(\mathbb{A})))$ on which $\left( \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right)_{\infty}$ acts by $\pm 1$.

∀$\gamma \in H^1(G(\mathbb{Q}), X(G(\mathbb{A})))$, it can be uniquely written as $\gamma = \gamma^+ + \gamma^-$, with $\gamma^\pm \in H^1(G(\mathbb{Q}), X(G(\mathbb{A})))^\pm$.

We denote by $\iota_{ES}^\pm(f)$ the element $\iota_{ES}(f)^\pm$. And this gives a decomposition of $\iota_{ES}$ with the form

$$\iota_{ES} = \iota_{ES}^+ + \iota_{ES}^-.$$
Outline

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- Description via group cohomology
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10 Cohomological representations
- Multiplicities
- Description of Π
Definition

Let $\Pi$ be a smooth, absolutely irreducible representation of $G(\mathbb{A}_f)$ defined over a number field $(\Pi)$. We say $\Pi$ is cohomological if there exists $(k, j) \in \mathbb{N} \times \mathbb{Z}$ such that

$$\text{Hom}_{G(\mathbb{A}_f)}(\Pi, \mathbb{Q}(\pi) \otimes \mathbb{Q} H^1_{\text{dR}, c}(W^{\text{dR}}_{k,j})) \neq 0.$$ 

The pair $(k, j)$ is uniquely determined by $\Pi$ and we say that $\Pi$ is cohomological of weight $(k, j)$.

Definition

If $\Pi$ is cohomological of weight $(k, j)$, and if $\text{truc} \in \{B, \text{dR}, \text{ét}\}$, we set

$$m_{\text{truc}}(\Pi) = \text{Hom}_{G(\mathbb{A}_f)}(\Pi, \mathbb{Q}(\Pi) \otimes \mathbb{Q} H^1_{\text{truc}, c}(W^{\text{truc}}_{k,j})).$$
Properties: étale side

- $\dim_{\mathbb{Q}(\Pi)} m_{dR}(\pi) = 2$
- we have the following isomorphism of comparison:

\[ m_{\text{ét}}(\Pi) = \mathbb{Q}_p \otimes_{\mathbb{Q}} m_B(\Pi), \]
\[ \mathbb{C} \otimes_{\mathbb{Q}} m_B(\Pi) \cong \mathbb{C} \otimes_{\mathbb{Q}} m_{dR}(\Pi), \]
\[ B_{dR} \otimes_{\mathbb{Q}} m_B(\Pi) \cong B_{dR} \otimes_{\mathbb{Q}} m_{dR}(\Pi). \]

- $m_{\text{ét}}(\pi)$ is a rank 2 $\mathbb{Q}_p \otimes \mathbb{Q}(\Pi)$-representation of $\text{Gal}_{\mathbb{Q}}$; its restriction to $\text{Gal}_{\mathbb{Q}_p}$ is a de Rham representation of Hodge-Tate weight $j + 1$ and $j - k$, and $D_{dR}(m_{\text{ét}}(\Pi)) = \mathbb{Q}_p \otimes m_{dR}(\Pi)$ as filtred $\mathbb{Q}_p \otimes \mathbb{Q}(\Pi)$-module.
Properties: de Rham side

- We denote by $m^+_{dR}(\Pi)$ the non-trivial step of the filtration on $m_{dR}(\Pi)$,
  \[m^+_{dR}(\Pi) = \text{Hom}_{\mathbb{G}(\mathbb{A}_f)}(\Pi, M_{k+2,j+1}(\mathbb{Q}(\Pi))),\]
  and $m^-_{dR}(\Pi)$ its quotient, so that the graduation associated to the filtration is
  \[\text{gr}(m_{dR}(\pi)) = m^+_{dR}(\Pi) \oplus m^-_{dR}(\Pi).\]

- After extending the scalar to $\mathbb{C}$, the Hodge theory gives a natural splitting of the Hodge filtration
  \[\mathbb{C} \otimes m_{dR}(\Pi) = m^{(-j-1,k-j)}(\pi) \oplus m^{(k-j,-j-1)}(\Pi),\]
  with
  \[m^{(-j-1,k-j)}(\Pi) = \mathbb{C} \otimes m^+_{dR}(\Pi) \text{ and } m^{(k-j,-j-1)}(\Pi) \sim \mathbb{C} \otimes m^-_{dR}(\Pi).\]
  Moreover, \((-1 \ 0 \ 0 \ 1)\infty\) exchanges $m^{(-j-1,k-j)}(\Pi)$ and $m^{(k-j,-j-1)}(\Pi)$.
The relation between the Betti cohomology and the cohomology of the topological fundamental group gives a natural isomorphism

\[ m_B(\Pi) \cong m(\Pi) := \text{Hom}_{G(\mathbb{A}_f)}(\Pi, H^1(G(\mathbb{Q}), LC(G(\mathbb{A}), \mathbb{Q}(\Pi)) \otimes W_{k,j})). \]

Let \( \iota^+_d \) be a base \( m^+_d \). Then \( \iota_{ES} \circ \iota_+ \) gives an element of \( m(\Pi) \).
Encardement: Kirillov model for finite adeles

**Definition**

If \( \Pi \) is a smooth representation of \( G(\mathbb{A}_f) \), a "encadrement" of \( \Pi \) is a Kirillov model for \( \pi \), i.e. a \( P(\mathbb{A}_f) \)-equivariant injection

\[
\Pi \hookrightarrow \text{LC}((f)^*, \mathbb{Q}(\Pi) \otimes \mathbb{Q}^{\text{cycl}})^{\hat{\mathbb{Z}}^*},
\]

where \((\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}) \in P(\mathbb{A}_f)\) acts by \(((\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}) \star \phi)(x) = e^{\text{infty}}(bx)\phi(ax)\), and the invariance by par \( \hat{\mathbb{Z}}^* \) can be translated by \( \phi(ax) = \sigma_a(\phi(x)) \), where \( a \in \hat{\mathbb{Z}}^* \) and \( x \in (A_f)^* \) (and \( \sigma_a \) acts on \( \mathbb{Q}^{\text{cycl}} \)).

- All the irreducible cohomological representation has an encadrement (unique up to a multiplication by an element of \( \mathbb{Q}(\Pi)^* \)): it is enough to compose \( \kappa \) with an element of \( m^+_{\text{dR}}(\pi) \).
- A representation encadrée of \( G(\mathbb{A}_f) \) is a sub-\( P(\mathbb{A}_f) \)-module

\[
\Pi \subset \text{LC}((A_f)^*, \mathbb{Q}(\Pi) \otimes \mathbb{Q}^{\text{cycl}})^{\hat{\mathbb{Z}}^*},
\]

equipped with a smooth action of \( G(\mathbb{A}_f) \) extending that of \( P(\mathbb{A}_f) \).
Definition

If \( \Pi \) is a smooth representation of \( G(\mathbb{A}_f) \), a "encadrement" of \( \Pi \) is a Kirillov model for \( \pi \), i.e. a \( P(\mathbb{A}_f) \)-equivariant injection

\[
\Pi \hookrightarrow LC((f)^*, \mathbb{Q}(\Pi) \otimes \mathbb{Q}_{cycl})\hat{\mathbb{Z}}^*,
\]

where \((\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}) \in P(\mathbb{A}_f)\) acts by \(((\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}) \star \phi)(x) = e^{\infty}(bx)\phi(ax)\), and the invariance by par \( \hat{\mathbb{Z}}^* \) can be translated by \( \phi(ax) = \sigma_a(\phi(x)) \), where \( a \in \hat{\mathbb{Z}}^* \) and \( x \in (A_f)^* \) (and \( \sigma_a \) acts on \( \mathbb{Q}_{cycl} \)).

- All the irreducible cohomological representation has an encadrement (unique up to a multiplication by an element of \( \mathbb{Q}(\Pi)^* \)): it is enough to compose \( \kappa \) with an element of \( m^+_{dR}(\pi) \).
- A representation encadrée of \( G(\mathbb{A}_f) \) is a sub-\( P(\mathbb{A}_f) \)-module

\[
\Pi \subset LC((A_f)^*, \mathbb{Q}(\Pi) \otimes \mathbb{Q}_{cycl})\hat{\mathbb{Z}}^*,
\]

equipped with a smooth action of \( G(\mathbb{A}_f) \) extending that of \( P(\mathbb{A}_f) \).
Encardement: Kirillov model for finite adeles

**Definition**

If $\Pi$ is a smooth representation of $G(\mathbb{A}_f)$, a "encadrement" of $\Pi$ is a Kirillov model for $\pi$, i.e. a $P(\mathbb{A}_f)$-equivariant injection

$$\Pi \hookrightarrow LC\left((f)^*, Q(\Pi) \otimes \mathbb{Q}^{cycl}\right)^{\hat{\mathbb{Z}}^*},$$

where \((\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix})\) $\in P(\mathbb{A}_f)$ acts by $((\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}) \star \phi)(x) = e^{\ln}(bx)\phi(ax)$, and the invariance by par $\hat{\mathbb{Z}}^*$ can be translated by $\phi(ax) = \sigma_a(\phi(x))$, where $a \in \hat{\mathbb{Z}}^*$ and $x \in (A_f)^*$ (and $\sigma_a$ acts on $\mathbb{Q}^{cycl}$).

- All the irreducible cohomological representation has an encadrement (unique up to a multiplication by an element of $Q(\Pi)^*$): it is enough to compose $\kappa$ with an element of $m^{+}_{dR}(\pi)$.
- A representation encadrée of $G(\mathbb{A}_f)$ is a sub-$P(\mathbb{A}_f)$-module

$$\Pi \subset LC\left((\mathbb{A}_f)^*, Q(\Pi) \otimes \mathbb{Q}^{cycl}\right)^{\hat{\mathbb{Z}}^*},$$

equipped with a smooth action of $G(\mathbb{A}_f)$ extending that of $P(\mathbb{A}_f)$. 
Local Wittaker model

**Definition**

If \( \Pi \) is a smooth representation of \( G(\mathbb{Q}_\ell) \) with central character \( \omega_\Pi \), a Wittaker model for \( \Pi \) is a \( B(\mathbb{Q}_\ell) \)-equivariant injection

\[
\pi \hookrightarrow LC(\mathbb{Q}_\ell^*, L \otimes \mathbb{Q}[\mu_{\ell\infty}])^{\mathbb{Z}_\ell^*}
\]

in the \( L \)-vector space consisting of locally constant functions \( \phi : \mathbb{Q}_\ell^* \to L[\mu_{\ell\infty}] \), satisfying \( \sigma_a(\phi(x)) = \phi(ax) \), for all \( x \in \mathbb{Q}_\ell^* \) and \( a \in \mathbb{Z}_\ell^* \) (where \( \sigma_a \) acts on \( \mu_{\ell\infty} \) by \( \zeta \mapsto \zeta^a \)), equipped the action of \( B(\mathbb{Q}_\ell) \) given by

\[
(a \ b \ 0 \ d \cdot \phi)(x) = \omega_\Pi(d) e_\ell \left(\frac{bx}{d}\right) \phi \left(\frac{ax}{d}\right),
\]

where \( e_\ell(x) = \exp(2i\pi x) \) and \( x \) is the image of \( x \) in \( \mathbb{Z}[\frac{1}{\ell}]/\mathbb{Z} = \mathbb{Q}_\ell/\mathbb{Z}_\ell \).

In the case where the centre character is not clear, we only the equivariant of the mirabolic subgroup.
Let $\chi_1, \chi_2$ be locally constant characters of $\mathbb{Q}_\ell^*$, we denote by $\chi_1 \otimes | \ell^{-1} \chi_2$ the character of $B(\mathbb{Q}_\ell)$ defined by $(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}) \mapsto \chi_1(a)\chi_2(d)|d|^{\ell^{-1}}$.

Ind$_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}(\chi_1 \otimes | \ell^{-1} \chi_2)$ the induced smooth representation, i.e.

$$\{\phi \in \text{LC}(G(\mathbb{Q}_\ell), L), \phi((\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix})x) = \chi_1(a)\chi_2(d)|d|^{\ell^{-1}}\phi(x), \forall x \in G(\mathbb{Q}_\ell), (\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}) \in B(\mathbb{Q}_\ell) \}$$

where the action of $G(\mathbb{Q}_\ell)$ is $(g*\phi)(x) = \phi(xg)$.

The Kirillov model of $\pi = \text{Ind}(\chi_{\ell,1} \otimes \chi_{\ell,2} | \ell^{-1})$ sends $\phi$ to $\kappa_\phi$ defined by

$$\kappa_\phi(y) = \int_{\mathbb{Q}_\ell} \phi((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix})e_\ell(-x)\,dx.$$

If $\chi_1$ and $\chi_2$ are unramified, $H^0(G(\mathbb{Z}_\ell), \text{Ind}_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}(\chi_1 \otimes | \ell^{-1} \chi_2))$ is of dimension 1.
Let $\chi_1, \chi_2$ be locally constant characters of $\mathbb{Q}_\ell^*$, we denote by $\chi_1 \otimes |^{-1}_\ell \chi_2$ the character of $B(\mathbb{Q}_\ell)$ defined by $(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}) \mapsto \chi_1(a)\chi_2(d)|d|^{-1}_\ell$.

$\text{Ind}_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}(\chi_1 \otimes |^{-1}_\ell \chi_2)$ the induced smooth representation, i.e.

$$\{\phi \in \text{LC}(G(\mathbb{Q}_\ell), L), \phi((\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix})x) = \chi_1(a)\chi_2(d)|d|^{-1}_\ell \phi(x), \forall x \in G(\mathbb{Q}_\ell), (\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}) \in B(\mathbb{Q}_\ell)\}$$

where the action of $G(\mathbb{Q}_\ell)$ is $(g \ast \phi)(x) = \phi(xg)$.

The Kirillov model of $\pi = \text{Ind}(\chi_{\ell,1} \otimes \chi_{\ell,2} |^{-1}_\ell)$ sends $\phi$ to $\kappa_\phi$ defined by

$$\kappa_\phi(y) = \int_{\mathbb{Q}_\ell} \phi((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}))e_\ell(-x) \, dx.$$ 

If $\chi_1$ and $\chi_2$ are unramified, $H^0(G(\mathbb{Z}_\ell), \text{Ind}_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}(\chi_1 \otimes |^{-1}_\ell \chi_2))$ is of dimension 1.
Let $\chi_1, \chi_2$ be locally constant characters of $\mathbb{Q}_\ell^*$, we denote by $\chi_1 \otimes |_{\ell}^{-1} \chi_2$ the character of $B(\mathbb{Q}_\ell)$ defined by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)|_{\ell}^{-1}$.

$\text{Ind}_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}(\chi_1 \otimes |_{\ell}^{-1} \chi_2)$ the induced smooth representation, i.e.

$$\{\phi \in \text{LC}(G(\mathbb{Q}_\ell), L), \phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} x) = \chi_1(a)\chi_2(d)|_{\ell}^{-1} \phi(x), \forall x \in G(\mathbb{Q}_\ell), \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(\mathbb{Q}_\ell)\}$$

where the action of $G(\mathbb{Q}_\ell)$ is $(g \ast \phi)(x) = \phi(xg)$.

The Kirillov model of $\pi = \text{Ind}(\chi_{\ell,1} \otimes \chi_{\ell,2} |_{\ell}^{-1})$ sends $\phi$ to $\kappa_\phi$ defined by

$$\kappa_\phi(y) = \int_{\mathbb{Q}_\ell} \phi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})(y \begin{pmatrix} x \\ 1 \end{pmatrix}) e_{\ell}(-x) \, dx.$$ 

If $\chi_1$ and $\chi_2$ are unramified, $H^0(G(\mathbb{Z}_\ell), \text{Ind}_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}(\chi_1 \otimes |_{\ell}^{-1} \chi_2))$ is of dimension 1.
unramified case

- Let $\chi_1, \chi_2$ be locally constant characters of $\mathbb{Q}_\ell^*$, we denote by $\chi_1 \otimes |\ell^{-1}\chi_2$ the character of $B(\mathbb{Q}_\ell)$ defined by $\left(\begin{array}{cc} a & b \\ 0 & d \end{array}\right) \mapsto \chi_1(a)\chi_2(d)|d|^{-1}_\ell$.

- $\text{Ind}_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}(\chi_1 \otimes |\ell^{-1}\chi_2)$ the induced smooth representation, i.e.

  $$\{\phi \in \text{LC}(G(\mathbb{Q}_\ell), L), \phi\left(\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)\left(\begin{array}{cc} y & x \\ 0 & 1 \end{array}\right)\right) = \chi_1(a)\chi_2(d)|d|^{-1}_\ell \phi(x), \forall x \in G(\mathbb{Q}_\ell), \left(\begin{array}{cc} a & b \\ 0 & d \end{array}\right) \in B(\mathbb{Q}_\ell)\}$$

  where the action of $G(\mathbb{Q}_\ell)$ is $(g \ast \phi)(x) = \phi(xg)$.

- The Kirillov model of $\pi = \text{Ind}(\chi_{\ell,1} \otimes \chi_{\ell,2} | |^{-1}_\ell)$ sends $\phi$ to $\kappa_\phi$ defined by

  $$\kappa_\phi(y) = \int_{\mathbb{Q}_\ell} \phi\left(\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)\left(\begin{array}{cc} y & x \\ 0 & 1 \end{array}\right)e(\ell(x)) \right) dx.$$  

- If $\chi_1$ and $\chi_2$ are unramified, $H^0(G(\mathbb{Z}_\ell), \text{Ind}_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}(\chi_1 \otimes |\ell^{-1}\chi_2))$ is of dimension 1.
In the Kirillov model, this space is generated by the function $v_{\mathcal{N}_{\ell}}$ defined by

$$v_{\mathcal{N}_{\ell}}(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Z}_{\ell}, \\ \chi_1(\ell)^n + \chi_1(\ell)^{n-1}\chi_2(\ell) + \cdots + \chi_2(\ell)^n & \text{if } x \in \ell^n\mathbb{Z}_{\ell}^* \text{ and } n \geq 0. \end{cases}$$

This is the image of the *normalized newvector* (i.e. by the condition $v_{\mathcal{N}_{\ell}}(1) = 1$) of $\mathcal{N}_{\ell}$; and it generates the space as a $P(\ell)$-module.
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