Holomorphic isometries from the unit ball into symmetric domains

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Outline

Bounded symmetric domain

Definition

A bounded domain D in the complex space $Z \subseteq \mathbb{C}^n$ is symmetric: for every $x \in D$, there is an biholomorphism s such that $s^2 = id_D$, and x is an isolated fixed point of s.

Let

$$G := Aut(D) = \{\Phi : D \to D \; biholomorphic\}$$

$$K := \{g \in G : g(0) = 0\},$$

Then $D = G/K$.

Bounded Symmetric Domain

Classification:

Classical cases:

$$D^{I}_{m,n} = \{Z \in M_{m,n}(\mathbb{C}) : \|Z\| < 1\}$$

$$D^{II}_{n} = \{Z \in M_{n}(\mathbb{C}) : \|Z\| < 1, Z^t = -Z\}$$

$$D^{III}_{n} = \{Z \in M_{n}(\mathbb{C}) : \|Z\| < 1, Z^t = Z\}$$

$$D^{IV}_{n} = \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n : |z| < 1, 1 - |z|^2 + |\sum_i z_i^2|^2 > 0\}$$

Exceptional Domains

D^{V}, dim 16

D^{VI}, dim 27

Example

Type I tube domain:
$D = \{z \in M_r(\mathbb{C}) : \|z\| < 1\} = SU_{r,r}(\mathbb{C})/S(U_r(\mathbb{C}) \otimes U_r(\mathbb{C}))$
with the action $\begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) := (az+b)(cz+d)^{-1}$
$K = S(U_r(\mathbb{C}) \otimes U_r(\mathbb{C}))$ with the action

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}(z) := azd^{-1}$$

$Rank(D) = \max\{rank(z) : z \in D\}$
$Rank(D) = 1 : D = \mathbb{B}^n$

Bergman space

Bergman space:

$$L_a^2(D) = \{f \in Hol(D) : \|f\|^2 = \int_D |f(z)|^2 dm(z) < \infty\}.$$

Bergman kernel:

By Riesz representation theorem, for fixed $z \in D$ there exists $K_z \in L_a^2(D)$ such that

$$f(z) = \langle f, K_z \rangle, \forall f \in L_a^2(D).$$

Bergman metric:

$$ds_{D,z}(u, v) = \partial_u \overline{\partial}_v \log K(z, z)$$

Holomorphic Isometry

Holomorphic mapping $f : (D, \lambda ds_D) \to (\Omega, ds_\Omega)$ is (λ-)isometry if

$$f^* ds_\Omega = \lambda ds_D.$$

local isometry on $z \in D$: $f^* ds_\Omega = \lambda ds_D$ near z.

Question

Classify holomorphic isometry between bounded symmetric domain.

Motivation:
can be traced back to Bochner and Calabi
Shimura varieties
Clozel-Ullmo work on commutants of modular correspondences

Mok, Ann. of Math. 1987 ; Mok, Wong, preprint

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded symmetric domain of rank≥ 2, $\Gamma \subseteq Aut(\Omega)$ be a torsion-free irreducible lattice and $M = \Omega \setminus \Gamma$ is the finite volume quotient. Let N be a complex manifold and $\tau : \tilde{N} \to N$ is its universal cover. Suppose $f : M \to N$ is a holomorphic map, and $F : \Omega \to \tilde{N}$ is the lifting of f. Assume $(M, N; f)$ satisfies the following non-degeneracy condition: for component Ω_k, there exists a bounded holomorphic function h_k on $\tilde{N}$ and an irreducible factor subdomain $\Omega'_k \subseteq \Omega_k$ such that h_k is non constant on $F(\Omega'_k)$.
Then there exists a bounded holomorphic map $R : \tilde{N} \to \mathbb{C}^n$ such that $R \circ F = id_\Omega$.

Mok, Ann. of Math. 1987;Clozel-Ullmo, Crelle's, 2003

Let $D \subseteq \mathbb{C}^n$ be a bounded symmetric domain of rank≥ 2, and $F : (D, \lambda ds_D) \to (\Omega, ds_\Omega)$ be a holomorphic isometry to another bounded symmetric domain Ω. Then F is a totally geodesic holomorphic embedding.

Example

$F : \mathbb{D} \to B_2$ defined by $F(z) = (z, 0)$

Theorem(Mok JEMS2012)

Let D, Ω be bounded symmetric domains in their Harish-Chandra realizations. Let λ be any positive real number and

$$f : (D, \lambda ds_D; 0) \to (\Omega, ds_\Omega; 0)$$

be a germ of holomorphic isometry with $f(0) = 0$. Then, the germ $Graph(f)$ extends to an affine-algebraic subvariety $T \subseteq \mathbb{C}^n \times \mathbb{C}^N$, such that $S := T \cap (D \times \Omega)$ is the graph of a proper holomorphic isometric embedding $F : (D, \lambda ds_D) \to (\Omega, ds_\Omega)$.

p-root

Theorem(Mok JEMS2012)

Let $\mathcal{H}$ be the upper half-plane, $p \geq 2$ be a positive integer and $\gamma = e^{\frac{i\pi}{p}}$. Then, the proper holomorphic mapping $f : \mathcal{H} \to \mathcal{H}^p$ defined by

$$f(z) = (z^{1/p}, \gamma z^{1/p}, \cdots, \gamma^{p-1} z^{1/p})$$

is a holomorphic isometric embedding.

p-root

Chan, PAMS 2016; Ng PAMS 2010;

For $p = 2$ or for p odd, if $f : D \to D^p$ is a holomorphic isometric embedding with sheeting number $n = p$, then f is the p-th root embedding up to reparametrization.

Complete Classification for holomorphic isometric $f : D \to D^p$:
Ng, PAMS 2010: p = 2, 3
Chan, Michigan Math. J. 2017 : p = 4 .

Chan, Mok, accepted by JDG

Let $f : \mathbb{D} \to \Omega$ be a ($\lambda-$)holomorphic isometric embedding, where Ω is a bounded symmetric domain in its Harish-Chandra realization. Then, f is asymptotically totally geodesic at a general point $b \in \partial\mathbb{D}$.

Application to Ax-Lindemann conjecture

Let Ω be a bounded symmetric domain in its Harish-Chandra realization, $\Gamma \in Aut(\Omega)$ be a not necessarily arithmetic torsion-free cocompact lattice. Write $X := \Omega \setminus \Gamma$ and $\pi : \Omega \to X$. Let Y be an irreducible subvariety of X, and $Z \subseteq \Omega$ be an irreducible component of $\pi^{-1}(Y)$. Suppose Z is an algebraic subset. Then, Z is a totally geodesic complex submanifold.

Theorem(Mok JEMS2012)

$F : (D, \lambda ds_D) \to (\Omega, ds_\Omega)$ is a totally geodesic holomorphic embedding if

- whenever each irreducible component of D is of *rank* ≥ 2;
- whenever $D = B^n$ with $n \geq 2$, and $\Omega = B^n \times \cdots \times B^n$.

Theorem(Yuan and Zhang, JDG2012)

$F : (D, \lambda ds_D) \to (\Omega, ds_\Omega)$ is a totally geodesic holomorphic embedding if $D = B^n$ with $n \geq 2$, and $\Omega = B^{N_1} \times \cdots \times B^{N_k}$.

Mok, PAMS, 2016

Let Ω be an irreducible bounded symmetric domain of rank ≥ 2 with genus p. Then, there exists a nonstandard proper holomorphic isometric embedding $F : (B_{p+1}, ds_{B_{p+1}}) \to (\Omega, ds_\Omega)$.

With the help of Mok mapping, Xiao-Yuan, Chan-Mok, Upmeier-Wang-Zhang classified the holomorphic isometric from B^p into D^{IV} of type IV

Jordan triple

Call $Z = \mathbb{C}^N$ a Jordan triple if Z has a Jordan triple product:

$$(u, v, w) \mapsto \{u, v, w\}$$

satisfying: linear on u, w, antilinear v,

$$\{u, v, w\} = \{w, v, u\}$$

$$\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} - \{u, \{y, x, v\}, v\} + \{u, v, \{x, y, w\}\}.$$

Classification:
Jordan algebra: Jordan, von Neumann, Wigner...

Jordan triple: Koecher, Loos, Satake...

Bounded symmetric domain

$z\Box w : Z \to Z$ is defined by

$$z\Box w(\xi) := \{z, w, \xi\}$$

Then $D = \{z \in Z : \|z\Box z\| < 1\}$ is bounded symmetric domain.
Let $Q_z(w) := \{z, w, z\}$, then Bergman operator

$$B(z, w) = I - 2z\Box w + Q_z Q_{\overline{w}}$$

is coinvariant property under $Aut(D)$-action.
Bergman kernel:

$$K^D(z, w) = detB(z, w)^{-1} = \triangle(z, w)^{-p}.$$

$\triangle(z, w)$ quais-determinant, p genus

Tripotent

c is a tripotent: $\{c, c, c\} = c$.

Peice decomposition with respect to c: $Z = Z_2(c) \oplus Z_1(c) \oplus Z_0(c)$ with

$$Z_\alpha(c) := \{z \in Z : \{c, c, z\} = \frac{\alpha}{2} z\}.$$

Frame: let $e_1, \cdots, e_r$ be a maximal set of orthogonal minimal tripotents, then

$$Z = \sum_{0 \le i \le j \le r} Z_{ij}$$

with $Z_{ii} = \mathbb{C} e_i, 1 \le i \le r; Z_{ij} = Z_1(e_i) \cap Z_1(e_j), 1 \le i < j \le r$.
Fact: $a = \dim Z_{12}, b = \dim Z_{01}$, then $p = 2 + a(r-1) + b$.

Example

For $r \leq s$, the matrix space $Z = \mathbb{C}^{r \times s}$ is a hermitian Jordan triple with triple product

$$\{u, v, w\} = \frac{1}{2}(uv^*w + wv^*u).$$

The associated bounded symmetric domain is the matrix unit ball

$$D = \{z \in \mathbb{C}^{r \times s} : \ 1 - zz^* > 0\}$$

of genus $p = r + s$. The Bergman operators have the form

$$B(z, w)\zeta = (1 - zw^*)\zeta(1 - w^*z)$$

and the quasi-determinant is $\Delta(z, w) = \det(1 - zw^*)$.

Example

Let $Z = \mathbb{C}^n$. The Jordan triple product

$$\{u, v, w\} = (uv^*)w + (wv^*)u - (uw^t)\overline{v}$$

makes Z into a hermitian Jordan triple of rank 2. The corresponding symmetric domain is D_n^{IV} of type IV. The quasi-determinant of the spin factor is given by

$$\Delta(z, w) = 1 - \langle z|w\rangle + \frac{1}{4}\langle z|\overline{z}\rangle\langle\overline{w}|w\rangle,$$

and the Bergman kernel $K(z, w) = \Delta(z, w)^{-p}$ with genus $p = n$.

Metric

Bergman metric ds_D at $z \in D$ is defined by

$$ds_{D,z}(u, v) = \partial_u \overline{\partial}_v \log K(z, z)$$

Or,

$$ds_{D,z}(u, v) = \langle B(z, z)^{-1} u, v \rangle.$$

Curvature $R_D(u, v)x = -\{u, v, x\}$

Mok JEMS2012

Let D and Ω be bounded complete circular domains. Let λ be any positive real number and $f : (D, \lambda ds_D; 0) \to (\Omega, ds_\Omega; 0)$ be a germ of holomorphic isometry with $f(0) = 0$. Then, such that for $z, w \in D$ sufficiently close to 0, we have

$$K_\Omega(f(z), f(z)) = K_D(z, z)^\lambda;$$

and hence

$$K_\Omega(f(z), f(w)) = K_D(z, w)^\lambda.$$

Theorem

Let $f : (B, \lambda ds_B^2; 0) \to f : (D, ds_D^2; 0)$ be a germ of holomorphic isometry with $f(0) = 0$ between bounded symmetric domain B, D. Then there exists a proper holomorphic isometric embedding $F : B \to D$ extending the germ of holomorphic map f.

proof: for $\lambda = 1$,
Step1: a isometry map $U : L_a^2(B) \to L_a^2(D)$ by

$$U(K_z^B) = K_{f(z)}^D$$

with z near 0. Note $\overline{span}\{K_z^B : z\, near\, 0\} = L_a^2(B)$.
Step 2: $U(K_z^B) = K_{F(z)}^D, z \in B$ for some map $F : B \to D$.

Assumption(maximal case)
Fix $\lambda = 1$ and $D = \mathbb{B}^d$. Consider the case $p_\Omega = p_{\mathbb{B}^d} = d + 1$.
Then $f : \mathbb{B}^d \to \Omega$ is an isometry if

$$K_\Omega(f(z), f(w)) = K_{\mathbb{B}^d}(z, w)$$

or

$$\triangle_\Omega(f(z), f(w)) = \triangle_D(z, w).$$

Observation
Fix a minimal tripotent c and the Peice decomposition with respect to c: $Z = Z_2 \oplus Z_1 \oplus Z_0$. Then $Z_2 \oplus Z_1$ is a complex subspace of dimension d and $\Omega \cap Z_2 \oplus Z_1 = \mathbb{B}^d$.

Umpeier-Wang-Zhang, IMRN 2019

The map

$$(u,v) \mapsto G_c(u,v) := \left(\frac{uc+v}{u+1}\right)^c = uc + v + \frac{1}{1+u} Q_c(v,v)$$

defines a holomorphic isometry from the unit ball $B \subset Z^2 \oplus Z^1$ into Ω. More precisely, the quasi-determinant satisfies

$$\triangle_\Omega(G_c(u,v), G_c(u',v')) = 1 - u\overline{u}' - \langle v|v'\rangle.$$

Therefore

$$K_\Omega(G_c(u,v), G_c(u',v')) = K_{\mathbb{B}^d}(uc+v, u'c+v').$$

proof

Let $z^w = B(z,w)^{-1}(z - \{z,w,z\})$ be the quasi-inverse. Then

$$B(z^c, z^c) = B^{-1}(z,c)B(c,v)B(a,c)B(v,c)B^{-1}(c,z),$$

where $z = uc + v \in Z^2 \oplus Z^1, a = u + \{c,u,c\} + \{v,v,c\}$.
This implies that

$$\triangle((u+v)^c, (u+v)^c) = |\triangle(u,c)|^{-2} \triangle(a,c)$$

Example 1: $Z = \mathbb{C}^{r\times s}$: assume that $c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$Z^2 \oplus Z^1 = \{ \begin{pmatrix} u & v \\ \tilde{v} & 0 \end{pmatrix} : u \in \mathbb{C},\ v \in \mathbb{C}^{1\times(s-1)},\ \tilde{v} \in \mathbb{C}^{(r-1)\times 1} \} \approx \mathbb{C}^{p-1}.$$

The map F is given by

$$G_c \begin{pmatrix} u & v \\ \tilde{v} & 0 \end{pmatrix} = \begin{pmatrix} u & v \\ \tilde{v} & \tilde{v}(u+1)^{-1}v \end{pmatrix}.$$

Example 2: Letting D^{d+1} be type IV of dimension $d+1$, then $f : B^d \to D^{d+1}$ is given by

$$f(z_1, z') = z_1 c_1 + q(z) c_2 + z', \quad z = z_1 c_1 \oplus 0 c_2 \oplus z'$$

where $\{c_1, c_2\}$ is a frame of minimal tripotents and $q(z) = \frac{(z', z')}{1+z_1}$.

Definition

An isometric embedding $F : B \to D \subset Z$ satisfying $F(0) = 0$ will be called a Mok embedding if F is equivalent to the map constructed by Mok.

Umpeier-Wang-Zhang, IMRN 2019

Let $F : B \to D \subset Z$ be an isometric embedding with $F(0) = 0$. Then F is a Mok embedding if and only if $\partial_e^0 F' = F''(0)(e, \cdot) = 0$ for some vector $e \in \mathbb{C}^d$.

Let $F, G : B \to D$ be two Bergman isometries preserving the origin, which agree up to order $n \geq 0$ at 0, i.e. $F^{(j)}(0) = G^{(j)}(0)$, $0 \leq j \leq n$. Then

$$\langle \partial^0_{u_0,\dots,u_m} F | \partial^0_{v_0,\dots,v_n} F \rangle = \langle \partial^0_{u_0,\dots,u_m} G | \partial^0_{v_0,\dots,v_n} G \rangle$$

for $m \leq n$ and vectors $u_i, v_j \in \mathbf{C}^d$, that is $F^{(n+1)}(0)^* F^{(m+1)} = G^{(n+1)}(0)^* G^{(m)} : \odot^m \mathbb{C}^d \to Z$. Therefore,

Lemma

Let $F : B \to D$ be a Bergman isometry which agrees up to order 2 with a Mok embedding G, then $F = G$.

$F''(0)(x, u)$ is the second fundamental form:

$$\langle \{u,v,x\}, y\rangle_B = \langle \{F'(0)u, F'(0)v, F'(0)x\}, F'(0)y\rangle_D + \langle F''(0)(x,u) | F''(0)(y,v)\rangle_D.$$

For any $p \in P_{1,1}(Z)$, the polar form $\tilde{p}$ satisfies

$$\tilde{p}(v, \partial_{v,v} F)(0) = 0.$$

for all $v \in Z^1$.
This implies that

$$\partial_{v,v} F(0) = \gamma \, \{v, c, v\}$$

for some $|\gamma| = 1$.

Umpeier-Wang-Zhang, IMRN 2019

Let $F : B^d \to D^{d+1}$ be a Bergman isometric embedding into the Lie ball, with $F(0) = 0$. then F is a Mok's embedding , or

$$F_\vartheta(u, v) = u\frac{e_1 + e_2}{\sqrt{2}} + v + \frac{e_2 - e_1}{\sqrt{2}}\left(1 - \sqrt{1 + u^2 + \langle \overline{e}_1 | e_2 \rangle \ \langle v | \overline{v} \rangle}\right).$$

with a frame of minimal tripotents $e_1, e_2 \in Z$ and a unimodular constant $\vartheta \in \mathbb{T}$.

proof.

case 1: $RanF''(0)$ consists of elements of only rank 1 .
Then F is a Mok map.

case 2: $RanF''(0)$ has elements of rank 2. Then F is irrational.

Yang, Ph.D. thesis 2017;Mok, Yang, in preparation

Let Ω be an irreducible bounded symmetric domain of rank 2 not biholomorphic to any type-IV domain or a type-I domain $D^I(3,q), q > 3$ or the 27-dimensional exceptional domain D^{VI}. Let $F : B^{p+1} \to \Omega$ be a holomorphic isometric embedding with $p := p(\Omega)$. Then F is congruent to the Mok embedding.

Thank you