The number of positive solutions to elliptic problems

Shuangjie Peng

(with Daomin Cao, Peng Luo, Shusen Yan, etc)

sjpeng@mail.ccnu.edu.cn

Nov. 11, 2020, 国家天元数学东南中心

Outline of Talk

- Motivation
- Necessary conditions
- Local uniqueness
- The number of bubble solutions
- A uniqueness result of bubble solutions

★ The problem:

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \varepsilon u, \ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

$\varepsilon > 0$ is a small parameter, $N \geq 3$.

★ The problem:

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \varepsilon u, \ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

$\varepsilon > 0$ is a small parameter, $N \geq 3$.

Determine the number of solutions of (1) as $\varepsilon > 0$ small.

Problem (1) has been investigated extensively.

♦ $\varepsilon = 0$:

Nonexistence :

Pohozaev (65') : Ω is star-shaped; Passasseo (93'), $\cdots$

Existence :

Coron(83'); Bahri-Coron(88'); Weiyue Ding(89'); $\ldots$

♦ $\varepsilon > 0$:

Existence :

Brezis-Nirenberg (83') : $\varepsilon \in (0, \lambda_1)$ for $N \geq 4$, $\varepsilon \in (\lambda_*, \lambda_1)$ for $N = 3$.

Struwe; Lions; $\cdots\cdots$

Asymptotic behavior as $\varepsilon \to 0^+$:

Brezis-Peletier(89'); Han(91'); Rey(90'); Y.-Y. Li(91'); $\cdots\cdots$

$$\begin{cases} -\Delta G(x,\cdot) = \delta_x, & \text{in } \Omega, \\ G(x,\cdot) = 0, & \text{on } \partial\Omega. \end{cases}$$

$G(x,y)$ has the form

$$G(x,y) = S(x,y) - H(x,y), \ (x,y) \in \Omega \times \Omega,$$

$S(x,y) = \dfrac{1}{(N-2)\omega_N |y-x|^{N-2}}$ is the singular part;

$H(x,y)$ is the regular part;

$R(x) := H(x,x)$ Robin function.

Rey(90') proved:

If a solution u_ε of (1) satisfies

$$|\nabla u_\varepsilon|^2 \rightharpoonup S^{N/2}\delta_{x_0}, \text{ as } \varepsilon \to 0, \tag{2}$$

then x_0 is a critical point of $R(x)$.

If x_0 is a nondegenerate critical point of $R(x)$ and $N \geq 5$, then (1) has a solution u_ε satisfying (2).

Uniqueness:

Glangetas(93') (degree argument): the solution u_ε of (1) satisfying (2) is unique for ε small enough.

Cao, Yan, et al.(99',03'),$\cdots\cdots$

Cao, Deng , Yan, Guo, Lin, Peng, $\cdots\cdots$, since 2017, Local Pohozaev identities argument

Multi-bubble solutions:

$$U_{x,\lambda}(y) = C_N \frac{\lambda^{(N-2)/2}}{(1+\lambda^2|y-x|^2)^{(N-2)/2}}$$

solves uniquely

$$-\Delta u = u^{\frac{N+2}{N-2}}, \ u > 0 \quad \text{in } \mathbb{R}^N.$$

Set

$$A = \int_{\mathbb{R}^N} U_{0,1}^{\frac{N+2}{N-2}}, \ B = \int_{\mathbb{R}^N} U_{0,1}^2.$$

$$\Psi_k(x,\lambda) = A^2\Big(M_k(x)\lambda^{(N-2)/2}, \lambda^{(N-2)/2}\Big) - B\sum_{j=1}^{k}\lambda_j^2,$$

$$\lambda^{(N-2)/2} = \Big(\lambda_1^{(N-2)/2}, \cdots, \lambda_k^{(N-2)/2}\Big)^T;$$

$M_k(x) = \Big(m_{ij}(x)\Big)_{1\le i,j\le k}$ is a matrix

$$m_{ii}(x) = R(x_i),\ m_{ij}(x) = G(x_i, x_j),\ \text{if } i \neq j.$$

Musso and Pistoia(02'): there exists a family of solutions to (1) satisfying

$$|\nabla u_\varepsilon|^2 \rightharpoonup S^{N/2} \sum_{i=1}^{k} \delta_{a_i}, \ \text{as } \varepsilon \to 0, \tag{3}$$

if $N \geq 5$, (a^k, Λ^k) is a nondegenerate critical point of Ψ_k with $a^k = (a_1, \cdots, a_k)$ and some $\Lambda^k = (\lambda_1, \cdots, \lambda_k)$.

Our aim: determine the exact number of positive solutions to (1) for small ε.

2. Necessary conditions

Theorem 2.1 [Cao-Luo-P, Tran. Amer. Math. Soc., to appear]

$u_\varepsilon(x)$ is a solution of (1) with (3), then $M_k(a^k)$ is a non-negative matrix with $a^k = (a_1, \cdots, a_k)$ and

$$u_\varepsilon = \sum_{j=1}^{k} PU_{x_{j,\varepsilon},\lambda_{j,\varepsilon}} + w_\varepsilon(x),$$

$$\lambda_{j,\varepsilon} = \Big(u_\varepsilon(x_{j,\varepsilon})\Big)^{\frac{2}{N-2}} \to +\infty, x_{j,\varepsilon} \to a_j, \ \|w_\varepsilon\| = o(1).$$

Theorem 2.1 [Cao-Luo-P, continuity]

Moreover, if $M_k(a^k)$ is a positive matrix, then

$$0 < C_1 \leq \varepsilon^{\frac{1}{N-4}} \lambda_{j,\varepsilon} \leq C_2 < +\infty. \tag{4}$$

Furthermore if we denote (by choosing subsequence)

$$\lambda_j := \lim_{\varepsilon \to 0} \left(\varepsilon^{\frac{1}{N-4}} \lambda_{j,\varepsilon} \right)^{-1}, \text{ for } j = 1, \cdots, k,$$

Then (a^k, Λ^k) is a critical point of Ψ_k with $a^k = (a_1, \cdots, a_k)$ and $\Lambda^k = (\lambda_1, \cdots, \lambda_k)$.

Remarks

1. (4) was usually proposed as assumptions, see Bahri-Li-Rey(95').

2. When Ω is convex, Grossi(10') proved that, if $k \geq 2$ it is impossible that

$$\nabla \Psi_k(a^k, \Lambda^k) = 0.$$

Theorem 2.1 implies that (1) has no solutions blowing-up at multi-points on convex domain.

3. By Caffarelli(85') and Cardaliaguet-Tahraoui(02'), in a convex domain, the Robin function $R(x)$ is convex and has a unique critical point, which is also non-degenerate if $N = 2$. But if $N \geq 3$, the non-degeneracy of the critical point is unknown. If this is ture, then problem (1) has a unique solution for ε small enough when $N \geq 5$ and Ω is a convex domain.

Ideas of the proof:

Step 1: Shape of u_ε away from the bubble points

$$u_\varepsilon(x) = A\Big(\sum_{j=1}^{k}\frac{G(x_{j,\varepsilon},x)}{(\lambda_{j,\varepsilon})^{(N-2)/2}}\Big)+O\Big(\frac{1}{\lambda_\varepsilon^{(N+2)/2}}+\frac{\varepsilon}{\lambda_\varepsilon^{(N-2)/2}}\Big)$$
$$+o\Big(\frac{\varepsilon}{\lambda_\varepsilon^2}\Big), \text{ in } C^1\Big(\Omega\backslash\bigcup_{j=1}^{k}B_{2d}(x_{j,\varepsilon})\Big)$$

Step 2: description of $\lambda_{j,\varepsilon}$

By Loacl Pohozaev identity

$$
\begin{aligned}
& -\int_{\partial\Omega'} \frac{\partial u_\varepsilon}{\partial \nu}\big\langle x - x_{j,\varepsilon}, \nabla u_\varepsilon \big\rangle \\
& + \frac{1}{2}\int_{\partial\Omega'} |\nabla u_\varepsilon|^2 \big\langle x - x_{j,\varepsilon}, \nu \big\rangle + \frac{2-N}{2}\int_{\partial\Omega'} \frac{\partial u_\varepsilon}{\partial \nu} u_\varepsilon \\
= & \frac{N-2}{2N}\int_{\partial\Omega'} u_\varepsilon^{\frac{2N}{N-2}} \big\langle x - x_{j,\varepsilon}, \nu \big\rangle + \frac{\varepsilon}{2}\int_{\partial\Omega'} u_\varepsilon^2 \big\langle x - x_{j,\varepsilon}, \nu \big\rangle \\
& - \varepsilon \int_{\Omega'} u_\varepsilon^2,
\end{aligned}
\tag{5}
$$

We find

$$\begin{aligned}&\Lambda_{j,\varepsilon}^{N-2}R(x_{j,\varepsilon})-\sum_{l\neq j}^{k}\Lambda_{j,\varepsilon}^{(N-2)/2}\Lambda_{l,\varepsilon}^{(N-2)/2}G(x_{j,\varepsilon},x_{l,\varepsilon})\\=&\frac{2B}{A^2(N-2)}\Lambda_{j,\varepsilon}^2+o\Big((\Lambda_\varepsilon^k)^{N-2}\Big),\end{aligned}\tag{6}$$

where

$$\Lambda_{j,\varepsilon}:=\Big(\varepsilon^{\frac{1}{N-4}}\lambda_{j,\varepsilon}\Big)^{-1}.$$

Using $\frac{1}{k}\sum_{l=1}^{k}\Lambda_{l,\varepsilon}^{N-2} \le (\Lambda_\varepsilon^k)^{N-2} \le \sum_{l=1}^{k}\Lambda_{l,\varepsilon}^{N-2}$, (6) says

$$\vec{\mu}_{k,\varepsilon}\Big(M_k(x_\varepsilon)+o(1)\Big)\vec{\mu}_{k,\varepsilon}^T = \frac{2B}{A^2(N-2)}\text{diag}\left(\Lambda_{1,\varepsilon}^2,\cdots,\Lambda_{k,\varepsilon}^2\right), \tag{7}$$

$\vec{\mu}_{k,\varepsilon} = \left(\Lambda_{1,\varepsilon}^{(N-2)/2},\cdots,\Lambda_{k,\varepsilon}^{(N-2)/2}\right)$, $x_\varepsilon = (x_{1,\varepsilon},\cdots,x_{k,\varepsilon})$.

(7) reads that $M_k(a^k)$ is a non-negative matrix.

Moreover, if $M_k(a^k)$ is a positive matrix, we obtain the estimate for $\lambda_{j,\varepsilon}$. Letting $\varepsilon \to 0$ in (6), we find

$$\nabla_\lambda \Psi_k(a^k,\Lambda^k) = 0.$$

Step 3: description of $x_{j,\varepsilon}$

By Loacl Pohozaev identity

$$-\int_{\partial\Omega'}\frac{\partial u_\varepsilon}{\partial\nu}\frac{\partial u_\varepsilon}{\partial x_i}+\frac{1}{2}\int_{\partial\Omega'}|\nabla u_\varepsilon|^2\nu_i = \frac{N-2}{2N}\int_{\partial\Omega'}u_\varepsilon^{\frac{2N}{N-2}}\nu_i + \frac{\varepsilon}{2}\int_{\partial\Omega'}u_\varepsilon^2\nu_i,$$

We have

$$\frac{\Lambda_{j,\varepsilon}^{N-2}}{2}\frac{\partial R(x_{j,\varepsilon})}{\partial x_i}-\sum_{l=1,l\neq j}^{k}\Lambda_{j,\varepsilon}^{(N-2)/2}\Lambda_{l,\varepsilon}^{(N-2)/2}\frac{\partial G(x_{j,\varepsilon},x_{l,\varepsilon})}{\partial x_i}$$

$$= O\Big(\frac{1}{\lambda_\varepsilon^2}\Big).$$

3. Local uniqueness

Theorem 3.1 [Cao-Luo-P, Tran. Amer. Math. Soc., to appear]

Let $N \geq 6$. Assume that $u_\varepsilon^i(x)\,(i = 1, 2)$ are two sequences of solutions of (1) with (3). If $M_k(a^k)$ is a positive matrix, (a^k, Λ^k) is a nondegenerate critical point of Ψ_k. Then $u_\varepsilon^1 = u_\varepsilon^2$ for $\varepsilon > 0$ sufficiently small.

Step 1: Blowing up

Assume $u_\varepsilon^{(i)}(x)\,(i=1,2)$ are different. Set

$$\xi_\varepsilon(x)=\frac{u_\varepsilon^{(1)}(x)-u_\varepsilon^{(2)}(x)}{\|u_\varepsilon^{(1)}-u_\varepsilon^{(2)}\|_{L^\infty(\Omega)}}, \tag{8}$$

then $\xi_\varepsilon(x)$ satisfies $\|\xi_\varepsilon\|_{L^\infty(\Omega)}=1$ and

$$-\Delta\xi_\varepsilon(x)=C_\varepsilon(x)\xi_\varepsilon(x)+\varepsilon\xi_\varepsilon(x), \tag{9}$$

where

$$C_\varepsilon(x)=\left(\frac{N+2}{N-2}\right)\int_0^1\Big(tu_\varepsilon^{(1)}(x)+(1-t)u_\varepsilon^{(2)}(x)\Big)^{\frac{4}{N-2}}dt.$$

Let $\xi_{\varepsilon,j}(x) = \xi_\varepsilon(\frac{x}{\lambda^{(1)}_{j,\varepsilon}} + x^{(1)}_{j,\varepsilon})$, then for any $R > 0$,

$$\xi_{\varepsilon,j}(x) \to \sum_{i=0}^{N} c_{j,i}\psi_i(x), \text{ uniformly in } C^1\big(B_R(0)\big), \quad (10)$$

where $c_{j,i}$, $i = 0, 1, \cdots, N$ are some constants and

$$\psi_0(x) = \frac{\partial U_{0,\lambda}(x)}{\partial \lambda}\Big|_{\lambda=1}, \quad \psi_i(x) = \frac{\partial U_{0,1}(x)}{\partial x_i}, \ i = 1, \cdots, N.$$

We need prove $c_{j,i} = 0$.

Step 2: $c_{j,0} = 0$, $j = 1, \cdots, k$.

$$\begin{aligned}
&\frac{1}{2}\int_{\partial\Omega'} \langle \nabla(u_\varepsilon^{(1)} + u_\varepsilon^{(2)}), \nabla\xi_\varepsilon\rangle\langle x - x_{j,\varepsilon}^{(1)}, \nu\rangle \\
&\quad - \int_{\partial\Omega'} \frac{\partial\xi_\varepsilon}{\partial\nu}\langle x - x_{j,\varepsilon}^{(1)}, \nabla u_\varepsilon^{(1)}\rangle + \frac{2-N}{2}\int_{\partial\Omega'} \frac{\partial\xi_\varepsilon}{\partial\nu} u_\varepsilon^{(1)} \\
&\quad - \int_{\partial\Omega'} \frac{\partial u_\varepsilon^{(2)}}{\partial\nu}\langle x - x_{j,\varepsilon}^{(1)}, \nabla\xi_\varepsilon\rangle + \frac{2-N}{2}\int_{\partial\Omega'} \frac{\partial u_\varepsilon^{(2)}}{\partial\nu}\xi_\varepsilon \\
&= \int_{\partial\Omega'} D_\varepsilon(x)\xi_\varepsilon\langle x - x_{j,\varepsilon}^{(1)}, \nu\rangle \\
&\quad + \varepsilon\int_{\partial\Omega'} (u_\varepsilon^{(1)} + u_\varepsilon^{(2)})\xi_\varepsilon\langle x - x_{j,\varepsilon}^{(1)}, \nu\rangle - \varepsilon\int_{\Omega'} (u_\varepsilon^{(1)} + u_\varepsilon^{(2)})\xi_\varepsilon.
\end{aligned}$$

$$
\begin{aligned}
&(N-10)d_{j,\varepsilon}\Big(\frac{R(x^{(1)}_{j,\varepsilon})}{(\lambda^{(1)}_{j,\varepsilon})^{N-2}}-\sum_{l\neq j}^{k}\frac{G(x^{(1)}_{j,\varepsilon},x^{(1)}_{l,\varepsilon})}{(\lambda^{(1)}_{j,\varepsilon})^{(N-2)/2}(\lambda^{(1)}_{l,\varepsilon})^{(N-2)/2}}\Big)\\
&+(N-2)\Big(\frac{d_{j,\varepsilon}R(x^{(1)}_{j,\varepsilon})}{(\lambda^{(1)}_{j,\varepsilon})^{N-2}}-\sum_{l\neq j}^{k}\frac{d_{l,\varepsilon}G(x^{(1)}_{j,\varepsilon},x^{(1)}_{l,\varepsilon})}{(\lambda^{(1)}_{j,\varepsilon})^{(N-2)/2}(\lambda^{(1)}_{l,\varepsilon})^{(N-2)/2}}\Big)\\
=&o\Big(\frac{1}{\bar\lambda_\varepsilon^{3(N-2)/2}}\Big).
\end{aligned}
$$

$$
\Rightarrow\quad \frac{(N-2)A^2c_{j,0}}{8(\lambda^{(1)}_{j,\varepsilon})^{(N-2)/2}}:=d_{j,\varepsilon}=o\Big(\frac{1}{\bar\lambda_\varepsilon^{N-2}}\Big).
$$

Step 3: $c_{j,i} = 0$, $j = 1, \cdots, k$, $i = 1, \cdots, N$.

$$
\begin{aligned}
& -\int_{\partial\Omega'} \frac{\partial \xi_\varepsilon}{\partial \nu} \frac{\partial u_\varepsilon^{(1)}}{\partial x_i} - \int_{\partial\Omega'} \frac{\partial u_\varepsilon^{(2)}}{\partial \nu} \frac{\partial \xi_\varepsilon}{\partial x_i} \\
& + \frac{1}{2} \int_{\partial\Omega'} \big\langle \nabla (u_\varepsilon^{(1)} + u_\varepsilon^{(2)}), \nabla \xi_\varepsilon \big\rangle \nu_i \\
= & \frac{N-2}{2N} \int_{\partial\Omega'} D_\varepsilon(x) \xi_\varepsilon \nu_i + \varepsilon \int_{\partial\Omega'} (u_\varepsilon^{(1)} + u_\varepsilon^{(2)}) \xi_\varepsilon \nu_i,
\end{aligned}
$$

$$\left(D_{xx}^2\Psi_k(x,\lambda)\right)_{(a^k,\Lambda^k)=(\bar{x}_\varepsilon,\bar{\lambda}_\varepsilon)}\widetilde{C}_{\varepsilon,k}=o(1),$$

$$\widetilde{C}_{\varepsilon,k}=\left(c_{\varepsilon,1,1},c_{\varepsilon,1,2}\cdots,c_{\varepsilon,k,N}\right)^T.$$

(a^k,Λ^k) is a nondegenerate critical point of Ψ_k, then

$$c_{\varepsilon,j,i}=o(1),j=1,\cdots,k,\ \ i=1,\cdots,N. \tag{11}$$

Main tasks:

1. The estimate for u_ε^i;
2. The estimate for ξ_ε :

$$\xi_\varepsilon(x) = \sum_{j=1}^{k} A_{\varepsilon,j} G(x_{j,\varepsilon}^{(1)}, x) + \sum_{j=1}^{k}\sum_{i=1}^{N} B_{\varepsilon,j,i}\partial_i G(x_{j,\varepsilon}^{(1)}, x) + \begin{cases} O\Big(\dfrac{\ln \bar\lambda_\varepsilon}{\bar\lambda_\varepsilon^4}\Big), & N=5, \\ O\Big(\dfrac{\ln \bar\lambda_\varepsilon}{\bar\lambda_\varepsilon^N}\Big), & N\geq 6, \end{cases} \text{ in } C^1\Big(\Omega\backslash \bigcup_{j=1}^{k} B_{2d}(x_{j,\varepsilon}^{(1)})\Big)$$

3. $\big|x_{j,\varepsilon}^{(1)} - x_{j,\varepsilon}^{(2)}\big| = o\Big(\dfrac{1}{\lambda_\varepsilon^2}\Big), \big|\lambda_j - \big(\varepsilon^{\frac{1}{N-4}}\lambda_{j,\varepsilon}\big)^{-1}\big| = O\Big(\dfrac{1}{\lambda_\varepsilon^2}\Big)$

4. The number of bubble solutions

$$S_k = \Big\{ (a^k, \Lambda^k) = (a_1, \cdots, a_k, \lambda_1, \cdots, \lambda_k), \\ \nabla_{x,\lambda} \Psi_k(a^k, \Lambda^k) = 0 \Big\}.$$

Theorem 4.1 [Cao-Luo-P, Tran. Amer. Math. Soc., to appear]

Let $N \geq 6$, for any given $(a^k, \Lambda^k) \in S_k$, $M_k(a^k)$ is positive, (a^k, Λ^k) is nondegenerate. Then the number of solutions to (1) satisfying (3) is $\sharp S_k$.

Assumption A: *The problem*

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}},\ u > 0, & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{12}$$

has no solutions.

Denote the largest number of blow-up points by k_0 and define

$$T_k = \Big\{ (a^k, \Lambda^k) \nabla_{x,\lambda} \Psi_k(a^k, \Lambda^k) = 0 \Big\}.$$

Theorem 4.2 [Cao-Luo-P, Tran. Amer. Math. Soc., to appear]

Let $N \geq 6$ and domain Ω satisfies **Assumption A**. For any $k \in [1, k_0]$ and any $(a^k, \Lambda^k) \in T_k$, suppose that $M_k(a^k)$ is positive, (a^k, Λ^k) is nondegenerate. Then

$$\text{the number of solutions to (1)} \quad = \sum_{k=1}^{k_0} \sharp T_k.$$

Remark: For the case $N = 4, 5$, our argument does not work !

5. A uniqueness result of bubble solutions

$$
\begin{cases} -\Delta u = u^{2^*-1}, \quad u > 0 & in \;\; \Omega_\varepsilon, \\ u = 0, & on \;\; \partial\Omega_\varepsilon, \end{cases} \tag{13}
$$

D is star-shaped with respect to some point $x_0 \in D$,

$\Omega_\varepsilon = D \setminus B_\varepsilon(p)$, and $p \in D$.

Let u_ε be a solution, satisfying

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \le C, \tag{14}$$

as $\varepsilon \to 0$. Then $u_\varepsilon \rightharpoonup 0$, since

$$-\Delta u = u^{2^*-1}, \ u \ge 0 \ \ in \ \ D, \quad u = 0, \qquad on \ \ \partial D,$$

has only zero solution. Therefore, standard blow-up argument yields to

$$I_\varepsilon(u_\varepsilon) \to \frac{k}{N} S^{\frac{N}{2}}, \quad \text{as } \varepsilon \to 0, \tag{15}$$

for some positive integer k, where

$$I_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 - \frac{1}{2^*}\int_{\Omega_\varepsilon} |u|^{2^*}.$$

By Coron's result, if $\varepsilon > 0$ is small enough, (13) has a solution u_ε, satisfying

$$I_\varepsilon(u_\varepsilon) \in \Big(\frac{1}{N}S^{\frac{N}{2}},\ \frac{2}{N}S^{\frac{N}{2}}\Big). \tag{16}$$

Moreover, as $\varepsilon \to 0$, it holds

$$I_\varepsilon(u_\varepsilon) \to \frac{1}{N}S^{\frac{N}{2}}. \tag{17}$$

If $\Omega_\varepsilon = B_1(0) \setminus B_\varepsilon(0)$, by the mountain pass lemma in radially symmetric space, (13) has a radially symmetric solution u_ε for any $\varepsilon \in (0, 1)$.

A natural question: any solution u_ε satisfying (14) as $\varepsilon \to 0$ is radial.

Theorem 5.1 [P-Yan,etc, 2020]

If D is star-shaped with respect to some point in D and $p \in D$, then, for $\varepsilon > 0$ small, solution u_ε of (13), satisfying (15) for $k = 1$, is unique.

Theorem 5.2 [P-Yan,etc, 2020]

D is bounded convex domain in $\mathbb{R}^N$ with smooth boundary. Let $p \in D$ be the unique minimum point of the Robin function $R(x)$. Then, (13) has no u_ε satisfying (15) for $k \geq 2$.

Theorem 5.3 [P-Yan,etc, 2020]

Under the same conditions as in Theorem 5.2, solution u_ε of (13), satisfying (14), is unique. In particular, if $\Omega_\varepsilon = B_1(0) \setminus B_\varepsilon(0)$, then any solution u_ε of (13), satisfying (14), must be radial.

Outline of the proof:

Step 1: the rough profile of u_ε:

Let u_ε satisfying $I_\varepsilon(u_\varepsilon) \to \frac{k}{N} S^{\frac{N}{2}}$. Take $x_\varepsilon \in D$ and $\mu_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Make the change of variable $y = \mu_\varepsilon(x - x_\varepsilon)$, then the domain $\Omega_{\varepsilon, x_\varepsilon, \mu_\varepsilon} := \left\{ y : \ \mu_\varepsilon^{-1} y + x_\varepsilon \in \Omega_\varepsilon \right\}$ approaches to either $\mathbb{R}^N$, or $\mathbb{R}^N_+$ or $\mathbb{R}^N \setminus B_\delta(x_0)$ for some $x_0 \in \mathbb{R}^N$ and $\delta > 0$. Moreover, as $\varepsilon \to 0$, $\mu_\varepsilon^{-\frac{N-2}{2}} u_\varepsilon(\mu_\varepsilon^{-1} y + x_\varepsilon) \to u$ weakly in H^1, and u is a non-negative solution of

$$-\Delta u = u^{2^*-1}.$$

The problem

$$-\Delta u = u^{2^*-1}, \quad u > 0, \quad u \in H_0^1(\Omega),$$

has no solution if $\Omega = \mathbb{R}^N_+$, or $\Omega = \mathbb{R}^N \setminus B_\delta(x_0)$, we find that $\Omega_{\varepsilon, x_\varepsilon, \mu_\varepsilon}$ must approach to $\mathbb{R}^N$ as $\varepsilon \to 0$. Hence

$$\mu_\varepsilon d(x_\varepsilon, \partial D) \to +\infty, \tag{18}$$

and

$$\mu_\varepsilon \varepsilon \to 0, \quad \text{if } \mu_\varepsilon d(x_\varepsilon, \partial B_\varepsilon(0)) \to t < \infty. \tag{19}$$

$$u_\varepsilon = \sum_{j=1}^{k} P_\varepsilon U_{\bar{x}_{\varepsilon,j},\bar{\mu}_{\varepsilon,j}} + \bar{\omega}_\varepsilon,$$

where $\bar{x}_{\varepsilon,j} \in D$, $\bar{\mu}_{\varepsilon,j} > 0$, and as $\varepsilon \to 0$, $\bar{\mu}_{\varepsilon,j} \to +\infty$,

$$\bar{\varepsilon}_{ij} := \left(\frac{1}{\frac{\bar{\mu}_{\varepsilon,j}}{\bar{\mu}_{\varepsilon,i}} + \frac{\bar{\mu}_{\varepsilon,i}}{\bar{\mu}_{\varepsilon,j}} + \bar{\mu}_{\varepsilon,i}\bar{\mu}_{\varepsilon,j}|\bar{x}_{\varepsilon,i} - \bar{x}_{\varepsilon,j}|^2} \right)^{\frac{N-2}{2}} \to 0.$$

$$J_0 = \{j : \ (\bar{x}_{\varepsilon,j}, \bar{\mu}_{\varepsilon,j}) : \bar{x}_{\varepsilon,j} \in D, \ d(\bar{x}_{\varepsilon,j}, \partial B_\varepsilon(0))\bar{\mu}_{\varepsilon,j} \le C\},$$

$$J_1 = \{j : \ (\bar{x}_{\varepsilon,j}, \bar{\mu}_{\varepsilon,j}) : \quad \bar{x}_{\varepsilon,j} \in \Omega_\varepsilon, \ \bar{x}_{\varepsilon,j} \to 0, \quad d(\bar{x}_{\varepsilon,j}, \partial B_\varepsilon(0))\bar{\mu}_{\varepsilon,j} >> 1\},$$

$$J_2 = \{j : \ (\bar{x}_{\varepsilon,j}, \bar{\mu}_{\varepsilon,j}) : \bar{x}_{\varepsilon,j} \in \Omega_\varepsilon, \ \bar{x}_{\varepsilon,j} \to x_j \ne 0\}.$$

Step 2: Adjustment of u_ε:

Consider

$$\inf\Big\{ \quad \|u_\varepsilon - \textstyle\sum_{j=1}^{k} P_\varepsilon U_{x_j,\mu_j}\|^2 : \\ \mu_j \in [c_0\bar{\mu}_{\varepsilon,j}, c_1\bar{\mu}_{\varepsilon,j}], |x_j - \bar{x}_{\varepsilon,j}| \le \frac{1}{c_0^2\bar{\mu}_{\varepsilon,j}}, \;\; j = 1, \cdots k\Big\}.$$

We can write

$$u_\varepsilon = \sum_{j=1}^{k} P_\varepsilon U_{x_{\varepsilon,j},\mu_{\varepsilon,j}} + \omega_\varepsilon, \quad \omega \in E_{x_\varepsilon,\mu_\varepsilon,k}, \tag{20}$$

and $\|\omega\| \to 0$.

The bubbles can be classified into three types:

$$J_0 = \{j : \ (x_{\varepsilon,j}, \mu_{\varepsilon,j}) : x_{\varepsilon,j} \in D, \ d(x_{\varepsilon,j}, \partial B_\varepsilon(P))\mu_{\varepsilon,j} \le C\},$$

$$\begin{aligned} J_1 = \{j : \ (x_{\varepsilon,j}, \mu_{\varepsilon,j}) : \quad & x_{\varepsilon,j} \in \Omega_\varepsilon, \ x_{\varepsilon,j} \to P, \\ & d(x_{\varepsilon,j}, \partial B_\varepsilon(P))\mu_{\varepsilon,j} >> 1\}, \end{aligned}$$

and

$$J_2 = \{j : \ (x_{\varepsilon,j}, \mu_{\varepsilon,j}) : x_{\varepsilon,j} \in \Omega_\varepsilon, \ x_{\varepsilon,j} \to x_j \ne P\}.$$

Step 3: Much more complicated:

If D is star-shaped with respect to some point in D, then, for $\varepsilon > 0$ small, solution u_ε of (13), satisfying (20) for $k = 1$, is unique. If D is convex and $p \in D$ is the unique minimum point of the Robin function $R(x)$, then, for $\varepsilon > 0$ small, (13) has no u_ε satisfying (20) for $k \geq 2$.

Thank you !