

# Transposed Poisson algebras

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arXiv:2005.01110

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Xiamen, October 29, 2020

- What is a transposed Poisson algebra?
- Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras
- Motivation (II): relationships with 3-Lie algebras
- Some open questions

# What is a transposed Poisson algebra?

- Notions.

Poisson algebra:

## Definition

Let  $L$  be a vector space equipped with two bilinear operations

$$\cdot, [ , ] : L \otimes L \rightarrow L.$$

The triple  $(L, \cdot, [ , ])$  is called a **Poisson algebra** if  $(L, \cdot)$  is a commutative associative algebra and  $(L, [ , ])$  is a Lie algebra that satisfy the compatibility condition

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z], \quad \forall x, y, z \in L. \quad (1)$$

Eq. (1) is called the **Leibniz rule** since the adjoint operators of the Lie algebra are derivations of the commutative associative algebra.

# What is a transposed Poisson algebra?

Transposed Poisson algebra:

## Definition

Let  $L$  be a vector space equipped with two bilinear operations

$$\cdot, [ , ] : L \otimes L \rightarrow L.$$

The triple  $(L, \cdot, [ , ])$  is called a **derivation-transposed Poisson algebra** or **transposed Poisson algebra** in short if  $(L, \cdot)$  is a commutative associative algebra and  $(L, [ , ])$  is a Lie algebra that satisfy the following compatibility condition

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y], \quad \forall x, y, z \in L. \quad (2)$$

Eq. (2) is called the **transposed Leibniz rule** because the roles played by the two binary operations in the Leibniz rule in a Poisson algebra are switched. Further, the resulting operation is rescaled by introducing a factor 2 on the left hand side.

# What is a transposed Poisson algebra?

Similar structure:

## Definition

Let  $L$  be a Lie algebra and  $\varphi : L \otimes L \rightarrow L$  be a bilinear operation. If  $\varphi$  satisfies

$$\varphi([x, y], z) = [x, \varphi(y, z)] + [\varphi(x, z), y], \quad (3)$$

$$\varphi(x, [y, z]) = [\varphi(x, y), z] + [y, \varphi(x, z)], \quad (4)$$

for all  $x, y, z \in L$ , then  $\varphi$  is called a **biderivation** on  $L$ .

If in addition,  $\varphi$  defines a commutative associative algebra on  $L$ , then it is biderivation on  $L$  if and only if

$$z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y], \quad \forall x, y, z \in L. \quad (5)$$

# What is a transposed Poisson algebra?

- Typical examples.

Poisson algebras;

## Example

Let  $(L, \cdot)$  be a commutative associative algebra and  $D_1, D_2$  be commuting derivations (that is,  $D_1D_2 = D_2D_1$ ). Then there is a Lie algebra  $(L, [ , ])$  defined by

$$[x, y] = D_1(x) \cdot D_2(y) - D_1(y) \cdot D_2(x), \quad \forall x, y \in L. \quad (6)$$

Moreover,  $(L, \cdot, [ , ])$  is a Poisson algebra.

# What is a transposed Poisson algebra?

Transposed Poisson algebras:

## Example

Let  $(L, \cdot)$  be a commutative associative algebra and let  $D$  be a derivation. Define the multiplication

$$[x, y] := x \cdot D(y) - D(x) \cdot y, \quad \forall x, y \in L. \quad (7)$$

Then  $(L, \cdot, [ , ])$  is a transposed Poisson algebra.

# What is a transposed Poisson algebra?

- Some basic properties.

Poisson algebras:

## Proposition

*Let  $(L_1, \cdot_1, [ , ]_1)$  and  $(L_2, \cdot_2, [ , ]_2)$  be two Poisson algebras. Define two operations  $\cdot$  and  $[ , ]$  on  $L_1 \otimes L_2$  by*

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = x_1 \cdot_1 y_1 \otimes x_2 \cdot_2 y_2, \quad (8)$$

$$[x_1 \otimes x_2, y_1 \otimes y_2] = [x_1, y_1]_1 \otimes x_2 \cdot_2 y_2 + x_1 \cdot_1 y_1 \otimes [x_2, y_2]_2, \quad (9)$$

*for all  $x_1, y_1 \in L_1, x_2, y_2 \in L_2$ . Then  $(L_1 \otimes L_2, \cdot, [ , ])$  is a Poisson algebra.*

# What is a transposed Poisson algebra?

Transposed Poisson algebras:

## Theorem

*Let  $(L_1, \cdot_1, [ , ]_1)$  and  $(L_2, \cdot_2, [ , ]_2)$  be two transposed Poisson algebras. Define two operations  $\cdot$  and  $[ , ]$  on  $L_1 \otimes L_2$  by Eqs. (8) and (9) respectively. Then  $(L_1 \otimes L_2, \cdot, [ , ])$  is a transposed Poisson algebra.*

# What is a transposed Poisson algebra?

The operad of Poisson algebras is self dual.

## Proposition

*The operad of transposed Poisson algebras is self dual.*

# What is a transposed Poisson algebra?

The compatibility relations of the Poisson algebra and those of the transposed Poisson algebra are independent in the following sense.

## Proposition

*Let  $(L, \cdot)$  be a commutative associative algebra and  $(L, [ , ])$  be a Lie algebra. Then  $(L, \cdot, [ , ])$  is both a Poisson algebra and a transposed Poisson algebra if and only if*

$$x[y, z] = [xy, z] = 0, \quad \forall x, y, z \in L. \quad (10)$$

# What is a transposed Poisson algebra?

- Identities in transposed Poisson algebras

## Theorem

*Let  $(L, \cdot, [ , ])$  be a transposed Poisson algebra algebra. Then the following identities hold.*

$$x[y, z] + y[z, x] + z[x, y] = 0, \quad (11)$$

$$[h[x, y], z] + [h[y, z], x] + [h[z, x], y] = 0, \quad (12)$$

$$[hx, [y, z]] + [hy, [z, x]] + [hz, [x, y]] = 0, \quad (13)$$

$$[h, x][y, z] + [h, y][z, x] + [h, z][x, y] = 0, \quad (14)$$

$$[xu, yv] + [xv, yu] = 2uv[x, y], \quad (15)$$

$$x[u, yv] + v[xy, u] + yu[v, x] = 0, \quad (16)$$

*for all  $x, y, z, h, u, v \in L$ .*

# What is a transposed Poisson algebra?

Applications of identities.

Eq. (12)  $\implies$

## Proposition

*Let  $(L, \cdot, [ , ])$  be a transposed Poisson algebra. For any  $h \in L$ , define a new bilinear operation  $[ , ]_h$  on  $L$  by*

$$[x, y]_h = h[x, y], \quad \forall x, y \in L. \quad (17)$$

*Then  $(L, \cdot, [ , ]_h)$  is a transposed Poisson algebra.*

# What is a transposed Poisson algebra?

Recall that a **Hom-Lie algebra** is a triple  $(\mathfrak{g}, [\ , \ ], \varphi)$  consisting of a linear space  $\mathfrak{g}$ , a skew-symmetric bilinear operation  $[\ , \ ] : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$  and a linear map  $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$  satisfying

$$[\varphi(x), [y, z]] + [\varphi(y), [z, x]] + [\varphi(z), [x, y]] = 0, \quad \forall x, y, z \in \mathfrak{g}. \quad (18)$$

If in addition,  $\varphi$  is an algebra homomorphism, then the Hom-Lie algebra  $(\mathfrak{g}, [\ , \ ], \varphi)$  is called **multiplicative**.

# What is a transposed Poisson algebra?

Eq. (13) and Eq. (15)  $\implies$

## Proposition

Let  $(L, \cdot, [ , ])$  be a transposed Poisson algebra. For any  $h \in L$ , define a linear map  $\varphi_h : L \rightarrow L$  by

$$\varphi_h(x) = hx, \quad \forall x \in L. \quad (19)$$

Then  $(L, [ , ], \varphi_h)$  is a Hom-Lie algebra. Moreover,  $\varphi_h$  satisfies

$$\varphi_h^2([x, y]) = [\varphi_h(x), \varphi_h(y)], \quad \forall x, y \in L. \quad (20)$$

Hence if  $\varphi_h^2 = \varphi_h$ , that is,  $hhx = hx$  for all  $x \in L$ , then  $(L, [ , ], \varphi_h)$  is a multiplicative Hom-Lie algebra.

# Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras

## Definition

A **Novikov-Poisson algebra** is a triple  $(L, \cdot, \circ)$ , where  $L$  is a vector space and  $\cdot, \circ$  are two bilinear operations on  $L$  satisfying the following conditions:

- (1)  $(L, \cdot)$  is a commutative associative algebra.
- (2)  $(L, \circ)$  is a Novikov algebra, that is, for all  $x, y, z \in L$ ,

$$(x \circ y) \circ z - (y \circ x) \circ z = x \circ (y \circ z) - y \circ (x \circ z), \quad (21)$$

$$(x \circ y) \circ z = (x \circ z) \circ y. \quad (22)$$

- (3) The following equations hold for all  $x, y, z \in L$ .

$$(xy) \circ z = x(y \circ z), \quad (23)$$

$$(x \circ y)z - (y \circ x)z = x \circ (yz) - y \circ (xz). \quad (24)$$

# Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras

## Theorem

Let  $(L, \cdot, \circ)$  be a Novikov-Poisson algebra. Define

$$[x, y] = x \circ y - y \circ x, \quad \forall x, y \in L. \quad (25)$$

Then  $(L, \cdot, [ , ])$  is a transposed Poisson algebra.

# Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras

## Lemma

Let  $(L, \cdot)$  be a commutative associative algebra and  $D$  be a derivation. Define a bilinear operation  $\circ$  on  $L$  by

$$x \circ y = xD(y), \quad \forall x, y \in L. \quad (26)$$

Then

- 1 (S. Gelfand)  $(L, \circ)$  is a Novikov algebra.
- 2 (Xu)  $(L, \cdot, \circ)$  is a Novikov-Poisson algebra.

# Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras

## Corollary

(=Typical Example) *Let  $(L, \cdot)$  be a commutative associative algebra and  $D$  be a derivation. Then  $(L, \cdot, [ , ])$  is a transposed Poisson algebra, where*

$$[x, y] = xD(y) - yD(x), \quad \forall x, y \in L. \quad (27)$$

# Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras

## Definition

A **Jacobi algebra** (or **generalized Poisson algebra**) is a quadruple  $A = (A, \cdot, [-, -], \partial)$ , where  $(A, \cdot)$  is a commutative and associative algebra,  $(A, [-, -])$  is a Lie algebra, and  $\partial$  is a derivation of  $A$  with respect to the product  $\cdot$  and the bracket  $[-, -]$  such that for any  $a, b, c \in A$ :

$$[ab, c] = a[b, c] + [a, c]b - ab\partial c.$$

## Lemma

Let  $A$  be a commutative associative algebra with a derivation  $\partial$ .

Define

$$a \cdot b = ab, \quad [a, b] = \partial ab - a\partial b, \quad \forall a, b \in A.$$

Then  $(A, \cdot, [-, -], \partial)$  is a Jacobi algebra.

# Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras

## Definition

A **pre-Lie Poisson algebra** is a triple  $(L, \cdot, \circ)$ , where  $L$  is a vector space and  $\cdot, \circ$  are two bilinear operations on  $L$  satisfying the following conditions.

- (1)  $(L, \cdot)$  is a commutative associative algebra.
- (2)  $(L, \circ)$  is a pre-Lie algebra, that is, Eq. (21) holds.
- (3) Eqs. (23) and (24) hold.

# Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras

## Proposition

*Let  $(L, \cdot, \circ)$  be a pre-Lie Poisson algebra. Then  $(L, \cdot, [ , ])$  is a transposed Poisson algebra, where the operation  $[ , ]$  is defined by Eq. (25):*

$$[x, y] = x \circ y - y \circ x, \quad \forall x, y \in L.$$

# Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras

Some related structures:

## Definition

- 1 A **pre-Lie commutative algebra (or PreLie-Com algebra)** is a triple  $(L, \cdot, \circ)$ , where  $(L, \cdot)$  is a commutative associative algebra and  $(L, \circ)$  is a pre-Lie algebra satisfying

$$x \circ (yz) - (x \circ y)z - y(x \circ z) = 0, \quad \forall x, y, z \in L. \quad (28)$$

- 2 A **differential Novikov-Poisson algebra** is a triple  $(L, \cdot, \circ)$ , where  $(L, \cdot)$  is a commutative associative algebra and  $(L, \circ)$  is a Novikov algebra satisfying Eqs. (23), (24) and (28).

# Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras

## Lemma

- 1 *A Novikov-Poisson algebra is a pre-Lie Poisson algebra;*
- 2 *A PreLie-Com algebra  $(L, \cdot, \circ)$  satisfying Eq. (23) is a pre-Lie Poisson algebra;*
- 3 *A differential Novikov-Poisson algebra is a PreLie-Com algebra satisfying Eq. (23) and hence is pre-Lie Poisson algebra.*

# Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras

## Corollary

*Let  $(L, \cdot, \circ)$  be a PreLie-Com algebra. If in addition, Eq. (23) holds, then  $(L, \cdot, [ , ])$  is a transposed Poisson algebra, where the operation  $[ , ]$  is defined by Eq. (25):  $[x, y] := x \circ y - y \circ x$  for all  $x, y \in L$ . In particular, the conclusion holds when  $(L, \cdot, \circ)$  is a differential Novikov-Poisson algebra.*

# Motivation (I): relationships with Novikov-Poisson algebras and pre-Lie Poisson algebras

## Definition

A **Gelfand-Dorfman algebra** is a vector space  $L$  equipped with two binary operations  $\circ$  and  $[\cdot, \cdot]$  such that  $(L, \circ)$  is a Novikov algebra,  $(L, [\cdot, \cdot])$  is a Lie algebra, and such that the two operations  $\circ$  and  $[\cdot, \cdot]$  satisfy the following compatibility relation

$$[a, b] \circ c - [a, c] \circ b + [a \circ b, c] - [a \circ c, b] - a \circ [b, c] = 0, \quad (29)$$

for all  $a, b, c \in L$ .

Note that a commutative associative algebra is a Novikov algebra.

## Corollary

*Any transposed Poisson algebra is a Gelfand-Dorfman algebra.*

## Definition

A **3-Lie algebra** is a vector space  $A$  together with a skew-symmetric linear map (3-Lie bracket)  $[\ , \ ] : \otimes^3 A \rightarrow A$  such that the following **Fundamental Identity (FI)** holds:

$$[[x, y, z], u, v] = [[x, u, v], y, z] + [[y, u, v], z, x] + [[z, u, v], x, y], \quad (30)$$

for  $x, y, z, u, v \in A$ .

## Motivation (II): relationships with 3-Lie algebras

Construction of 3-Lie algebras from strong Poisson algebras"

### Definition

A Poisson algebra  $(L, \cdot, [ , ])$  is called **strong** if

$$[h, x][y, z] + [h, y][z, x] + [h, z][x, y] = 0, \quad \forall x, y, z, h \in L. \quad (31)$$

### Definition

A **Poisson 3-Lie algebra** is a vector space  $L$  with a bilinear operator  $\cdot$  and a ternary operator  $[ , , ]$  such that  $(L, \cdot)$  is a commutative associative algebra,  $(L, [ , , ])$  is a 3-Lie algebra and the following equation holds.

$$[x, y, uv] = u[x, y, v] + [x, y, u]v, \quad \forall x, y, u, v \in L. \quad (32)$$

Thus the adjoint action of  $x, y$  on the commutative associative product is a derivation.

### Proposition

Let  $(L, \cdot, [ , ])$  be a strong Poisson algebra. Suppose that  $D$  is a derivation of  $(L, \cdot)$  and  $(L, [ , ])$ . Define a ternary operation by

$$[x, y, z] := D(x)[y, z] + D(y)[z, x] + D(z)[x, y], \quad \forall x, y, z \in L. \quad (33)$$

Then  $(L, \cdot, [ , , ])$  is a Poisson 3-Lie algebra.

## Motivation (II): relationships with 3-Lie algebras

### Example

Let  $(L, \cdot)$  be a commutative associative algebra and  $D_1, D_2$  be two commuting derivations. Let  $(L, [ , ])$  be the Lie algebra defined by Eq. (6) through  $D_1, D_2$ . Let  $D_3$  be a third derivation of  $(L, \cdot)$  commuting with  $D_1$  and  $D_2$ . It is straightforward to show that  $D_3$  is a derivation of  $(L, [ , ])$ . Hence there is a 3-Lie algebra defined by Eq. (33). Explicitly,

$$[x, y, z] := \det \begin{pmatrix} D_1(x) & D_1(y) & D_1(z) \\ D_2(x) & D_2(y) & D_2(z) \\ D_3(x) & D_3(y) & D_3(z) \end{pmatrix} = (D_1 \wedge D_2 \wedge D_3)(x, y, z), \quad (34)$$

for all  $x, y, z \in L$ .

This is one of the first and also one of the most important examples of 3-Lie algebras.

## Motivation (II): relationships with 3-Lie algebras

Constructions of 3-Lie algebras from transposed Poisson algebras:

### Theorem

Let  $(L, \cdot, [ , ])$  be a transposed Poisson algebra and let  $D$  be a derivation of  $(L, \cdot)$  and  $(L, [ , ])$ . Define a ternary operation on  $L$  by Eq. (33):

$$[x, y, z] := D(x)[y, z] + D(y)[z, x] + D(z)[x, y], \quad x, y, z, \in L. \quad (35)$$

Then  $(L, [ , , ])$  is a 3-Lie algebra.

## Motivation (II): relationships with 3-Lie algebras

### Corollary

Let  $(L, \cdot)$  be a commutative associative algebra and  $D_1, D_2$  be two commuting derivations. Then there exists a 3-Lie algebra defined by

$$[x, y, z] := D_2(x)(yD_1(z) - zD_1(x)) + D_2(y)(zD_1(x) - xD_1(z)) + D_2(z)(xD_1(y) - yD_1(x)), \quad \forall x, y, z \in L. \quad (36)$$

We can rewrite Eq. (36) as

$$[x, y, z] := \det \begin{pmatrix} x & y & z \\ D_1(x) & D_1(y) & D_1(z) \\ D_2(x) & D_2(y) & D_2(z) \end{pmatrix} = (\text{id} \wedge D_1 \wedge D_2)(x, y, z), \quad (37)$$

for all  $x, y, z \in L$ .

## Proposition

Let  $(L, \cdot, [ , ])$  be a transposed Poisson algebra. Suppose that  $D$  is a derivation of  $(L, \cdot)$  and  $(L, [ , ])$ . Then for the 3-Lie algebra defined by Eq. (35), the triple  $(L, \cdot, [ , , ])$  is a **transposed Poisson 3-Lie algebra** in the sense that the following additional equation holds.

$$3u[x, y, z] = [xu, y, z] + [x, yu, z] + [x, y, zu], \quad \forall x, y, z, u \in L. \quad (38)$$

## Definition

Let  $n \geq 2$  be an integer. A **transposed Poisson  $n$ -Lie algebra** is a triple  $(L, \cdot, \mu)$  where  $(L, \cdot)$  is a commutative associative algebra and  $(L, \mu)$  is an  $n$ -Lie algebra satisfying the following condition.

$$nw\mu(x_1, \dots, x_n) = \sum_{i=1}^n \mu(x_1, \dots, wx_i, \dots, x_n), \quad \forall w, x_1, \dots, x_n \in L.$$

## Motivation (II): relationships with 3-Lie algebras

It is reasonable to expect that a transposed Poisson  $(n + 1)$ -algebra can be obtained from a transposed Poisson  $n$ -Lie algebra with a derivation:

### Conjecture

Let  $n \geq 2$  be an integer. Let  $(L, \cdot, \mu_n)$  be a transposed Poisson  $n$ -Lie algebra and let  $D$  be a derivation of  $(L, \cdot)$  and  $(L, \mu_n)$ . Define an  $(n + 1)$ -ary operation

$$\mu_{n+1}(x_1, \dots, x_{n+1}) := \sum_{i=1}^{n+1} D(x_i) \mu_n(x_1, \dots, \check{x}_i, \dots, x_{n+1}),$$

for all  $x_1, \dots, x_{n+1} \in L$ , where  $\check{x}_i$  means that the  $i$ -th entry is omitted. Then  $(L, \cdot, \mu_{n+1})$  is a transposed Poisson  $(n + 1)$ -Lie algebra.

# Motivation (II): relationships with 3-Lie algebras

This conjecture gives a natural interpretation of the scalar 2 in the compatibility condition of a transposed Poisson algebra: it is simply the arity of the operation of the Lie algebra.

## Motivation (II): relationships with 3-Lie algebras

Another construction of 3-Lie algebras from transposed Poisson algebras:

### Theorem

*Let  $(L, \cdot, [ , ])$  be a transposed Poisson algebra. Let  $f$  be an endomorphism of  $(L, \cdot)$  satisfying  $f^2 = \text{id}$ , that is  $f$  is an involutive endomorphism of  $(L, \cdot)$  and*

$$f([x, y]) = -[f(x), f(y)], \quad \forall x, y \in L. \quad (39)$$

*Define a ternary operation on  $L$  by*

$$[x, y, z] := f(x)[y, z] + f(y)[z, x] + f(z)[x, y], \quad \forall x, y, z \in L. \quad (40)$$

*Then  $(L, [ , , ])$  is a 3-Lie algebra.*

## Corollary

Let  $(L, \cdot)$  be a commutative associative algebra and  $D$  be derivation. Let  $(L, [ , ])$  be the Lie algebra defined by Eq. (26). If  $f$  is an involutive endomorphism and  $Df = -fD$ , then Eq. (39) holds. Hence there is a 3-Lie algebra  $(L, [ , , ])$  defined by Eq. (40). Moreover,

$$\begin{aligned} [x, y, z] &= f(x)[y, z] + f(y)[z, x] + f(x)[y, z] \\ &= \det \begin{pmatrix} D(x) & D(y) & D(z) \\ f(x) & f(y) & f(z) \\ x & y & z \end{pmatrix} \\ &= (D \wedge f \wedge \text{Id})(x, y, z), \quad \forall x, y, z \in L. \end{aligned}$$

# Some open questions

- Free transposed Poisson algebras: It was firstly conjectured by L. Guo that the free transposed Poisson algebras might be “free Lie algebras on free commutative associative algebras”, similar as free Poisson algebras being “free commutative associative algebras on free Lie algebras”. But it is not true by L. Guo.

# Some open questions

- Quantization deformation theory: We have known that
- ① Poisson algebras are the semi-limit of quantization deformation of commutative associative algebras into associative algebras.
- ②  $F$ -manifold algebras are the semi-limit of quantization deformation of commutative associative algebras (or commutative pre-Lie algebras) into pre-Lie algebras.

It is natural to ask **into which algebras, the transposed Poisson algebras are the semi-limit of quantization deformation of commutative algebras (should be the same as or at least included in “commutative associative algebras”)**.

# Some open questions

- The possible geometry. Whether there is the so-called “integration” for transposed Poisson algebras?

**Thank You!**