交通拥堵收费问题的试验-校正方法

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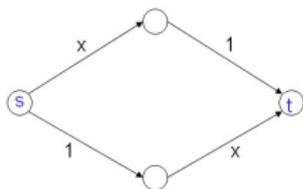
数 学 科 学 学 院

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Outline

UE & SO

- ► Users select routes with least cost
- ► The manager tries to minimize the whole costs of the network
- ► **Different total system cost**

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Braess's Paradox

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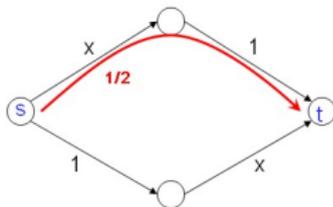

Braess's Paradox

average latency= 1+0.5 =1.5

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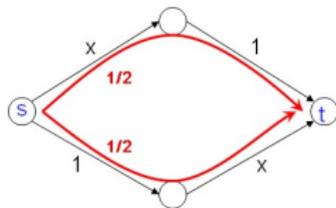
- ► **Different total system cost**
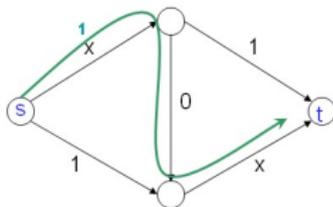
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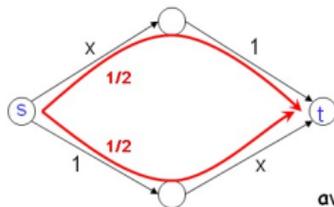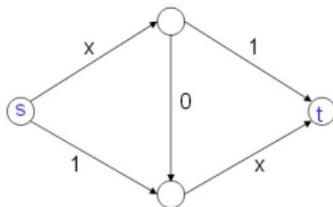**Braess's Paradox**
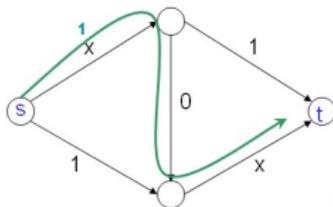
average latency= 1+1 =2

Two Basic Questions

How much?

Two Basic Questions

 How much?

 How to drive UE to SO?

Price of Anarchy

Find the upper bound of the ratio between the system cost occurred at UE and at SO

$$PoA := \frac{\bar{x}^T t(\bar{x})}{\tilde{x}^T t(\tilde{x})} \leq ?$$

- $\bar{x}$: Path flow at UE;
- $\tilde{x}$: Path flow at SO.

Mathematical Model for UE and SO

$G = (V, E)$: a directed network, Δ: edge-path incidence matrix. x: path flow vector f: edge flow vector

$$f = \Delta x, \ x \geq 0.$$

Γ: path-OD pair incidence matrix

$$d = \Gamma x, \ x \geq 0.$$

$$t_p(\bar{x}) \begin{cases} = \mu_w, & \text{if } \bar{x}_p > 0, \\ \geq \mu_w, & \text{if } \bar{x}_p = 0, \end{cases} \quad \forall w \in W,$$

where μ_w is the minimum cost between OD pair $w \in W$.

Mathematical Model for UE and SO

User Equilibrium

$$\begin{cases} t_p(\bar{x}) = \mu_w, & \text{if } \bar{x}_p > 0, \ p \in P, \\ t_p(\bar{x}) \geq \mu_w, & \text{if } \bar{x}_p = 0, \ p \in P, \\ D_w^{-1}(\bar{d}) = \mu_w, & \text{if } \bar{d}_w > 0, \ w \in W, \\ D_w^{-1}(\bar{d}) \geq \bar{\mu}_w, & \text{if } \bar{d}_w = 0, \ w \in W. \end{cases}$$

Variational Inequality Problem:

$$t(\bar{x})^{\top}(x - \bar{x}) - D^{-1}(\bar{d})^{\top}(d - \bar{d}) \geq 0, \ \ \forall (x, d) \in \Omega, \tag{1}$$

System Optimum

$$\min_{(x,d) \in \Omega} \quad t(x)^{\top} x - \sum_{w \in W} \int_0^{d_w} D_w^{-1}(\omega) d\omega \tag{2}$$

Tolled System

Under tolled network, users select routes with least generalized costs

$$\begin{cases} t_p(\bar{x}) + \tau_p = \mu_w, & \text{if } \bar{x}_p > 0, \ p \in P, \\ t_p(\bar{x}) + \tau_p \geq \mu_w, & \text{if } \bar{x}_p = 0, \ p \in P, \\ D_w^{-1}(\bar{d}) = \mu_w, & \text{if } \bar{d}_w > 0, \ w \in W, \\ D_w^{-1}(\bar{d}) \geq \bar{\mu}_w, & \text{if } \bar{d}_w = 0, \ w \in W. \end{cases}$$

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If $\tau = \nabla t(x^*)x^*$, then UE under such toll scheme is x^*, SO.

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If $\tau = \nabla t(x^*)x^*$, then UE under such toll scheme is x^*, SO.

Bad News: Demand functions unknown, impossible to find x^*

Good News: Responses of the users can be observed.

Task

Mathematically, the problem is to design a toll scheme τ, such that the UE under such toll scheme is the SO. Given τ_k, one can get solution $\bar{x}^k$ of the variational inequality problem

$$(t(\bar{x}_k) + \tau_k)^\top (x - \bar{x}_k) - D^{-1}(\bar{d}_k)^\top (d - \bar{d}_k) \geq 0, \quad \forall (x, d) \in \Omega. \tag{4}$$

Then, how to updates τ_{k+1}, such that

$$\lim_{k \to \infty} \bar{x}_k = x^*?$$

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$\{(\tau_k, \bar{x}_k)\} \Longrightarrow D^{-1} \Longrightarrow$ **Solve SO** $\Longrightarrow x^* \Longrightarrow \tau^*$**?**

Method of Successive Averaging (Yang et al 04)

Step 0. Initialization. $\{v_a^0\}$ feasible link flows, $k = 0$.

Step 1. Estimate Link tolls.

$$\tau_k = \nabla t(v^k)v^k.$$

Step 2. Observe link flows $\bar{v}^k$ and set

$$v^{k+1} = v^k + \alpha_k(\bar{v}^k - v^k).$$

Set $k := k + 1$ and go to Step 1.

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► **Convergence condition:** $0 < \alpha_k \leq 1$, $\sum \alpha_k = +\infty$, $\sum \alpha_k^2 < +\infty$.

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- **Convergence condition:** $0 < \alpha_k \leq 1$, $\sum \alpha_k = +\infty$, $\sum \alpha_k^2 < +\infty$.
- **Converges very slowly!**

Enhanced Trial-and-Error (H. & Yang 09)

Step 1. Estimate Link tolls: $\tau_k = \Gamma(v^k) := \nabla t(v^k) v^k$.

Step 2. Observe link flows $\bar{v}^k$ and set

$$v^{k+1} = \max\{v^k - \alpha_k h(v^k), 0\},$$

where

$$h(v^k) = 2\mu(v^k - \bar{v}^k) - (\Gamma(v^k) - \Gamma(\bar{v}^k)),$$

and

$$\alpha_k = \frac{\mu\|v^k - \bar{v}^k\|^2 - (v^k - \bar{v}^k)^\top (\Gamma(v^k) - \Gamma(\bar{v}^k))}{\|h(v^k)\|^2}.$$

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Set $k := k + 1$ and go to Step 1.

Still Slow!

Splitting Method for VI (H., Xu, Yang 08)

Find a vector $x^* \in \Omega$ such that

$$(x - x^*)^\top F(x^*) \geq 0, \quad \forall x \in \Omega.$$

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Available Information: For given x^k, we have $\bar{x}^k$, satisfying

$$f(\bar{x}^k)^\top (x - \bar{x}^k) + g(x^k)^\top (x - \bar{x}^k) \geq 0, \quad \forall x \in \Omega.$$

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Scheme:

$$x^{k+1} + \beta g(x^{k+1}) = x^k + \beta g(x^k) - \alpha(x^k - \bar{x}^k),$$

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Scheme:

$$x^{k+1} + \beta g(x^{k+1}) = x^k + \beta g(x^k) - \alpha(x^k - \bar{x}^k),$$

$$x^{k+1} + \beta g(x^{k+1}) \approx x^k + \beta g(x^k) - \alpha(x^k - \bar{x}^k).$$

Algorithm Description

Step 1. Get $\bar{x}^k$ via

$$f(\bar{x}^k)^\top (x - \bar{x}^k) + g(x^k)^\top (x - \bar{x}^k) \geq 0, \ \ \forall x \in \Omega.$$

Step 2. Choose $1 > \alpha_k \geq \alpha_0$ and solve

$$x^{k+1} + \beta_k g(x^{k+1}) = x^k + \beta_k g(x^k) - \alpha_k (x^k - \bar{x}^k).$$

$$x^{k+1} + \beta g(x^{k+1}) \approx x^k + \beta_k g(x^k) - \alpha (x^k - \bar{x}^k).$$

Step 3. Choose $\beta_{k+1} \in [\frac{1}{1+\tau_k}, (1 + \tau_k)\beta_k]$. Set $k := k + 1$, and go to Step 1.

Convergence

f is strongly monotone with modulus $\mu > 0$ and g is monotone. There is a small constant $\epsilon > 0$ such that $\beta_L \mu (1 - \epsilon) > 1/2$ and $1/5 > \alpha_k \geq \alpha_0 > 0$ for all k.

$$\|x^{k+1} - x^* + \beta_k(g(x^{k+1}) - g(x^*))\|^2$$
$$\leq \left(1 + \frac{4\eta_k^2}{\alpha_k}\right) \|x^k - x^* + \beta_k(g(x^k) - g(x^*))\|^2$$
$$- 2\alpha_k(\beta_k \mu(1 - 8\beta_k\mu\theta^2\eta_k^2) - 1/2)\|\tilde{x}^k - x^*\|^2$$
$$- \alpha_k(1 - 4\theta^2\eta_k^2(\beta_k L)^2)\|x^k - x^*\|^2$$
$$- \left(\frac{1}{4}\alpha_k - \alpha_k^2 - \eta_k^2 - 2\alpha_k\eta_k\right) \|x^k - \tilde{x}^k\|^2.$$

Computational Comparison

β	DPRV		HLW		Han		Alg. 4.1	
	It.	CPU (Sec.)	It.	CPU (Sec.)	It.	CPU (Sec.)	It.	CPU (Sec.)
10^{-5}	—	—	60	21.11	57	11.82	7	1.46
10^{-4}	—	—	48	16.73	46	10.87	7	1.59
10^{-3}	602	422.55	50	16.05	48	12.25	7	2.38
10^{-2}	487	366.36	61	20.96	55	15.03	8	2.11
10^{-1}	765	422.25	56	19.98	54	13.79	10	2.08
1	—	—	61	22.24	59	15.07	12	3.03
10	—	—	69	26.22	68	17.36	15	3.59
100	—	—	76	28.11	74	17.16	18	5.28

'—' means the iteration number > 1000 and CPU time > 500 Sec.

Bounded Flows Control

Sometimes, link flows need to be restrained to satisfy environment capacity constraints.

Mathematically, this is to find $(v^*, d^*) \in S$, such that

$$t(v^*)^\top (v - v^*) - D^{-1}(d^*)^\top (d - d^*) \geq 0, \ \forall (v, d) \in S,$$

$$S = \{(v, d) \in \Omega | v_a \leq C_a, a \in A\},$$

$$\Omega = \{(v, d) | v = \Delta x, \Gamma x = d, x \geq 0\}.$$

If the upper bound C_a, $a \in A$, are roads physical bound, then the situation is just to remove traffic queues by setting tolls equivalent to the observing queuing delay at bottleneck links.

Yang, He, Xu, and Meng 10

Step 0. Initialization. $\{\tau_a^0\}$ initial link tolls, $k = 0$.

Step 1. Observe link flows. v^k.

Step 2. Update toll charges. Set intermediate toll

$$\bar{\tau}^k = \max\{\tau^k - \beta_k(C - \tau^k), 0\},$$

and again observe the revealed link flows $\bar{v}^k$ until

$$\beta_k \|v^k - \bar{v}^k\| \leq \frac{\gamma}{\mu} \|\tau^k - \bar{\tau}^k\|, \gamma \in (0, 1).$$

$$\tau^{k+1} = \max\{\tau^k - \alpha_k\beta_k(C - \bar{\tau}^k), 0\},$$

α_k is the step size.

Step 3. Check convergence. Set $k := k + 1$ and go to Step 1.

Revealed Link Flows

Given τ^k, the revealed link flow v^k is in fact a part of $(v^k, d^k) \in \Omega$, which is the solution of

$$(t(v^k) + \tau^k)^\top (v - v^k) - D^{-1}(d^k)^\top (d - d^k) \geq 0, \ \forall (v, d) \in \Omega.$$

- Why "Exact" revealed link flows?
  - Difficult to have.
  - Loss of efficiency.
- Approximate link flows

Inexact Method: H., Xu and Yang 10

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Step 1. Observe link flows. v^k.

Step 2. Update toll charges. Set intermediate toll

$$\bar{\tau}^k = \max\{\tau^k - \beta_k(C - \tau^k), 0\},$$

and an approximate equilibrium link flows $\bar{v}^k$ until

$$\beta_k \|v^k - \bar{v}^k\| \leq \frac{\gamma}{\mu}\|\tau^k - \bar{\tau}^k\|, \gamma \in (0, 1).$$

$$\tau^{k+1} = \max\{\tau^k - \alpha_k\beta_k(C - \bar{\tau}^k), 0\},$$

α_k is the step size.

Step 3. Check convergence. Set $k := k + 1$ and go to Step 1.

Number of Iterations

Table 4: Comparison of the number of outer iterations

Update scheme	Method	No. of outer iter. for $\varepsilon =$		
		10^{-2}	10^{-3}	10^{-4}
I	Exact	212	315	411
	Inexact	68	91	112
II	Exact	159	232	282
	Inexact	59	75	101

Inner Iterations

Outer iter. No.	Exact method		Inexact method	
	No. of inner trials	Iter. No. of each trial	No. of inner trials	Iter. No. of each trial
1	4	62 127 501 421	5	1 1 3 11 2
21	1	290	1	6

Summary

- In absence of demand function, optimal link tolls can go in a trial-and-error manner.
- How to go in an efficient manner is a problem deserving more research effort.
- How to find second-best tolls in absence of demand functions?
- For lower bounded flow control problems, several researcher suggest to provide subsidies, which, however, seems to be impossible.
    - Is it possible to insist on positive tolls?
    - How?