iQuantum Groups

Xiaomen, 10/8/20 (9AM)

Quantum Groups (1985 –)

—big impacts on rep. theory, quantum topology, math. physics

A Top 10 list

1	Definition	Drinfeld, Jimbo
2	(quasi) R matrix, YB equation	Drinfeld
3	Canonical basis	Lusztig, Kashiwara
4	Canonical basis character formula [of type A]	KL, LLT, Ariki, Brundan, Cheng-Lam-WW
5	Modular RT	Lusztig, AJS, KT, Williamson
6	Geometric RT, flag & quiver varieties	BLM, Nakajima, Vasserot
7	Categorification	KLR
8	Hall algebra	Ringel, Lusztig, Xiao, Bridgeland
9	Braid group action	Lusztig
10	Affine QG	Drinfeld, Chari, Kashiwara, FR, HL

// QG

i-Quantum Groups (iQG) $=$ QG in 21th century!

ALL QG 1–10 have i-counterparts initiated

[much remains to be done]

1 Definition [Gail Letzter '99, Kolb '14, Lu-WW '19]

2-7 iCanonical Basis + ··· Huanchen Bao, WW + Collaborators [2013-]
(鲍焕辰) + Kolb

8 iHall algebra

9 Braid group action on iQG

10 Affine iQG

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Goal: (Introduction to iQG)

Serre/Drinfeld type presentations for [affine] rank 1 or 2 iQG

QG $U=U_q(\mathfrak{g})$

e.g.

$$\begin{tikzcd} U=U_q(\mathfrak{sl}_2)=\langle E,F,K,K^{-1}\rangle & K^{-1} \\ \tilde{U}=\tilde{U}_q(\mathfrak{sl}_2)=\langle E,F,K,K'\rangle \arrow[u] & K' \arrow[u, mapsto] \end{tikzcd}$$

Drinfeld double $\quad \boxed{EF-FE=\frac{K-K'}{q-q^{-1}}} \quad KK'$: central

$U, \tilde{U}$ assoc. to (symmetric) Cartan matrix $A=(a_{ij})_{i,j\in I}$

e.g., $A=\begin{pmatrix} 2 & a \\ a & 2 \end{pmatrix}$, $a=-1$: $\mathfrak{sl}_3$, $a=-2$: $\widehat{\mathfrak{sl}}_2$, $a<-2$: KM

$U=\langle E_i, F_i, K_i, K_i^{-1}\rangle_{i\in I}$, $\quad$ Comult. $\Delta: U\to U\otimes U$

q-Serre: $\displaystyle\sum_{r=0}^{1-a}(-1)^r F_i^{(1-a-r)}F_jF_i^{(r)}=0.$

$(i\neq j)$ $\quad a=a_{ij}$ $\qquad$ Here $F_i^{(r)}=\frac{F_i^r}{[r]!}$ divided powers

iQuantum Groups (iQG)

[split] iQG $\widetilde{U}^\imath \leq \widetilde{U}$ subalgebras

$U^\imath \leq U$

[Lu-W'19]

$\widetilde{U}^\imath :=$ subalgebra of $\widetilde{U}$ generated by

$\boxed{B_i = F_i + E_i K_i'}$, $\mathbb{K}_i := K_i K_i'$ $(i \in I)$

central

$$\begin{tikzcd}
\widetilde{U}^\imath \arrow[d] & \mathbb{K}_i \arrow[d, mapsto] \\
U^\imath & \varsigma_i \in \mathbb{Q}(q)
\end{tikzcd}$$

$\leq U$ Parameters

- $(U, U^\imath)$ is a $\overset{\text{(split)}}{\text{quantum symmetric pair}}$ [Letzter]

(i) $\Delta : U^\imath \to U^\imath \otimes U$

(ii) $U^\imath$ is a q-deformation of "$U(\mathfrak{g}^\theta)$"

$\mathfrak{g}$: Semisimple Lie$/\mathbb{C}$ $\circlearrowright$ θ : involution

$(\mathfrak{g}, \mathfrak{g}^\theta)$: symmetric pairs $\overset{1:1}{\longleftrightarrow}$ real forms in $\mathfrak{g}$

Examples (1) $(\mathfrak{sl}_n , \mathfrak{so}_n = (\mathfrak{sl}_n)^{\omega = -\mathrm{tr}})$

[quasi-split] $(\omega \otimes 1)\circ \Delta : U \hookrightarrow U \otimes U$

(2) $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}^\Delta) \rightsquigarrow$ QSP $(U \otimes U, U)$

$\updownarrow$

$\mathbb{C}$-Lie group viewed as real group

$\boxed{QG \subseteq iQG}$

I. Serre presentation for iQG

Example split rank 2 of Kac-Moody type: $U^{\imath} = \langle B_1, B_2 \rangle$ $(\varsigma_i = q^{-1})$

[Letzter]
$$\left\{\begin{array}{lll} a_{12} = a = -1 & B_1^2 B_2 - [2] B_1 B_2 B_1 + B_2 B_1^2 = B_2 & (1) \\ \quad = -2 & B_1^3 B_2 - \cdots \quad \cdots & = [2]^2 (B_0 B_1 - B_1 B_0) \\ \quad = -3 & B_1^4 B_2 - \cdots & = \vdots\vdots\vdots \; ! \\ \quad \leq -4 & ??? & \end{array}\right.$$

Theorem 1 [陈新红-卢明-WW, CLW'18] $\forall\ a_{ij} = a \leq 0,\ i \neq j \in I$

i-Serre:
$$\boxed{\sum_{r=0}^{1-a} (-1)^r B_{i,\overline{a}}^{(1-a-r)} B_j B_{i,\overline{0}}^{(r)} = 0}, \quad \overline{a} \in \mathbb{Z}/2\mathbb{Z}$$

This leads to a Serre presentation for (quasi-) split iQG of $\forall$ KM type.

Remark iSerre $\rightsquigarrow$ higher Serre relations for iQG [CLW'20]

$\rightsquigarrow$ braid group action [forthcoming]

What is $B_{\bullet}^{(r)}$?

$\underset{\shortparallel}{\text{i-Divided powers}}$ $B^{(n)}_{odd}, B^{(n)}_{ev} \in \overset{\overset{\mathbb{Q}(q)[B]}{\shortparallel}}{U^{\imath}(sl_2)}$ [Bao-W'13, Berman-W'18]

i-canonical basis of rank 1, depending on parity $\{odd, ev\} = \mathbb{Z}/2\mathbb{Z}$

$$B^{(1)}_{odd} = B, \quad B^{(2)}_{odd} = \frac{B^2-1}{[2]}, \quad B^{(3)}_{odd} = \frac{B(B^2-1)}{[3]!}, \quad B^{(4)}_{odd} = \frac{(B^2-1)(B^2-[3]^2)}{[4]!}, \ \ldots$$

$$B^{(1)}_{ev} = B, \quad B^{(2)}_{ev} = \frac{B^2}{[2]}, \quad B^{(3)}_{ev} = \frac{B(B^2-[2]^2)}{[3]!}, \ \ldots$$

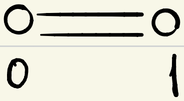
e.g. $a_{12}=-1$: $B_1^2B_2 - [2]B_1B_2B_1 + B_2B_1^2 = B_2$

$\Longleftrightarrow$

$$B^{(2)}_{1,odd}B_2 - B_1B_2B_1 + B_2B^{(2)}_{1,ev} = 0$$

Similarly, for $a_{ij}=-2,-3$, our i-Serre $\Longleftrightarrow$ Letzter's.

II. Affine type (A_1)



affine Lie algebra $\widehat{sl}_2 = \langle e_i, f_i, h_i \mid i=1,2 \rangle$

or loop algebra $Lsl_2 = sl_2 \otimes \mathbb{C}[t^{\pm 1}]$

(ignore the central ext.) $x(n) = x \otimes t^n$

Affine QG $\widetilde{U}$, $U = \langle E_i, F_i, K_i^{\pm 1},\ i=0,1 \rangle$

- U admits a Drinfeld [loop] presentation

$\xrightarrow{\text{Appl.}}$ $\begin{cases} \text{f.-d. RT of } U & \text{Chari, Kashiwara, FR, HL} + \cdots \\ \text{geometric RT} & \text{Nakajima, Vasserot} \end{cases}$

Hall algebra of projective line Kapranov, ...

Affine iQG of type A_1 [= q- Onsager algebra]

$\widetilde{U}^\imath = \langle B_0, B_1, \mathbb{K}_0, \mathbb{K}_1 \rangle$, $\mathbb{K}_i$ Central. $\widetilde{U}^\imath \twoheadrightarrow U^\imath$

rel^n $\begin{cases} B_0^3 B_1 - [3] B_0^2 B_1 B_0 + [3] B_0 B_1 B_0^2 - B_1 B_0^3 = -q^{-1} [2]^2 (B_0 B_1 - B_1 B_0) \mathbb{K}_0 \\ 0 \leftrightarrow 1 \end{cases}$

Classical limit of $U^\imath = (Lsl_2)^{\hat{\omega}}$

basis: $b(r) := f(r) + e(-r), r \in \mathbb{Z}$, $\theta_n = h_n - h_{-n}$ $(n > 0)$

- $\widetilde{U}^\imath$ admits (braid group) automorphisms T_0, T_1 [$\nexists$ restriction from QG]

a diagram autom $\dagger$: $0 \leftrightarrow 1$

- $T_\omega = \dagger \circ T_1$ shift automorphism

- Real root vectors $\quad B(r) := T_\omega^{-r}(B_1), \quad r \in \mathbb{Z}$

- Imaginary root vectors [Baseilhac-Kolb'20] on $\mathcal{U}^\imath$

Θ_n $(n \geq 1)$ is a (lin. comb. of) q-bracket of real root Vectors

Theorem 2 [Lu-W'20] $\widetilde{\mathcal{U}}^\imath$ has a new presentation:

generators : $B(r)$, Θ_n $(r \in \mathbb{Z}, n \geq 1)$, $\mathbb{K}_1$, C $[= \mathbb{K}_0\mathbb{K}_1]$

Set $B(z) = \sum_{r \in \mathbb{Z}} B(r) z^r$, $\Theta(z) = 1 + (q-q^{-1}) \sum_{n \geq 1} \Theta_n z^n$, $\Delta(z) = \sum_{n \in \mathbb{Z}} C^n z^n$

Relations : $\mathbb{K}_1$, C central

$$\Theta(z)\Theta(w) = \Theta(w)\Theta(z) \qquad (1)$$

$$\Theta(z)B(w) = \frac{(1-q^{-2}zw^{-1})(1-q^2zw\overset{0}{\cancel{C}})}{(1-q^2zw^{-1})(1-q^{-2}zw\underset{0}{\cancel{C}})} B(w)\Theta(z) \qquad (2)$$

$$(q^2z-w)B(z)B(w) + (q^2w-z)B(w)B(z) \qquad (3)$$

$$= \frac{q^{-2}\mathbb{K}_1}{q-q^{-1}} \underset{''0}{\Delta(\cancel{z}w)} \left((q^2z-w)\Theta(w) + (q^2w-z)\Theta(z)\right)$$

Remarks:

(a) Set $\Theta(z) = \exp\left(1+(q-q^{-1})\sum_{n \geq 1} H_n z^n\right)$, (2) $\Longleftrightarrow [H_n, B(r)] = \frac{[2n]}{n}\left(B(r+n) - B(r-n)C^n\right)$

(b) setting "$C=0$" above $\rightsquigarrow$ Drinfeld presentation for half of $\widehat{sl}_2$

(c) Thm 2 extends to split ADE type ••• & beyond [Lu-W] & Weinan Zhang

(d) 阮诗佺, 卢明: iHall algebra of weighted projective lines $\approx \widetilde{\mathcal{U}}^\imath$ ADE.