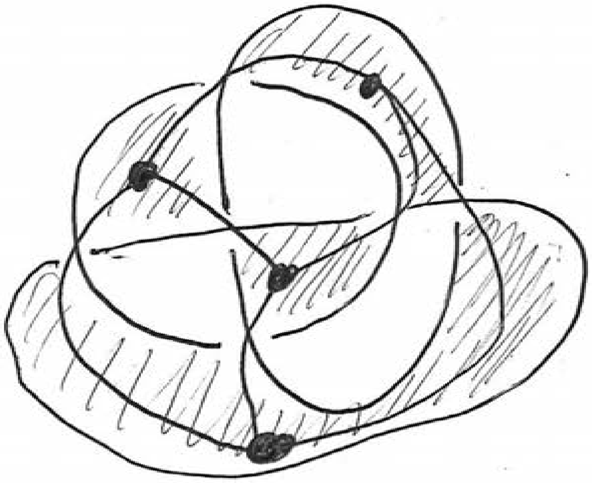
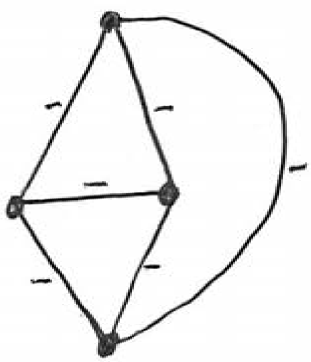


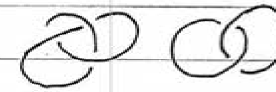


Xiamen – June 2016

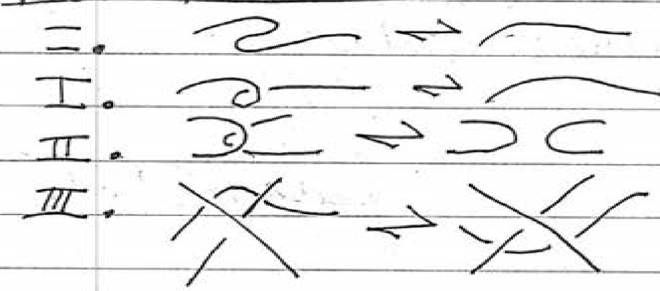
Notes on Combinatorial Knot Theory

Louis H. Kauffman UIC

I. Reidemeister Moves and Generalizations, Graphs and Diagrams

 diagrams

Reid Moves



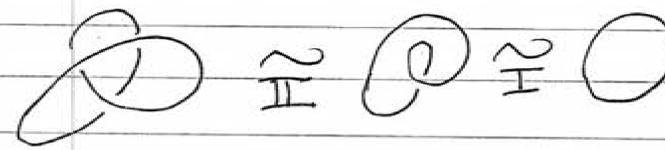
Theorem (Reidemeister).
$K, K' \subset R^3$ links embedded in R^3. D and D′ proj diagrams for K, K′. Then K ambient isotopic to K′ iff $D \underset{RM}{\sim} D'$.

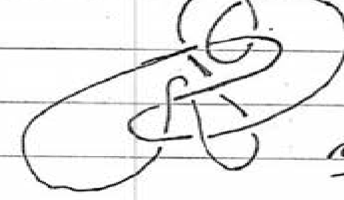
These notes begin with basics of combinatorial knot theory and end in an area of new problems and relationships between virtual knot theory, topological graph theory, knot polynomials and generalized Tutte polynomials.

LK

July 15, 2016

Examples.



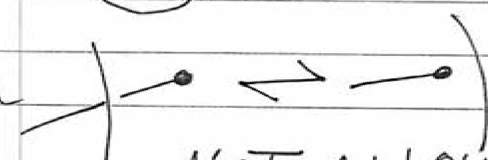


unknotted but needs complexifying moves.

descending diagrams are unknotted

Note. Can use open ended diagrams.

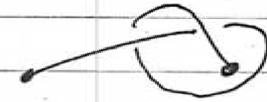
Then



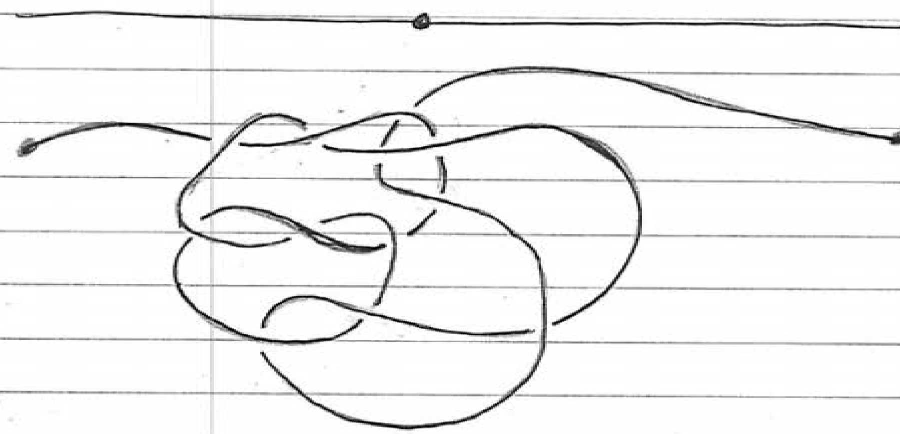
NOT ALLOWED

(2)

We can consider diagrams whose ends are not in same region. These are called knotoids.

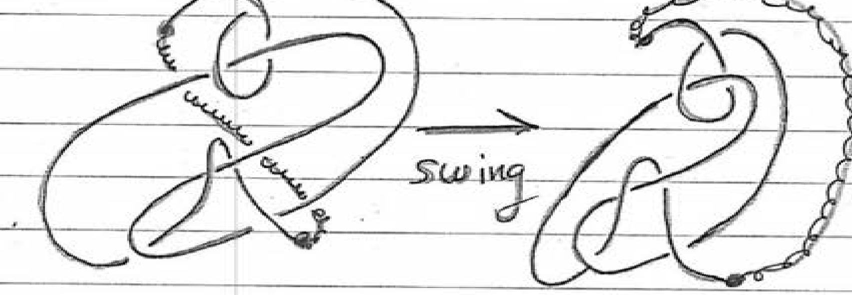


This is an example of a non-trivial knotoid. (We will prove it later).

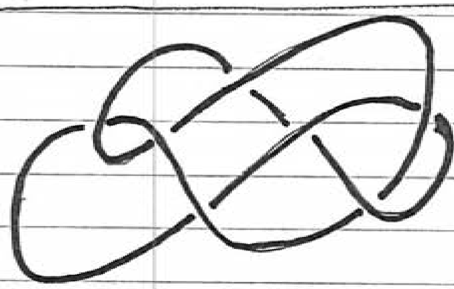


Exercise: Unknot the (standard) Knotoid (above) by using Reidemeister moves.

<u>Hint</u>: It is often useful to use <u>swing moves</u>:

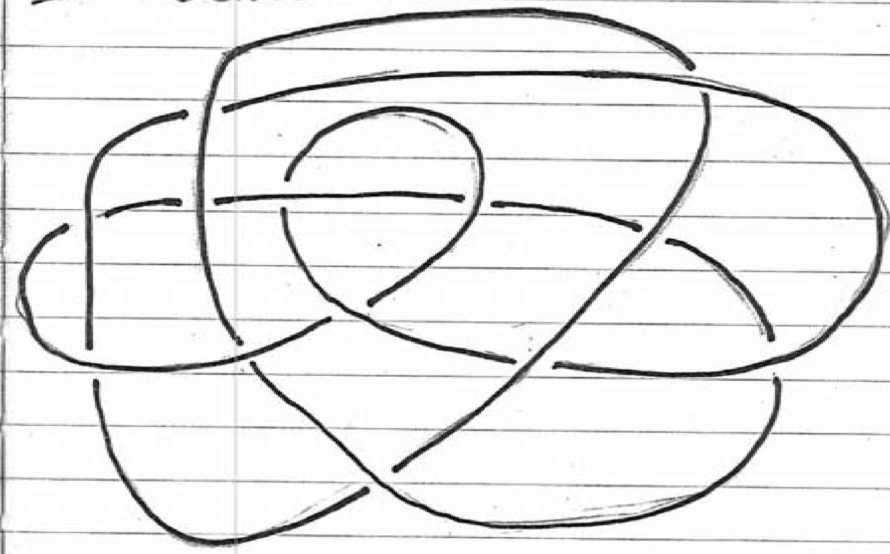


In a swing move we have an arc with all under-crossings or all overcrossings & we move it to a different position. A swing move can be factored into a sequence of Reidemeister moves.

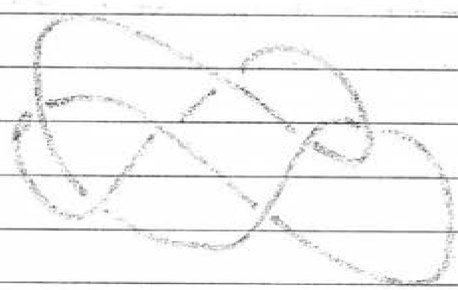


<u>Exercise</u>. Unknot this diagram

Prof. Ochiai's Unknot

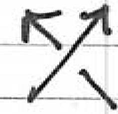


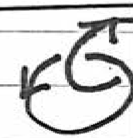
No individual swing move reduces crossing number, but you will find that some combinations of swing moves do reduce crossing number.

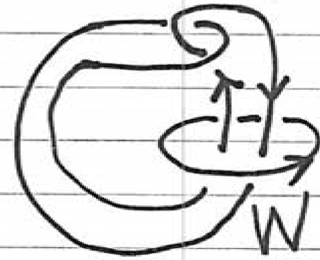


Linking Number (Invar under RM)

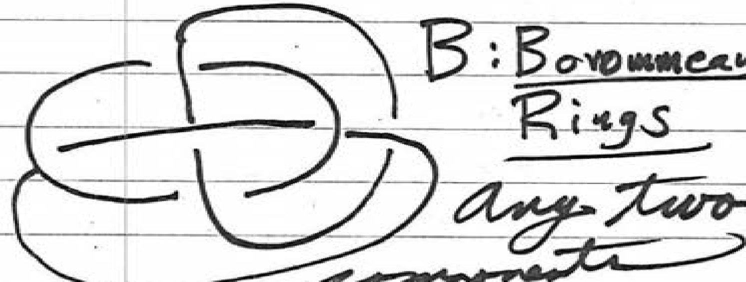
$$Lk(A,B) = \frac{1}{2} \sum_{c \in \text{Inter}(A,B)} \varepsilon(c)$$

 $\varepsilon = +1$  $\varepsilon = -1$

 $lk = +1$  $lk = -1$



$lk = 0$

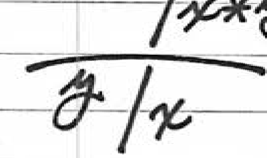


B: <u>Borommean Rings</u>

any two components are unlked.

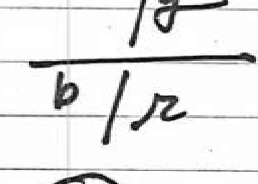
All linky #'s $= 0$.

3-coloring

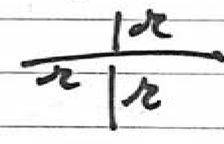


*	r	g	b
r	r	b	g
g	b	g	r
b	g	r	b

e.g.



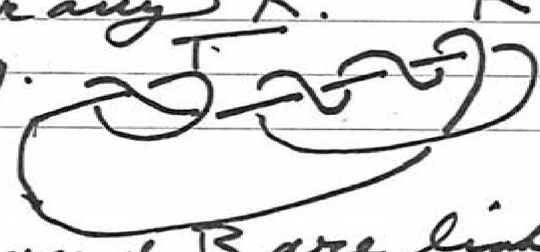
or



(3) ∞ (1)

Check that 3-coloring is preserved under Reidemeister moves.
Use to prove
- trefoil T knotted
- T # K is knotted for any K.

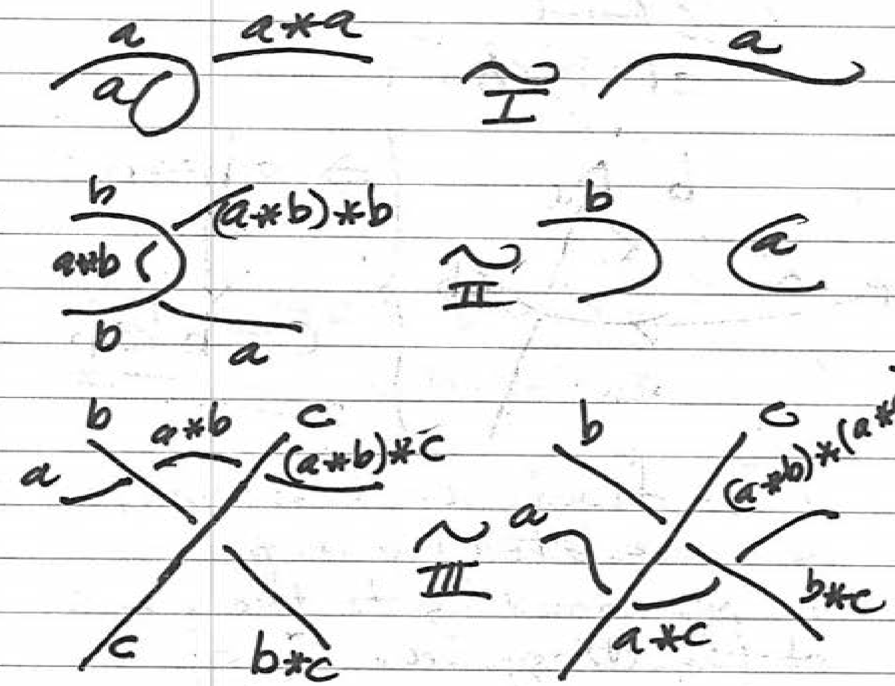
e.g.



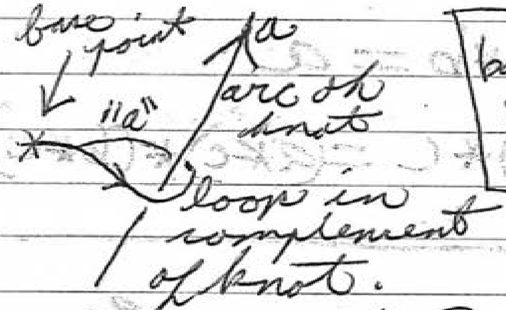
- W and B are linked.

Quandle Axioms

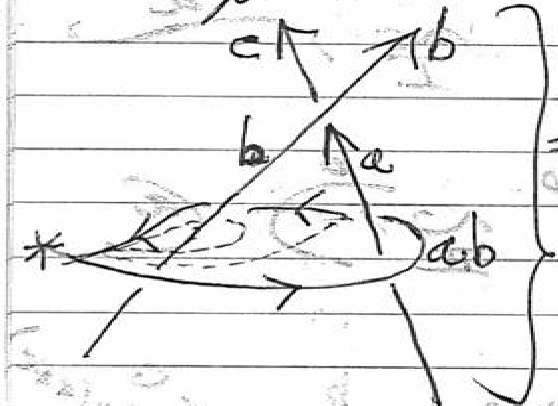
1. $a * a = a$
2. $(a * b) * b = a$
3. $(a * b) * c = (a * c) * (b * c)$



Fundamental Group



basepoint is above the "page"



$\Rightarrow ab = bc$

$\Updownarrow$

$c = b^{-1}ab$

The fundamental group is generated by loops corresponding to the arcs of the diagram. One relation at each crossing gives the generating set of relations.

Examples of Quandles

1) $a * b = 2b - a$ over $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$, n a natural no.

2) G any group, let

$$a * b = b a^{-1} b.$$

Check:

$$a * a = a a^{-1} a = a \checkmark$$

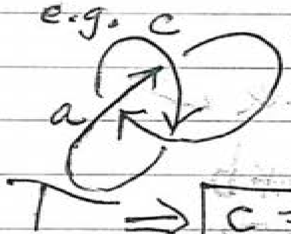
$$(a * b) * b = (b a^{-1} b) * b$$
$$= b (b a^{-1} b)^{-1} b$$
$$= b b^{-1} a b^{-1} b = a \checkmark$$

$$(a * c) * (b * c) = (c a^{-1} c) * (c b^{-1} c)$$
$$= c b^{-1} c \, (c a^{-1} c)^{-1} \, (c b^{-1} c)$$
$$= c b^{-1} c c^{-1} a c^{-1} c b^{-1} c$$
$$= c b^{-1} a b^{-1} c = c \, (b a^{-1} b)^{-1} c$$
$$= (a * b) * c \checkmark$$

Fundamental Group

When $a * b = b^{-1} a b$
$\quad\quad a \bar{*} b = b a b^{-1}$
then the resulting presented group is isomorphic to the fundamental group of the knot (or link) complement. $\boxed{\pi_1(S^3 - K)}$

e.g.



$b = a * c = c^{-1} a c$
$c = b * a = a^{-1} b a$
$a = c * b = b^{-1} c b$

$\Rightarrow \boxed{\begin{array}{l} c = a^{-1}(c^{-1}ac)a \\ a = (c^{-1}ac)^{-1} c (c^{-1}ac) \end{array}}$

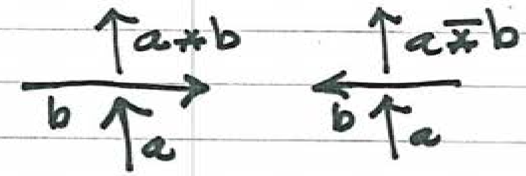
$\Leftrightarrow \begin{array}{l} c = a^{-1}c^{-1}aca \\ a = c^{-1}a^{-1}cac \end{array}$

$\Leftrightarrow \boxed{cac = aca}$

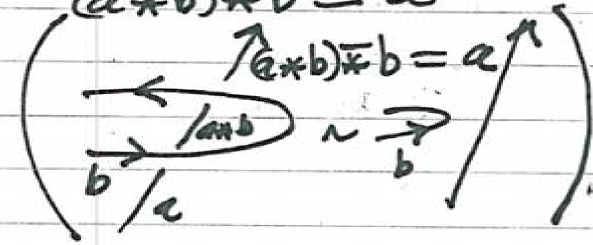
$\pi_1(S^3 - T) \cong (a, c \mid cac = aca)$

Exercise. Prove that this group is non-abelian.
Hint: Represent $\pi_1(S^3 - T)$ to the symmetric group on three letters, S_3.

Oriented Quandle



Then $a * a = a$, $a \bar{*} a = a$
$(a * b) \bar{*} b = a$



$(a \bar{*} b) * b = a$
$(a * b) * c = (a * c) * (b * c)$
$(a \bar{*} b) \bar{*} c = (a \bar{*} c) \bar{*} (b \bar{*} c)$

---

example. G any group

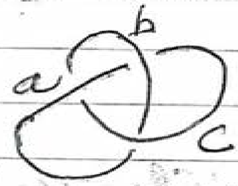
$a * b = b^{-1} a b$
$a \bar{*} b = b a b^{-1}$

---

example. M a module over $\mathbb{Z}[t, t^{-1}]$

$a * b = t a + (1 - t) b$
$a \bar{*} b = t^{-1} a + (1 - t^{-1}) b$
"Alexander Module"

Just using axioms for unoriented quandle:

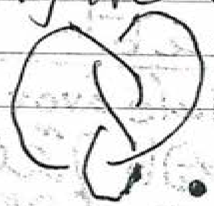


$a * b = c$
$c * a = b$
$b * c = a$

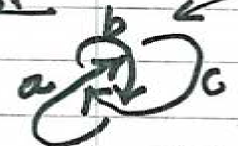
$\Rightarrow a = (a*b)*b = c*b$
$c = (c*a)*a = b*a$
$b = (b*c)*c = a*c$

*	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

Exercise. Work out a five element abstract quandle table for the figure eight knot.



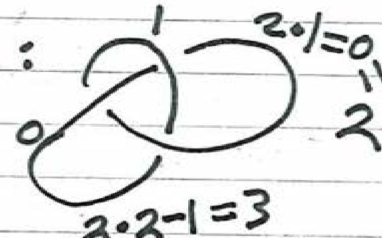
Unoriented Quandle for Trefoil Knot



$a*b = 2b - a$

$2b - a = a*b = c$
$2a - c = c*a = b$
$2c - b = b*c = a$

Try $a=0,\ b=1$:



$\Rightarrow 0 \equiv 3$ so $\mathbb{Z}/3\mathbb{Z}$.

In $\mathbb{Z}/3\mathbb{Z}$.

This is same as our 3-col.

In fact, abstractly.

*	a	b	c
a	a	c	[b]
b	[c]	b	a
c	b	[a]	c

$a*b = c$
$\Rightarrow (a*b)*b = c*b$
$\Rightarrow a = c*b$
& similarly.

Exercise.

Let G be any (multiplicative) group with composition $a, b \mapsto ab$.

Define: $a * b = b a^{-1} b$

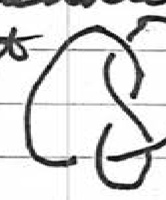
Show that this gives the group the structure of an unoriented quandle.

Note that if you abelianize G (above) then with $[x]$ the equiv class of x in $Abel(G)$ written additively we have

$$[a] * [b] = [b] - [a] + [b]$$
$$= 2[b] - [a].$$

So the trefoil knot has a unique 3-element unoriented quandle + this is our 3-color algebra.

Exercise. Work out unoriented quandle for figure eight knot



. Show it has five elements and that it is isomorphic with the $\mathbb{Z}/5\mathbb{Z}$ quandle.

$a * b = 2b - a$ algebra.
Write all equations.
Get matrix.
Reduce matrix by using one var = ϕ. Take out a row and a column corresp. to this + let resulting matrix be called M_0
(see next page)

1. $b = a * c = 2c - a$
2. $c = b * a = 2a - b$
3. $a = c * b = 2b - c$

	a	b	c
1	−1	−1	2
2	2	−1	−1
3	−1	2	−1

$$M = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}$$

<u>Note</u>: $\mathrm{Det}(M) = 1 + 2 = 3.$

Given $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have
$\tilde{M} = \mathrm{adjoint}(M)$ s.t. $\tilde{M}M = dI$
$d = \mathrm{Det}(M)$. For 2x2
$\tilde{M} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$M\tilde{M} = dI \Rightarrow M\tilde{M} = 0 \bmod d.$

∴ THE COLUMNS OF THE ADJOINT MATRIX OF M are colorings of K modulo $d = \mathrm{Det}(M)$.

<u>Exercise</u>. Do examples!
Show colorings for Fig8 knot.

Oriented Alexander Quandle (14)

Same matrix pattern

but use $a * b = ta + (1-t)b$.

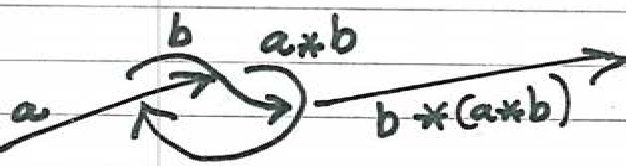
$a \bar{*} b = t^{-1}a + (1-t^{-1})b$.

$\Delta_K(t) \doteq \mathrm{Det}(M)$ is

the Alexander polynomial.

($a \doteq b$ means $a = \pm t^N b$)

This is not Alexander's original definition. See LK "On Knots" or "Knots and Physics" for more details.



$$b*(a*b) = tb + (1-t)(ta + (1-t)b)$$
$$= tb + (t-t^2)a + (1-t)^2 b$$

$a = b*(a*b)$

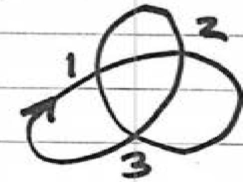
$$\Rightarrow (1 - t + t^2)a = (1 - 2t + t^2 + t)b$$

So need "$(1 - t + t^2) = 0$".

$\Delta_K(t) \doteq t^2 - t + 1$ if you check the determinant.

II. Gauss Codes

A) Bare Gauss Codes



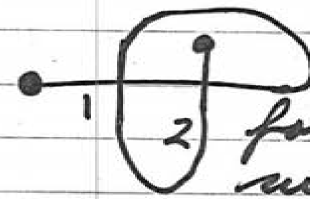
123123

In order for a code to realize in the plane it must be evenly intersticed.

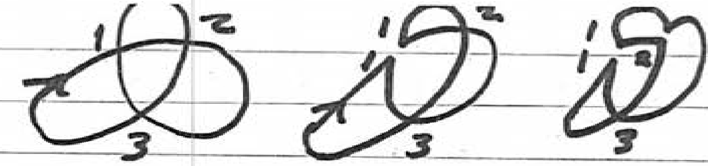
i xyz... w i

(xyz... w: even # of symbols.)

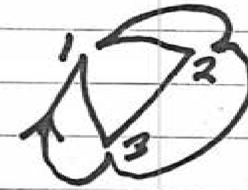
e.g. 1212 is not evenly intersticed

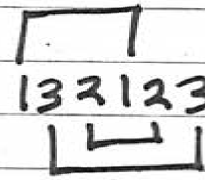


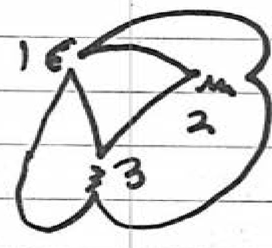
1212 is code for open-ended curve but not for a closed curve.

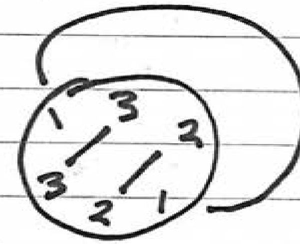


123123   132123   132123









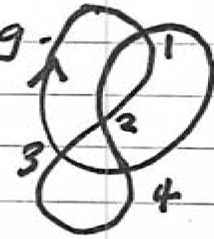
Check planarity by the <u>reveral algorithm</u>

~ i ABC . . . Z i ~

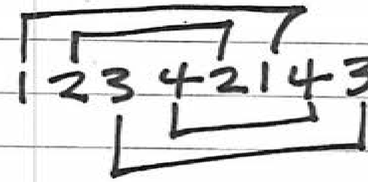
⟼ ~ i Z . . . CBA i ~

Do for $i = 1, 2, \ldots, n$.
Then check to see if chords partition into <u>inside</u> and <u>outside</u> <u>classes</u>.

e.g.



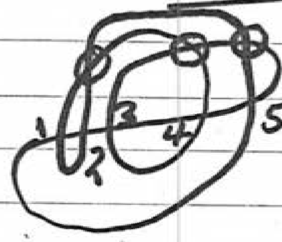
1234 2143
12432 143
13342 143
1234 1243
123421 43



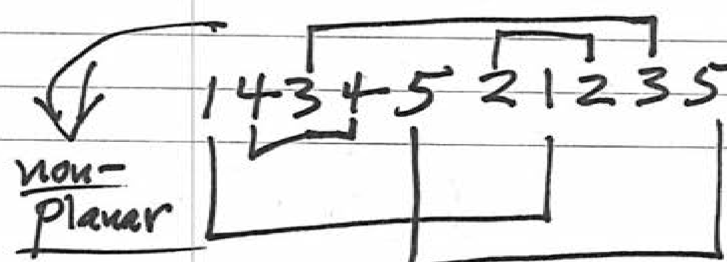
✓

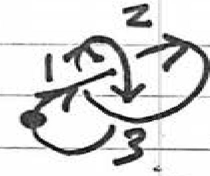
G = 123453 4125

G is evenly interstitched but not planar.



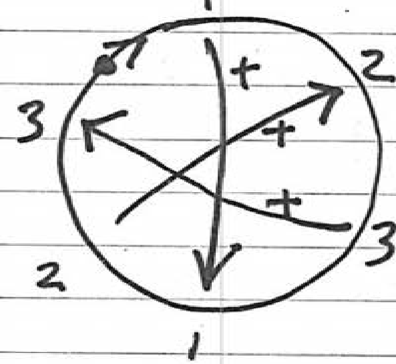
12345 34125
14354 32125
14345 32125
14345 21235



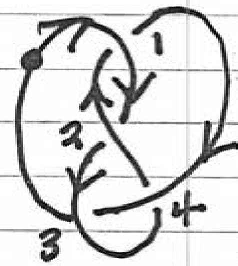


O1+ U2+ O3+
U1+ O2+ U3+

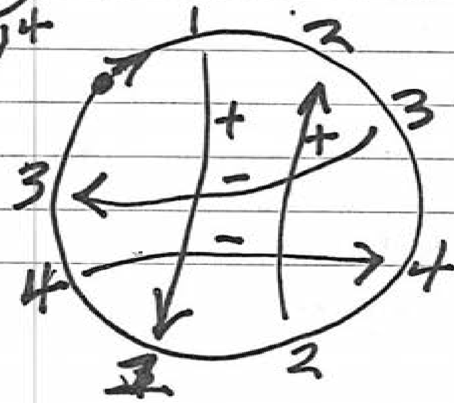
oriented Gauss code.



Gauss Diagram

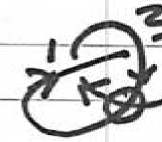


O1+ U2+ O3− U4− O2+
U1+ O4− U3−



III. Virtual Knot Theory

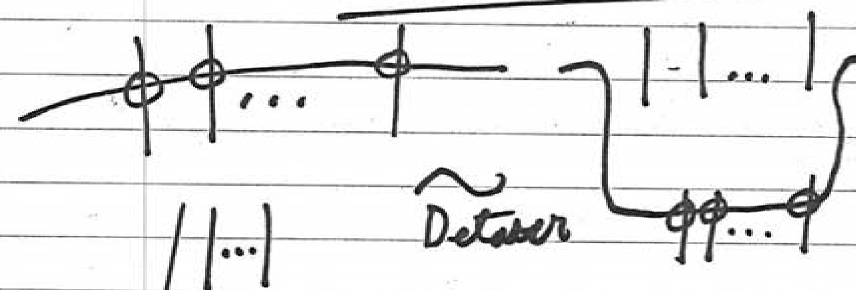
Add virtual crossings ⊗.



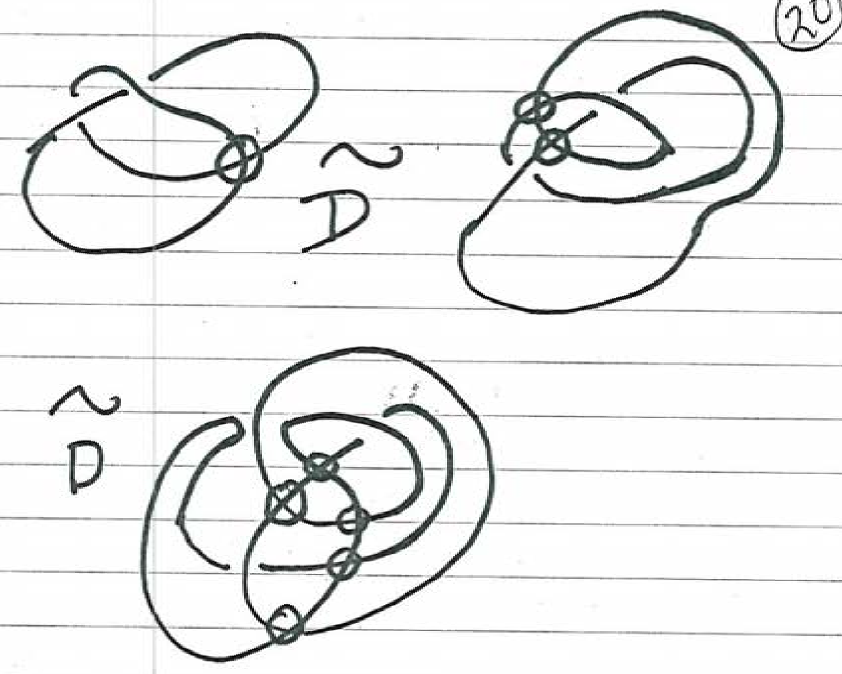
a virtual knot diagram.

Note the non-planar code
O1+U2+U1+O2+
(1212)

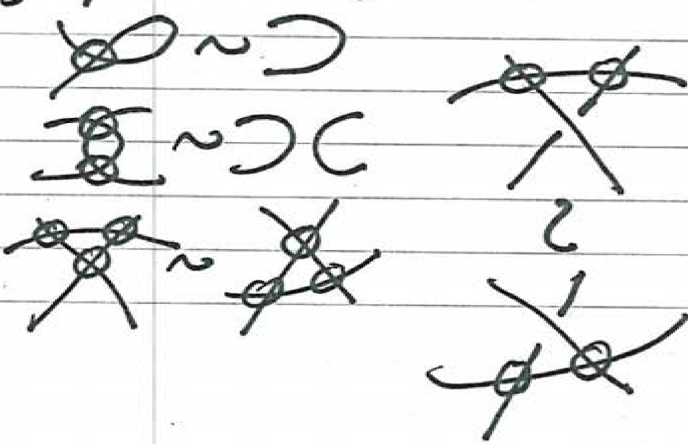
Moves = Reidemeister Moves + Detour Move



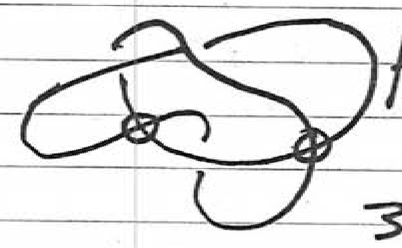
Excise a consequtive sequence of virtual crossings and reconnect elsewhere using virtual crossings.



Detour Move is generated by following local moves type

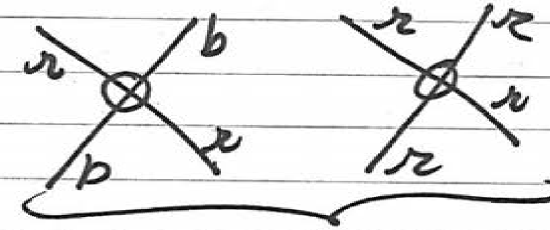


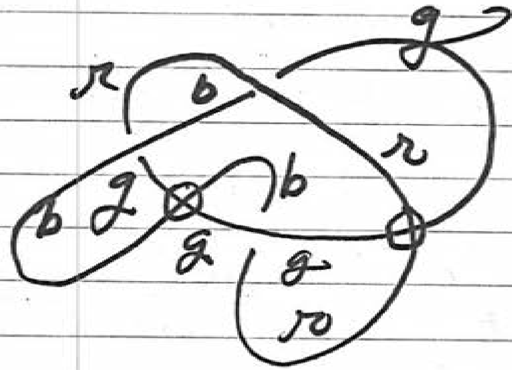
Coloring works for virtual knots. Hence



K is non-trivial since it can be 3-colored.

Note



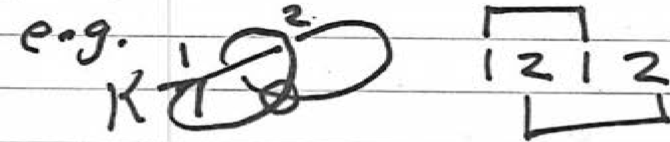


Note.

is only trivially colorable.

<u>Defn.</u> A crossing in a (virtual) knot is <u>odd</u> if in the bare Gauss code it has an odd interstice.

e.g.



Both 1 & 2 are odd.

Define the <u>Odd Writhe</u>

$$J(K) = \sum_{c \text{ an odd crossing}} \varepsilon(c)$$

when K is oriented.

e.g. $J(K) = 2$ above.

<u>Proposition.</u> 1) $J(K)$ is an invariant of virtual knots.

2) $J(K) = 0$ if $K \sim$ classical knot.

3) $J(K^*) = -J(K)$ when K^* = mirror image obtained by reversing all crossings.

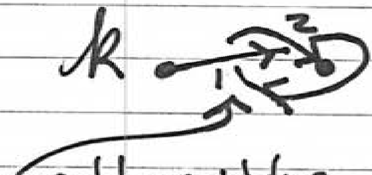
Exercise. Prove this proposition. (h.)

Note that $J(K) = 2$

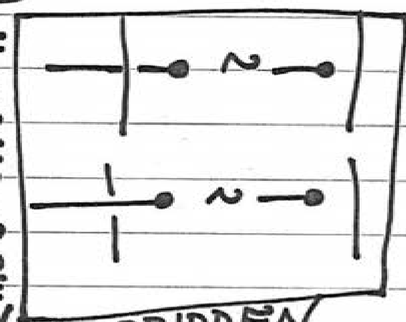
$\Rightarrow$ K is non-trivial.
$K \not\sim K^*$
K is not classical.

Knotoids

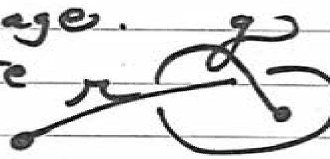
k



odd writhe shows this knotoid is non-trivial and non-classical. And not ~ its mirror image.

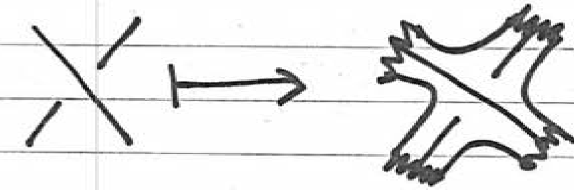


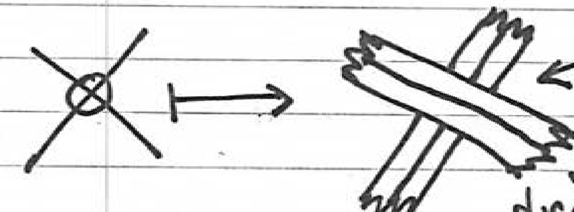
Note

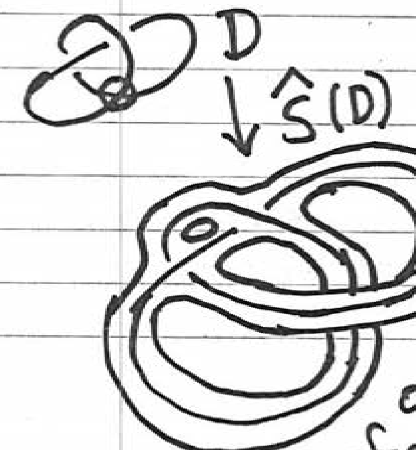


$\Rightarrow$ k is non-trivial.

Given a virtual link diagram D there is a uniquely defined orientable closed surface S(D) on which D is lifted to a diagram without virtual crossings.

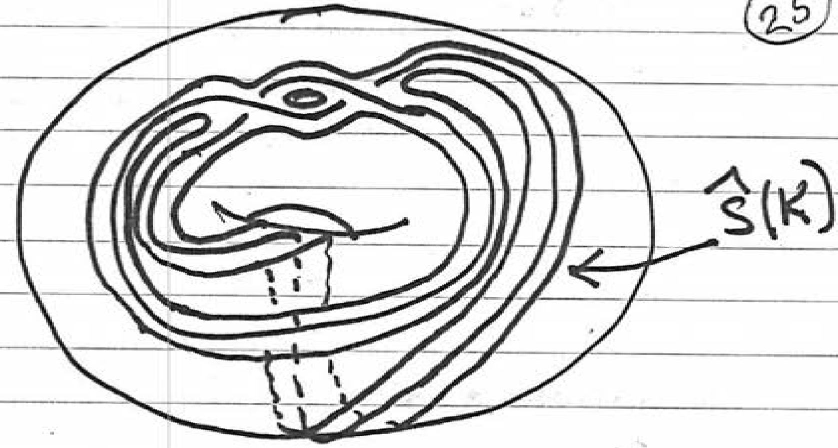


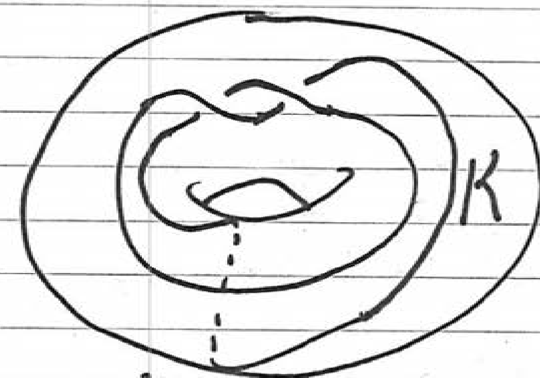




$S(D) = \hat{S}(D) \cup$ <u>discs</u>

Add discs to boundary of $\hat{S}(D)$ to form closed surface.

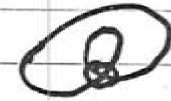




If K is a diagram in a closed surface, then we can construct $\hat{S}(K)$ and, from it, obtain a virtual diagram in the plane.

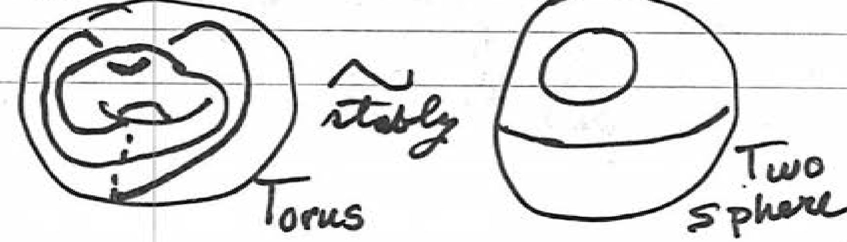
Note that genus($S(K)$) is <u>not</u> invariant under Reid Moves!

e.g.  → genus 1 as above.

 → genus 0.

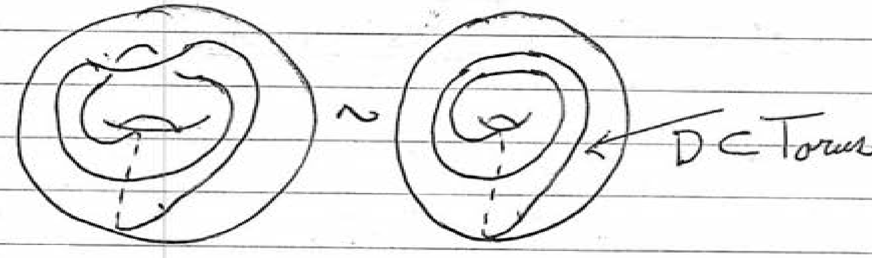
We say $K \subset S$ is <u>stably equiv</u> to $K' \subset S'$ if one can be obtained from the other by RM's on surfaces + <u>forming</u> neighborhoods $\hat{S}(K)$ or $\hat{S}(K')$ and adding boundary surfaces (orientable + not neccessarily disks).

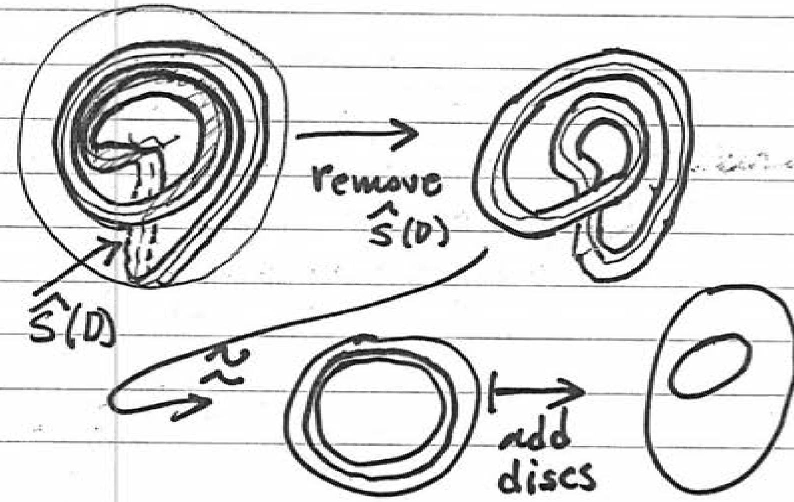
Thus



Thm. Virtual Knot Theory is same as stable classes of link diagrams in orientable surfaces.

Note how the previous example works.





Theorem (Kuperberg). Any virtual knot or link corresponds to a unique linktype in its topologically minimal representing surface.

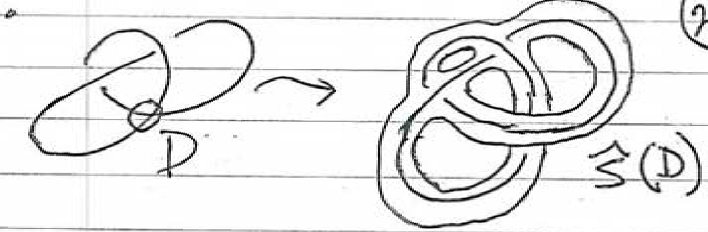
The proof involves elementary surgery techniques in 3 dimensional topology. We interpret a diagram in a surface as a link or knot embedded in the surface $\times [0,1]$. Proof is omitted here. Note also: in principle one can determine the minimal genus surface for a virtual link by 3d techniques but there is no simple combinatorial algorithm known for this.

For a given diagram D, we can determine the genus of $S(D)$. Use Euler characteristic:

$$\chi(F) = 2 - 2g$$
$$\chi(F) = v - e + f$$

where F is a closed surface.

e.g.



The knot embedded in $S(D)$ as a diagram has $v = \#$ crossings, $e = \#$ edges in the diagram ($= 2\#$ crossings)

So $e = 2v = 2c(D) = \#$ crossings.

$f = \#$ 2-cells $= \lambda(D) = \#$ of boundary components of $\hat{S}(D)$.

$$\therefore 2-2g = v - e + f$$

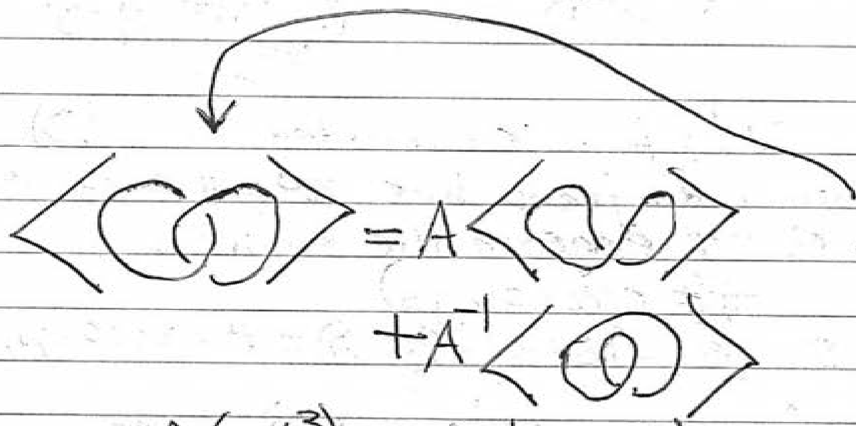
$$2-2g = v - 2v + \lambda$$

$$2-2g = -v + \lambda$$

$$2g = 2 + v - \lambda$$

$$\boxed{g = 1 + \frac{v - \lambda}{2}}$$

Here $g = 1 + \frac{2-2}{2} = 1$ ✓

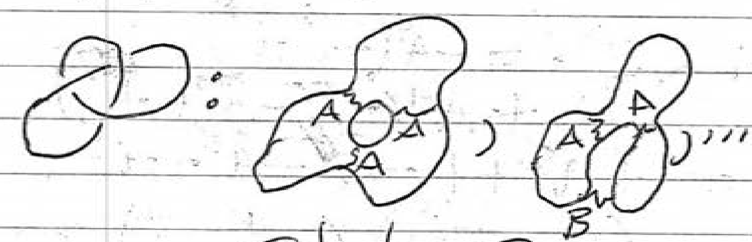


$$= A(-A^3) + A^{-1}(-A^{-3})$$
$$= -A^4 - A^{-4}$$

Since $g(S(D))$ is least genus for the diagram D, we can often guess the genus of a given virtual knot or link.

II. Bracket Polynomial and Jones Polynomial

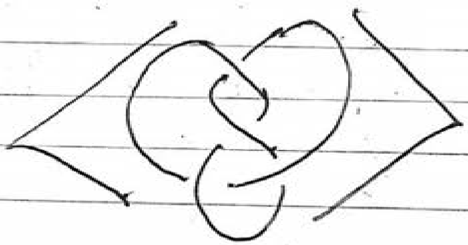
$$\langle OK \rangle = d \langle K \rangle$$
$$\langle O \rangle = 1$$

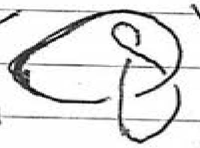
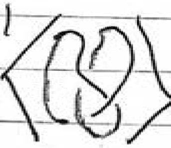


States S.

$$\langle K \rangle = \sum_S \langle K|S \rangle d^{||S||-1}$$

$\langle K|S \rangle$ = product of A's and B's.
$||S|| = \#(\text{loops}) - 1$.

$\langle$  $\rangle = \langle E \rangle$

$= A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$= A(-A^3) \langle$  $\rangle + A^{-1} \langle$  $\rangle$

~~[illegible]~~

In two pages we'll see that

$\langle$  $\rangle = -A^5 - A^{-3} + A^{-7}$

$\therefore \langle E \rangle = -A^4(-A^4 - A^{-4})$
$\quad + A^{-1}(-A^5 - A^{-3} + A^{-7})$

$= +A^8 + 1 - A^{-4} - A^{4} + A^{-8}$

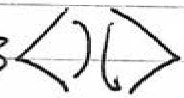
$= (A^8 + A^{-8}) + (-A^4 - A^{-4}) + 1$

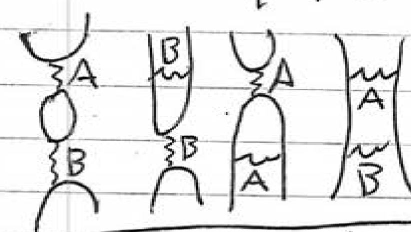
$w(E) = 0$

$\therefore f_E = \langle E \rangle$ ✓

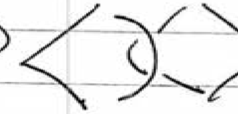
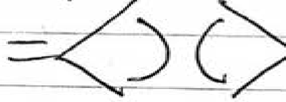
$f_E(A) = f_E(A^{-1})$.

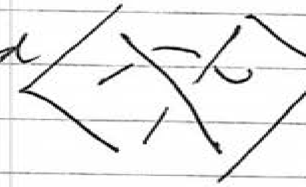
<u>Exercise</u>: Show $E \cong E^*$, its mirror image.

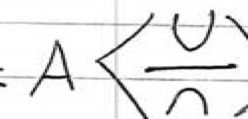
$\langle$  $\rangle = (Ad + A^2 + B^2) \langle$  $\rangle$
$\quad + AB \langle$  $\rangle$.

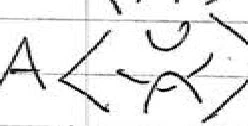
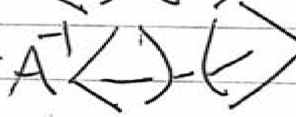


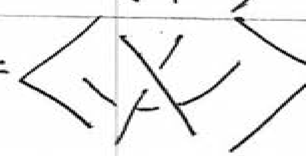
Take $B = A^{-1}$, $d = -A^2 - A^{-2}$

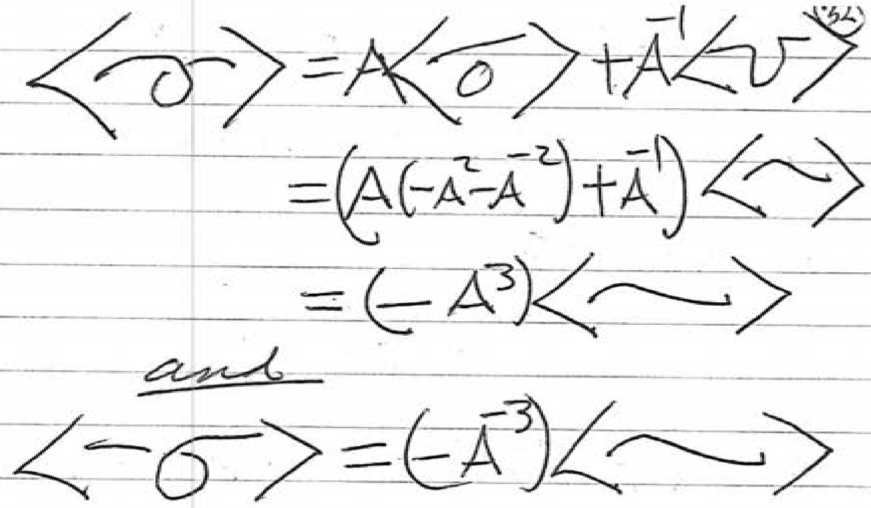
$\Rightarrow \langle$  $\rangle = \langle$  $\rangle$

and $\langle$  $\rangle = A \langle$  $\rangle$
$\quad + A^{-1} \langle$  $\rangle$

$= A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$= A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$= \langle$  $\rangle$.



Normalize:

$$f_K = (-A^3)^{-w(K)} \langle K \rangle$$

$$w(K) = \sum_{c \in \text{Crossings}(K)} \varepsilon(c)$$

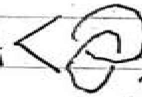
(K oriented)

- f_K invar under RM.
- $f_{K^*}(A) = f_K(A^{-1})$

Conjecture: K a knot, $f_K = 1$

$\Rightarrow K \underset{RM}{\sim} O$.

Calculation $\langle T \rangle$, T = 

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$= A [A \langle$  $\rangle + A^{-1} \langle$  $\rangle]$

$+ A^{-1} \langle$  $\rangle$

$= A^2(-A^3) + (-A^{-3}) + A^{-1}(-A^{-3})^2$

$= -A^5 - A^{-3} + A^{-7}$

$w(T) = +3$ 

$\therefore f_T = (-A^3)^{-3}(-A^5 - A^{-3} + A^{-7})$

$f_T = A^{-4} + A^{-12} - A^{-16}$

Since $f_T(A) \neq f_T(A^{-1})$,

this implies that

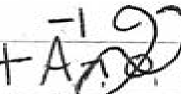
$T \not\cong T^*$

 $\not\cong$  .

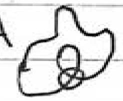
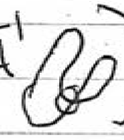
$\langle K \rangle$ defined same way

& f_K for virtual links.

All loops (even with virtual crossings) evaluate to $\delta = -A^2 - A^{-2}$.

e.g.  $\rightarrow A$  $+ A^{-1}$ 

K

$\Rightarrow A \Big(A$  $+ A^{-1}$  $\Big)$

$+ A^{-1}(-A^{-3})$ 

$\therefore \langle K \rangle = A^2 + 1 - A^{-4}$

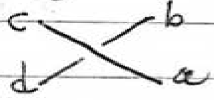
$f_K = (-A^3)^{-2}(A^2 + 1 - A^{-4})$

$f_K = A^{-4} + A^{-6} - A^{-10}$

$\therefore$ K non-trivial and

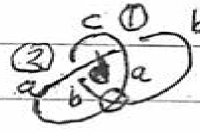
$K \not\cong K^*$ via bracket.

Encoding by Crossing and Edge Information

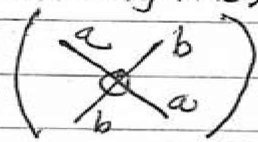


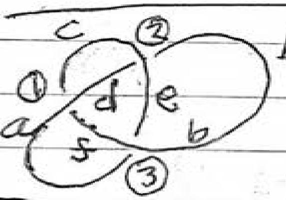
$\leftrightarrow [a,b,c,d]$ or $[c,d,a,b]$

(counterclockwise from overcrossing line)

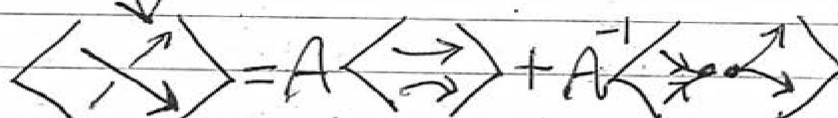


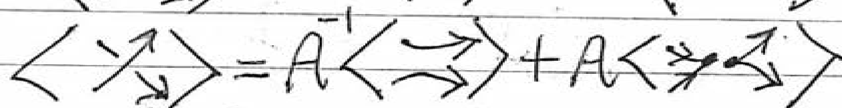
① $[a,b,c,d]$
② $[d,c,a,b]$





① $[d,c,a,f]$
② $[c,d,e,b]$
③ $[b,e,f,a]$





$\delta = O = -A^{2} - A^{-2}$

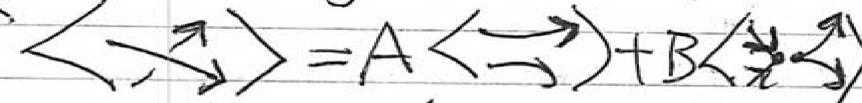
This method of encoding unoriented knots, links and virtual links is complete.

Oriented Diagrams & Arrow Poly

we can ignore this disorientation, or analyze it.
Here we analyze it, by allowing state diagrams to have cusps

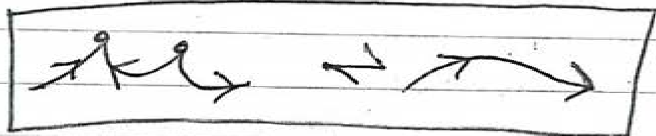
 and  and redoing our analysis of the Reidemeister moves.

So start again.

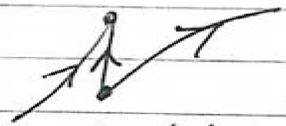


But we steer a midle course, taking $A' = A^{-1}$, $B' = A$.
$\delta = -A^{2} - A^{-2}$ as usual. $B = A^{-1}$

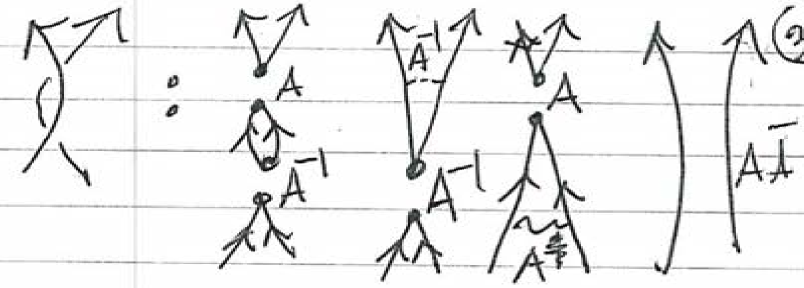
We explore what rules are needed for invariance and find

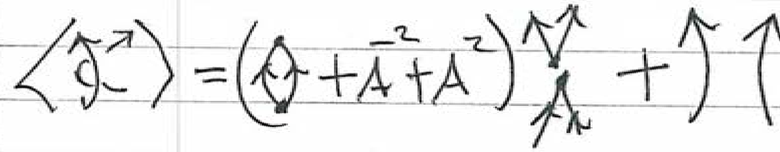


but the zig-zag can persist!

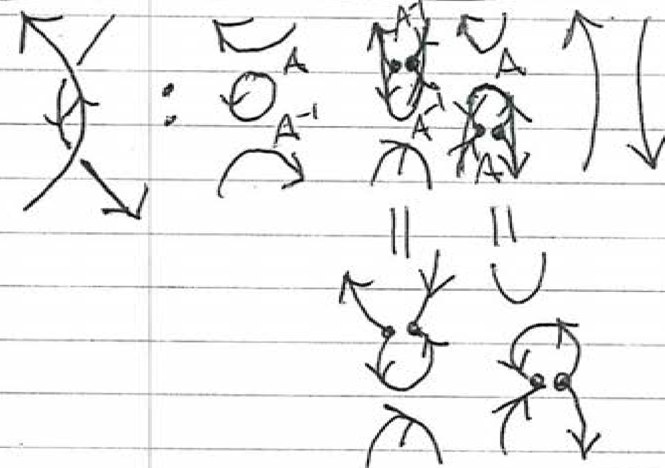


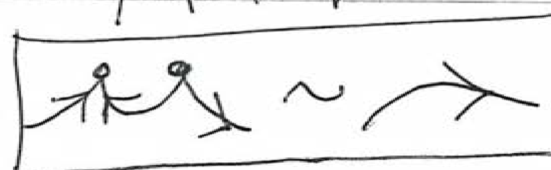
For virtual knots this results in potentially infinitely many new variables in the resulting generalization of the bracket polynomial.



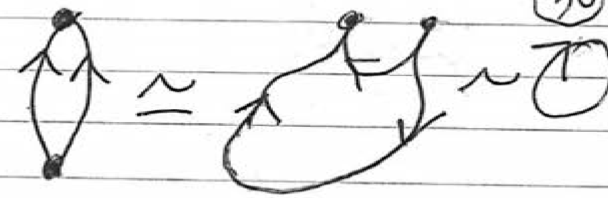


∴ Need  $= \delta = -A^{2} - A^{-2}$

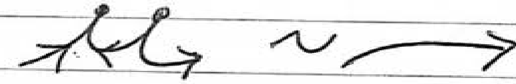


Need 

Note:

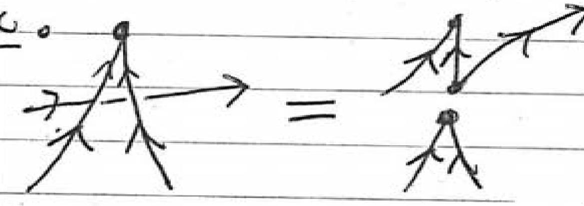


via

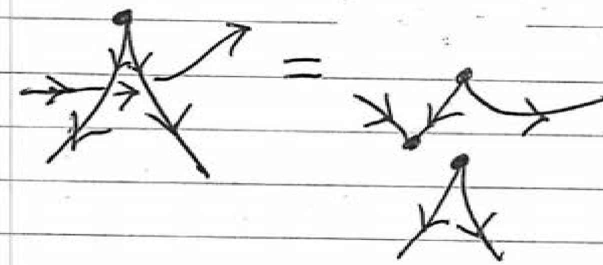


Lemma. (Bracket Evals)

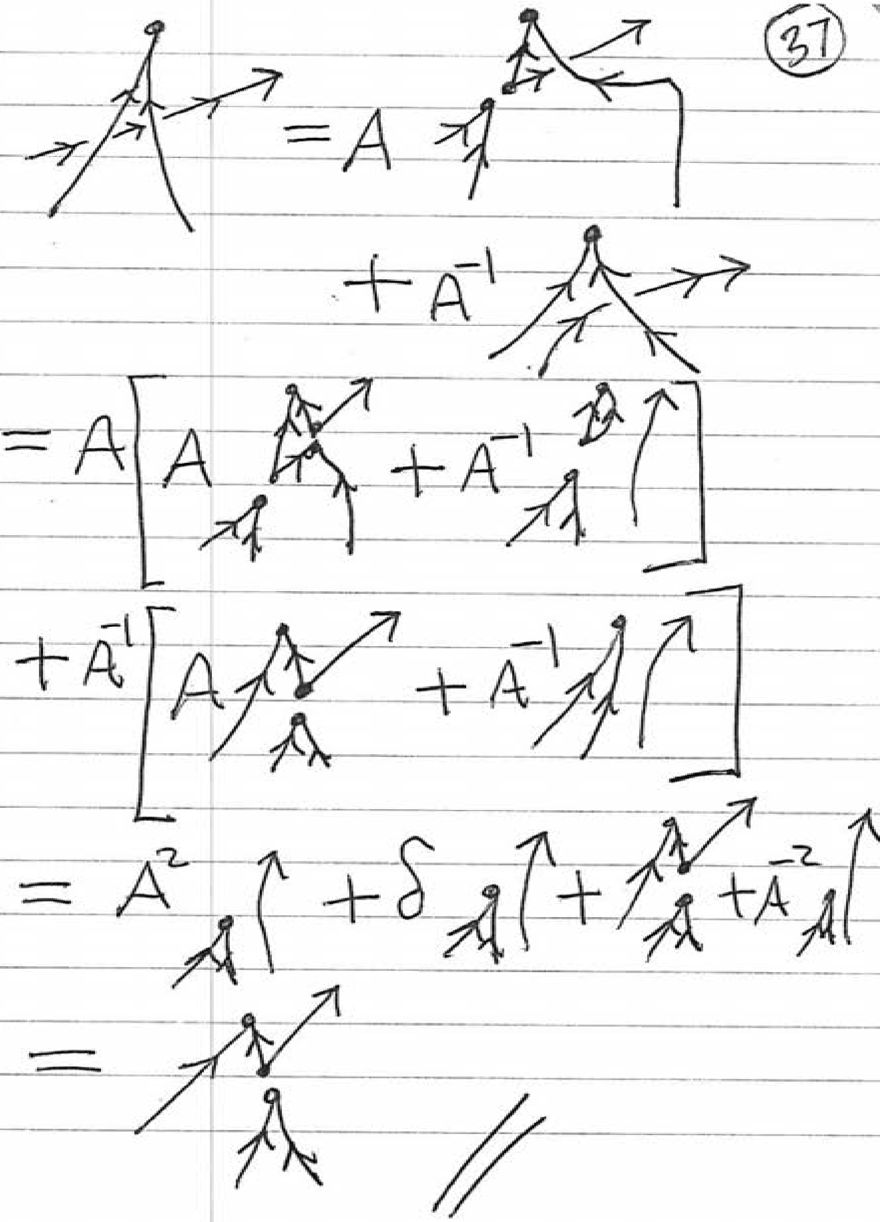
a)



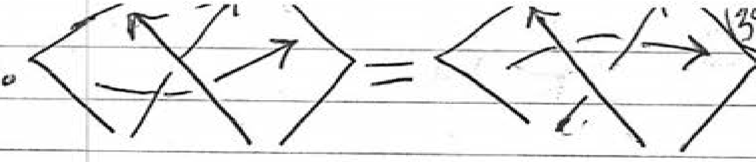
b)



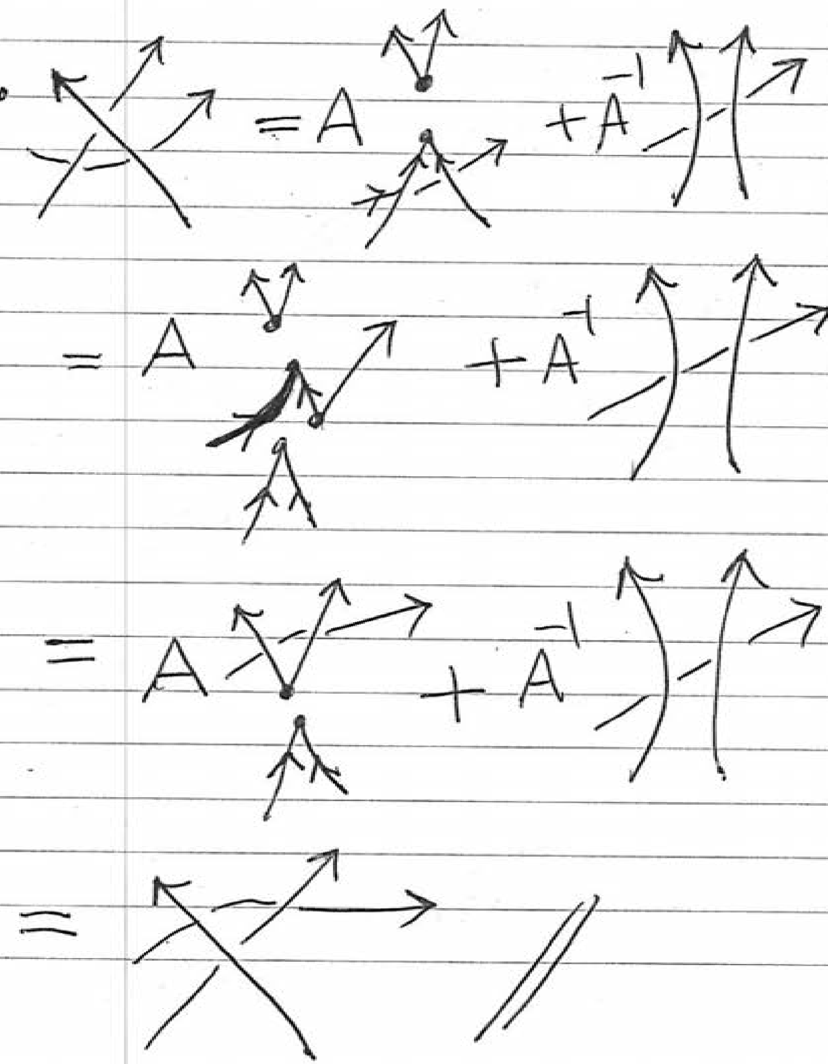
Proof: Next page.

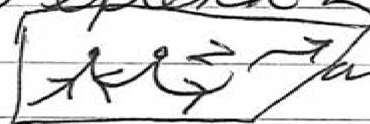


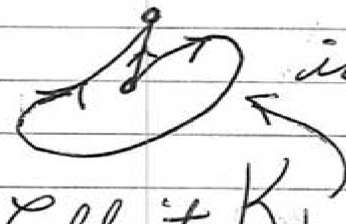
Prop.



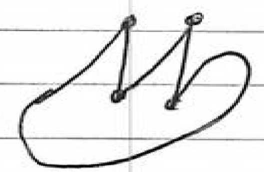
Pf.



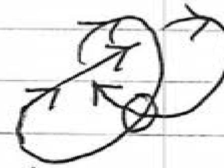
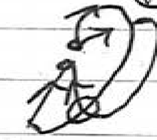
This means that we can extend ⟨K⟩ via just  and keep track of the zig-zags.

A state circle like  is distinct from 

Call it K_1.

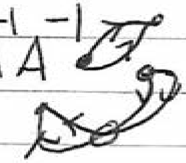
 K_2 etc.

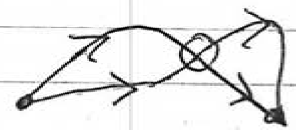
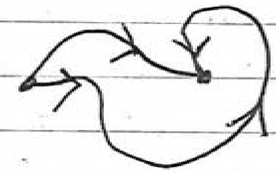
Such states <u>can only appear for virtual links</u>.

 $= A$  $+ A^{-1}$ 

K

$= A^2$  $+ AA^{-1}$ 

$+ A^{-1}A$  $+ A^{-1}A^{-1}$ 

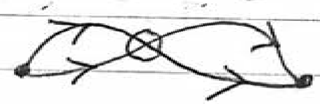
 $\simeq$  $= K_1$

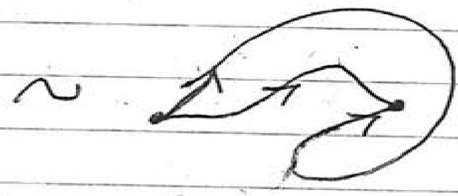
$$\therefore \langle K \rangle = A^2 + 2K_1 + \delta K_1$$

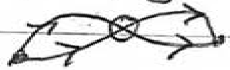
$$= A^2 + (2+\delta) K_1$$

This proves that
K_1 is non-classical.

Hint. Note that





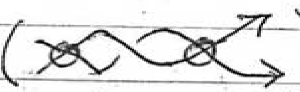
is a zig-zag and hence  $= K_1$.

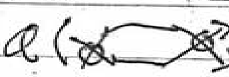
You can see how the presence of K_1 forces a virtual crossing in the diagram.

We call this $\langle K \rangle$ for virtuals the Arrow Polynomial and denote it by $\mathcal{A}(K)$.

Theorem. The virtual crossing number of K is bounded below by the maximal degree of the K_i in $\mathcal{A}(K)$, where $\deg(K_i) = i$, $\deg(K_i K_j) = i+j$ etc.

Proof. omitted //

Note: $\mathcal{A}($  $)$

$= A\mathcal{A}($  $) + A^{-1}\mathcal{A}($  $)$

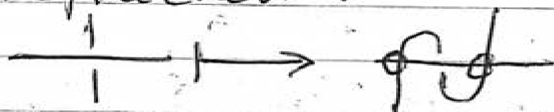
$= A\mathcal{A}($  $) + A^{-1}\mathcal{A}($  $)$

This part can have non-trivial contribution.

Remark. This example of a virtual knot K with $f_K = 1$ is not an accident.

We can make infinitely many of them by

1. choosing first a classical non-trivial knot k, and a subset $S \subset \text{Crossings}(k)$ s.t. if k is switched at all crossings in S to a new diagram k', then k' is unknotted.
2. For each crossing in S make the following replacement:

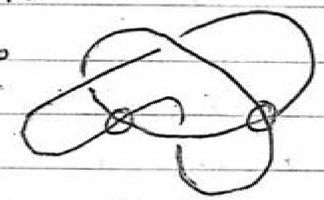


Call the resulting virtual knot K.

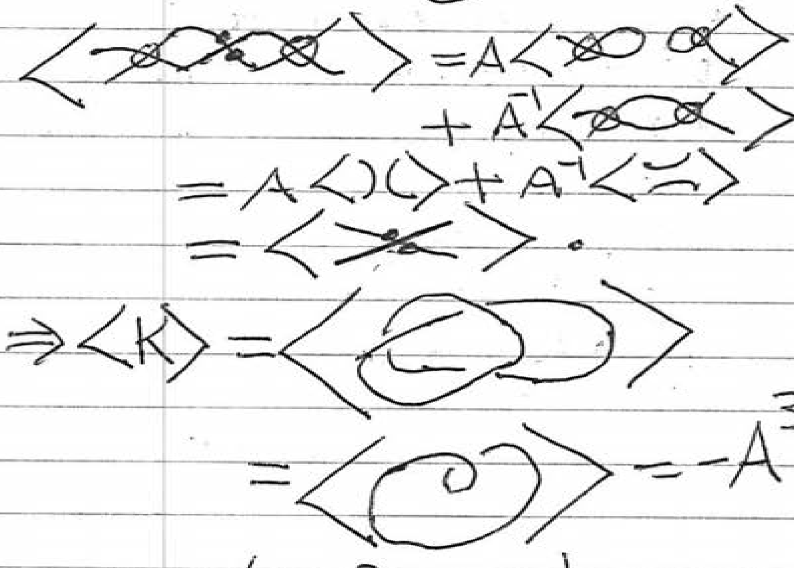
Then K is non-trivial & $f_K = 1$.

There are virtual knots with trivial bracket polynomial that are detected by the arrow polynomial.

Example.



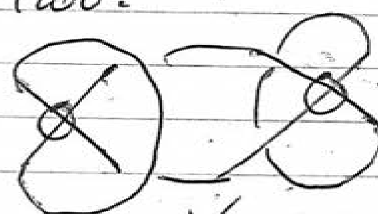
K



$\therefore f_K = 1$

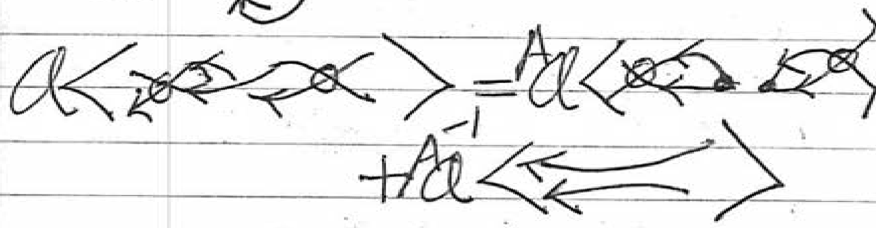
Since K is 3-colorable, we have K non-trivial and $f_K = 1$.

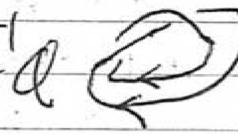
<u>Exercise</u>. Use the Arrow Polynomial to prove that the Kishino Diagram (below) represents a non-trivial virtual knot of genus two & virtual crossing number two.

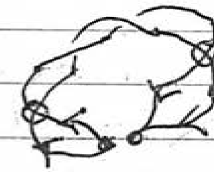


Kishino

Now consider $\mathcal{A}(K)$ (43)



$\mathcal{A}(K) = A\mathcal{A}$ $+ A^{-1}\mathcal{A}$ 



We leave it as an exercise to finish the calculation and show that K has vitrual crossing number equal to two.

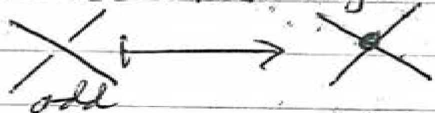
(Note that K is <u>represented</u> on a surface of genus 1.)

Manturov's Parity Bracket Poly

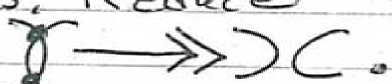
A crossing in a virtual knot is <u>even</u> if it encloses an even number of symbols in the Gauss code. e.g.

1212 : both crossing are <u>odd</u>.

<u>Parity bracket</u> : <u>Graphify</u> all odd crossings.



Then expand all even crossings: 

The resulting state sum contains graphs. Reduce graphs via 

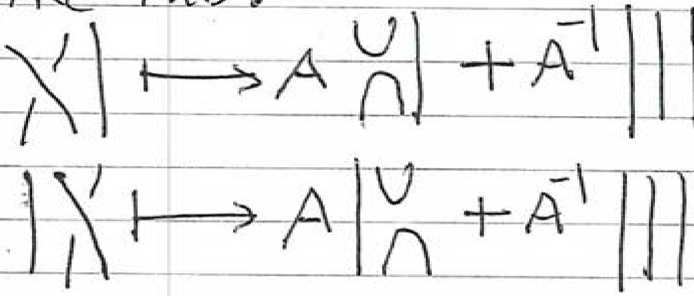
Use usual δ for  etc.

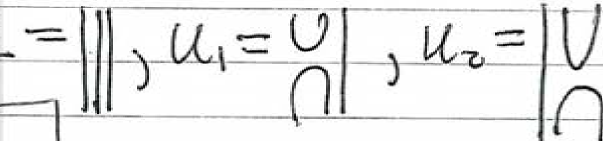
<u>Show</u>: Call [K] this state sum. Prove [K] is an invariant of regular isotopy.

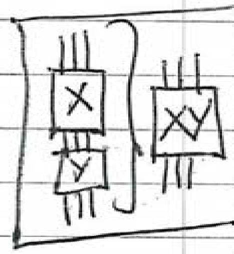
<u>Use</u> : Use [K] to prove that Kishino diagram is non-trivial virtual knot.

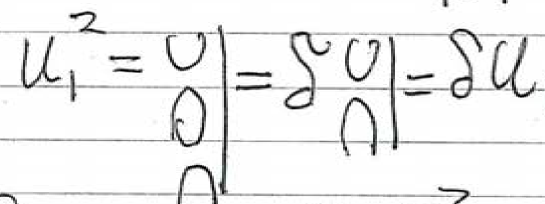
III. Temperley-Lieb Algebra and Jones polynomial

If we expand a braid via the bracket, we get diagrams like this:

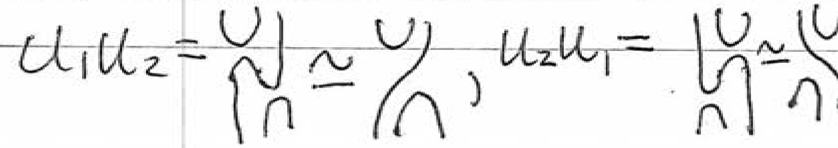


Let 





$\delta = \bigcirc = -A^2 - A^{-2}$.

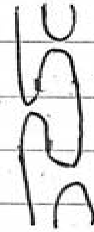


The diagrams

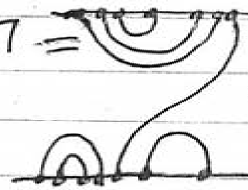
$\{\delta, \mathbb{1}, U_1, U_2, U_1U_2, U_2U_1\}$

form a closed algebra.
We take $\delta X = X\delta$ for all X
and regard $\delta = \bigcirc = -A^2 - A^{-2}$.

Note $U_1U_2U_1 =$  $=$  $= U_1$.

$U_2U_1U_2 =$  $=$  $= U_2$

For the 3-strand case
these relations
$U_1^2 = \delta U_1$, $U_2^2 = \delta U_2$
$U_1U_2U_1 = U_1$, $U_2U_1U_2 = U_2$
are complete.

$\Gamma =$  $\in TL_7$

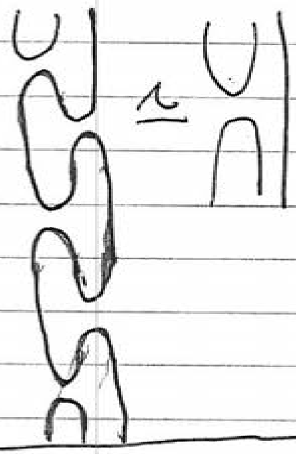
<u>Exercise</u> (a) Show directly that $\Gamma^2 \simeq \Gamma$ in TL_7.

(b) Write Γ as a product involving $U_1, U_2, U_3, U_4, U_5, U_6$.

(c) Using the result in (b) prove <u>algebraically</u> that $\Gamma^2 = \Gamma$ in TL_7. That is, use only the relations $U_i U_{i\pm1} U_i = U_i$, $U_i U_j = U_j U_i$ $|i-j|>1$, in your proof. (You will not need $U_i^2 = \delta U_i$.)

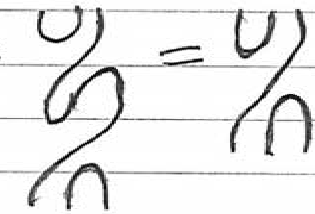
(d) Characterize the multiplicative elements P in TL_n such that $P^2 = P$.

e.g. $U_1 U_2 U_1 U_2 U_1$
$= U_1 U_2 U_1 = U_1$



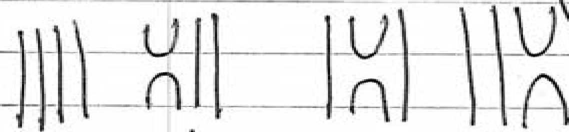
$(U_1 U_2)(U_1 U_2)$
$\|$
$(U_1 U_2 U_1) U_2$
$\|$
$U_1 U_2$

 $= U_1 U_2 =$ 

$(U_1 U_2)^2 =$  $= (U_1 U_2)$

We generalize to n-strands and call this the <u>n-strand Temperley Lieb Algebra</u>.

e.g. $n=4$ (TL4)



$\mathbb{1}_4 \quad U_1 \quad U_2 \quad U_3$

TL_n is generated by $\{\mathbb{1}_n, U_1, U_2, \ldots, U_{n-1}\}$

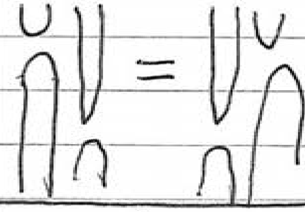
$U_i^2 = \delta U_i$

$U_i U_{i\pm 1} U_i = U_i$

$U_i U_j = U_j U_i, \ |i-j|>1.$

---

e.g.



$U_1 U_3 = U_3 U_1$

Exercise: In TL_5, let $\alpha = U_2 U_3 U_1 U_4 U_3 U_2 U_3$. Show that $\alpha^2 = \alpha$.

We are allowed to freely (48)
add elements of the
Temperley Lieb algebra.

Thus $A \overset{\cup}{\cap}| + A^{-1} ||| = AU_1 + A^{-1}\mathbb{1}$

$\in TL_3$.

The Artin Braid group
B_n is generated by
the elementary braids

$||\cdots| = \mathbb{1}_n$

$X|\cdots| = \sigma_1$, $X||\cdots| = \sigma_1^{-1}$

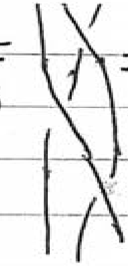
$\cdots$

$||\cdots|X = \sigma_{n-1}$, $||\cdots|X = \sigma_{n-1}^{-1}$

and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i , \ |i-j| > 1.$$

$\sigma_1\sigma_2\sigma_1 =$  $\overline{\overline{R3}}$  $= \sigma_2\sigma_1\sigma_2$

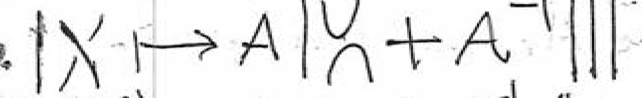
$\rho : B_n \longrightarrow TL_n$

(over $\mathbb{Z}[A, A^{-1}]$, $\delta = -A^2 - A^{-2}$)

$\rho(\sigma_i) = AU_i + A^{-1}\underline{1}$

$\rho(\sigma_i^{-1}) = A^{-1}U_i + A\underline{1}$

This gives a representation of the braid group to the Temperley Lieb algebra. It is an algebraic version of the bracket.

e.g. 

$\rho(\sigma_2) = AU_2 + A^{-1}\underline{1}$

This is a long-standing conjecture, that
$\rho : B_n \longrightarrow TL_n$
is a faithful representation of the Artin Braid group.

$\rho : B_n \longrightarrow TL_n$ satisfies 50

$$\rho(\sigma_i)^{-1} = \rho(\sigma_i^{-1})$$

$$\rho(\sigma_i)\rho(\sigma_{i+1})\rho(\sigma_i) = \rho(\sigma_{i+1})\rho(\sigma_i)\rho(\sigma_{i+1}).$$

This corresponds directly to the invariance of bracket under R2 and R3 and you can verify it again (exercise!) by pure algebra. It is interesting to see how the TL relations are related to the braid relations.

Conjecture:
$\rho : B_n \longrightarrow TL_n$
is injective for all n.

Define $tr : TL_n \longrightarrow \mathbb{Z}[A, A^{-1}]$

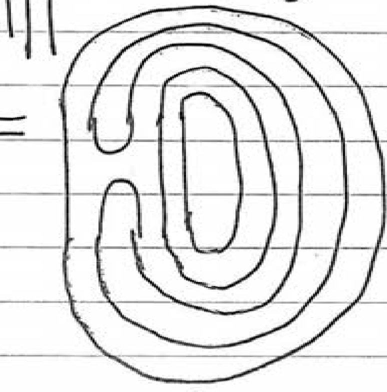
by $tr(x+y) = tr(x) + tr(y)$

tr(any product γ of generators)

$$= \delta^{\|\overline{\gamma}\|}$$

where $\overline{\gamma}$ = closure of γ

$U_2 =$  $\in TL_5$

$\overline{U_2} =$  closure of U_2

$tr(U_2 \in TL_5) = \delta^4$

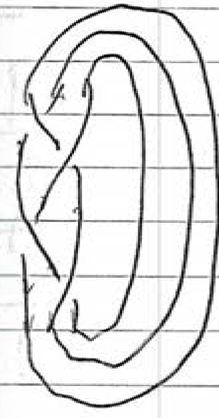
Exercise. $a, b \in TL_n$

$\Longrightarrow tr(ab) = tr(ba)$.

Exercise.

$$\begin{array}{ccc} B_n & \xrightarrow{\ \rho\ } & TL_n \\ \downarrow & & \downarrow tr \\ \overline{B_n} & \xrightarrow{\ \delta\langle - \rangle\ } & \mathbb{Z}[A, A^{-1}] \end{array}$$

$= b$



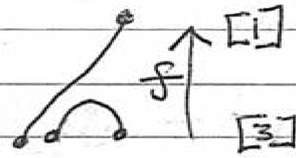
$\overline{b}$

That is, show that if $b \in B_n$ then $\delta\langle \overline{b} \rangle = tr(\rho(b))$

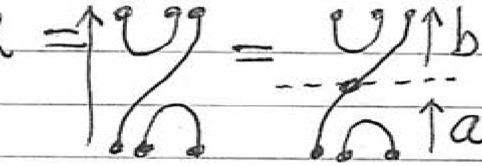
Remark. Vaughan Jones originally defined his polynomial invariant by by composing a representation of the braid group with a "trace" function on the

Temperley Lieb algebra).
Our representation and basis are equivalent to his, but we use a diagrammatic version of the algebra. He worked entirely algebraically).

Remark. It is useful to generalize the TL algebra to a TL-Category.

Then we can have elements like 

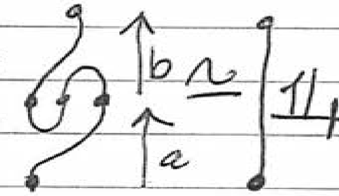
[n] denotes the number of dots. We regard  as a morphism taking [3] ⟶ [1].
The elements of TLn are morphisms from [n] ⟶ [n].

example. $\alpha =$ 

$$\alpha = b\,a$$

$$\alpha : [3] \longrightarrow [3]$$

$$a \searrow \quad [1] \quad \nearrow b$$

Note that $ab =$ 

Thus we have

$$\alpha^2 = (ba)(ba) = b(ab)a$$

$$= b\,\mathbb{1}_1\,a = ba = \alpha.$$

You can make other elements γ with $\gamma^2 = \mathbb{1}_n$ in TL_n by using this same idea.

Incidentally, we can translate to the skein relation for the Jones poly as follows:

$$\langle \times_+ \rangle = A\langle \rightrightarrows \rangle + A^{-1}\langle \asymp \rangle$$

$$\langle \times_- \rangle = A^{-1}\langle \rightrightarrows \rangle + A\langle \asymp \rangle$$

$$f_K = (-A^3)^{-w(K)}\langle K \rangle.$$

This means multiply each + crossing with $(-A^3)^{-1}$ & multiply each − crossing with $(-A^3)^{+1}$. ∴

$$f_{\times_+} = -A^{-3}A\, f_{0} - A^{-3}A^{-1} f_{\infty}$$

$$f_{\times_+} = -A^{-2} f_{0} - A^{-4} f_{\infty}$$

$$\& \; f_{\times_-} = -A^{2} f_{0} - A^{4} f_{\infty}$$

$$\therefore \; A^4 f_{\times_+} - A^{-4} f_{\times_-} = (-A^2 + A^{-2}) f_{0}$$

Let $t^{-1} = A^4$ & get

$V(t) = f(t^{-1/4})$ Jones Poly

$$t^{-1} V_{\times_+} - t V_{\times_-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{0}$$

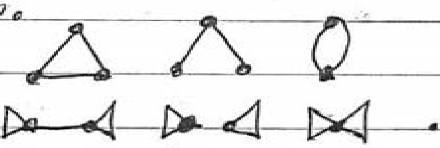
IV. Dichromatic Polynomial, Tutte Polynomial and the Potts Model

Chromatic Polynomial

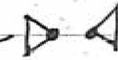
$C(G, q)$ = # of ways to color vertices of graph G with q colors s.t. if $v, w \in \mathrm{Vert}(G)$ and $\exists$ edge $\langle v, w \rangle$ (), then color(v) $\neq$ color(w).

Then let  stand for G, focusing on one edge.  delete edge,  contract edge.

e.g.



Then

C  $=$ C  $-$ C 

$C_{\bullet \sqcup G} = q\, C_G$

(1st identity is Logic: Different = Any − Same)

e.g. $C_{\bullet\!-\!\bullet} = C_{\bullet\ \bullet} - C_{\bullet}$
$= q^2 - q$
$= q(q-1)$.

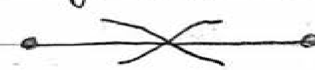
In the dichromatic polynomial $Z_G(q, v)$ we replace (−1) by v.

$$Z_{\triangleright\!-\!\triangleleft} = Z_{\triangleright\ \triangleleft} + v Z_{\triangleright\!\triangleleft}$$
$$Z_{\bullet \sqcup G} = q Z_G$$

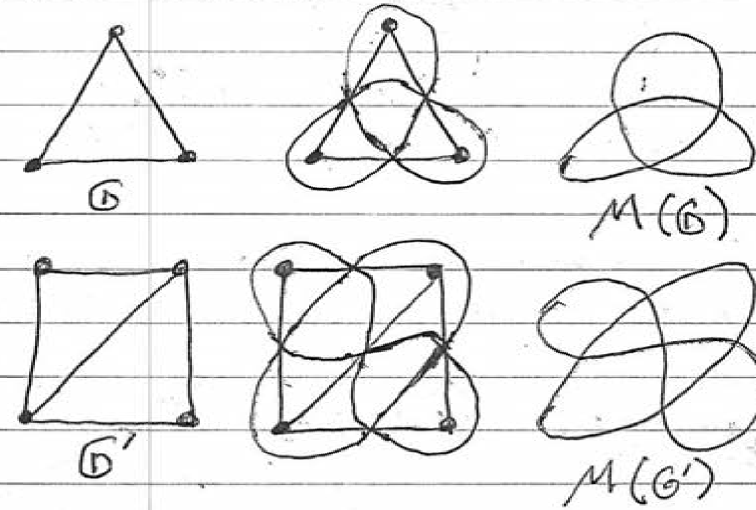
e.g. $Z_{\bullet\!-\!\bullet} = Z_{\bullet\ \bullet} + v Z_{\bullet}$
$= q^2 + vq$.

The dichromatic polynomial is also called the Tutte polynomial. We will return to the Tutte poly shortly, formulating it in terms of different variables.

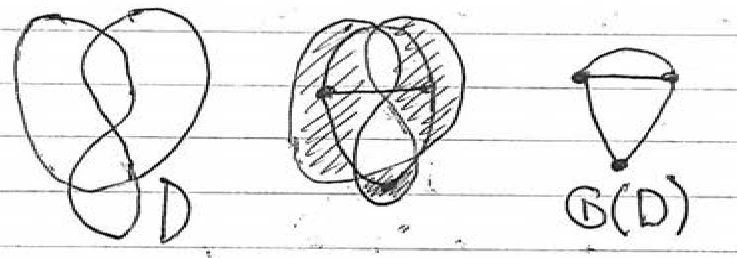
We now show how to translate the dichromatic polynomial to a bracket state sum. Given a planar graph G, define the medial graph $M(G)$ as follows: For each edge of G form a 4-regular node as shown:



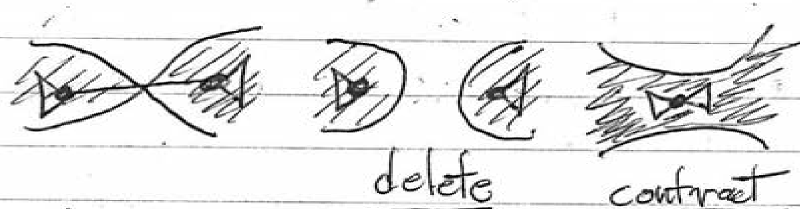
Connect these around each face as indicated.



Conversely, given a 4-regular (59)
plane graph D (i.e. a flat knot or link diagram), checkerboard shade it, assign one ~~node~~ to each region of the shading and one edge to each crossing (4 regular node) in common to two shaded regions. Call this graph $\mathbb{G}(D)$. Then $M(\mathbb{G}(D)) \simeq D$.



Now examine deletion and contraction in this context:



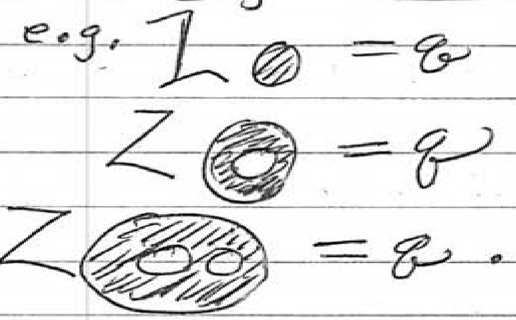
The two bracket smoothings correspond to deletion and contraction

(60)
This means that we can use flat, shaded link diagrams to compute the dichromate:

$$Z_{R \cup D} = q Z_D$$

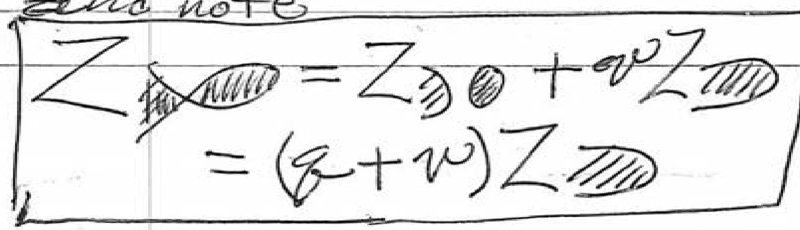
where R is a connected, shaded region with no crossings.

e.g.

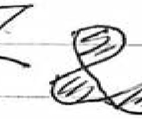


e.g.

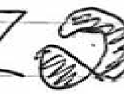
and note



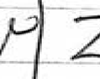
(61)

Thus Z  $= (q+v)^2 Z$ 

$= (q+v)^2 q$

Z  $= Z$  $+ vZ$ 

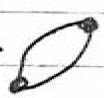
$= (q+v)q +$

$v[Z$  $+ vZ$  $]$

$= (q+v)q + v[q + vq]$

$= q^2 + 2vq + v^2 q$

Compare with

Z  $= Z$  $+ vZ$ 

$= (q+v)q + vZ_\circ + v^2 Z_\circ$

$= (q+v)q + vq + v^2 q$

$= q^2 + 2vq + v^2 q$

---

(62)

We can turn our <u>shaded bracket</u> expression for the dichromate into an <u>unshaded bracket</u> by obtaining a relationship between the number of shaded components of a state S, <u>call this $Comp(S)$</u>, and the

$\|S\|$ = # of loops in S

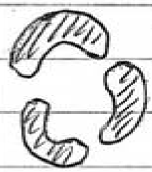
and

$\alpha(S)$ = # of shaded smoothings  ($\leftrightarrow$ "contractions")

---

e.g.  D

N = # of shaded regions in D ($\leftrightarrow$ # nodes in graph)

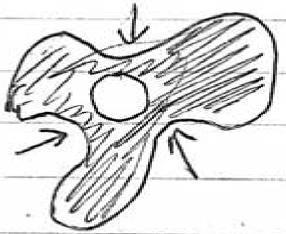
S 

$\alpha(S) = 0$
$\|S\| = 3$
$e(S) = Comp(S) = 3$
$N = 3$

S' 

$\alpha(S') = 1$
$\|S'\| = 2$
$e(S') = 2$
$N = 3$

S''



$\alpha(S'') = 3$

$\|S''\| = 2$

$e(S'') = 1$

$N = 3$

Lemma. For any shaded state S,

$$2e(S) = \|S\| + N - \alpha(S).$$

Proof. Exercise. (Hint — use Euler's formula for plane graphs) //

Now we have

$$Z_D = \sum_S \nu^{\alpha(S)} q^{e(S)}$$

(This is the state sum definition of the shaded diagram dichromate.)

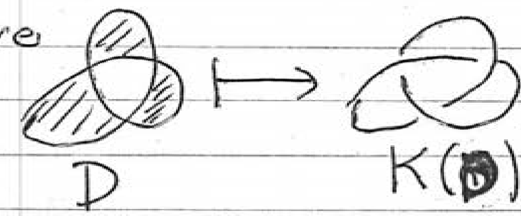
Thus

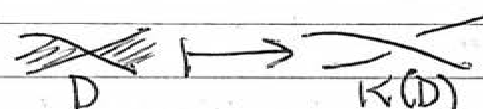
$$Z_D = \sum_S \nu^{\alpha(S)} q^{\left(\frac{\|S\| + N - \alpha(S)}{2}\right)}$$

$$= q^{N/2} \sum_S \nu^{\alpha(S)} q^{\left(\frac{\|S\| - \alpha(S)}{2}\right)}$$

$$= q^{N/2} \{K(D)\}$$

where





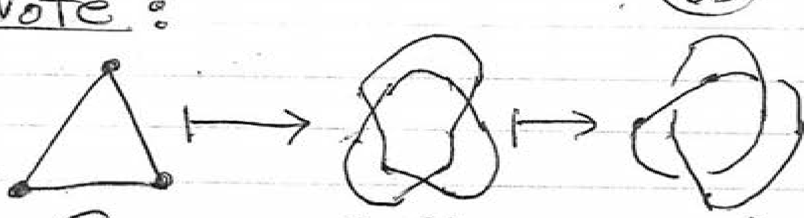
---

and $\{\times\} = \{\supset\subset\} + q^{-\frac{1}{2}}\nu\{\asymp\}$

$$\{O\} = q^{1/2}$$

(Just note $\{K(D)\} = \sum_S \nu^{\alpha(S)} q^{\left(\frac{\|S\| - \alpha(S)}{2}\right)}$.)

Note:



$G \qquad \mathcal{M}(G) = D \qquad K(D)$

Define $K(G) = K(\mathcal{M}(G))$.

Then we associate an alternating link diagram to each planar graph G and we have proved that

Theorem. $N = \#$ nodes of G, G planar, then the dichromate Z_G has bracket formula

$$Z_G = q^{N/2}\{K(G)\}.$$

where

$$\{\times\} = \{\supset\subset\} + q^{-\frac{1}{2}} v \{\asymp\}$$

$$\{O\} = q^{1/2}$$

This gives us a complete translation of the dichromatic polynomial as a bracket function on link diagrams that are alternating.

We can now go backwards and show how the topological bracket polynomial of an alternating link diagram can be expressed via the dichromate.

Lets start with the usual topological bracket:

$$\langle \times \rangle = A^{-1}\langle \supset\subset \rangle + A\langle \asymp \rangle$$

$$\langle O \rangle = -A^{2} - A^{-2}$$

(we do <u>not</u> normalize O to 1 here)

Make a new $[K]$ via (67)

$$[K] = A^{c(K)}[K]$$

where $c(K) = \#$ crossings of K. Then

$$[\times] = [\supset\,\subset] + A^{2}[\asymp]$$
$$[O] = -A^{2} - A^{-2}$$

We see that this could $= \{K\}$ if

$$A^{2} = q^{-1/2}v$$
$$-A^{2} - A^{-2} = q^{1/2}$$

Thus we need

$$\sqrt{q} = -A^{2} - A^{-2}$$
$$v = -A^{4} - 1$$

or

$$q = (A^{2} + A^{-2})^{2}$$
$$v = -A^{4} - 1$$

We can then state (68) that if K is an alternating link diagram, then

$$[K] = q^{-N/2} Z_{G(K)}(v, q)$$

where $G(K)$ is the plane graph for K obtained by the checkerboard construction and $q = (A^{2} + A^{-2})^{2}$, $v = -A^{4} - 1$.

Exercise. Give a corresponding formula for the normalized bracket $f_K(A)$.

This shows that the bracket and the dichromate are both closely related to both contraction/deletion and to knot theory.

We will generalize this (69) result by generalizing the dichromate (Tutte) polynomial to a Tutte polynomial for signed graphs. But first, a digression about statistical mechanics.

Partition Function

Physical System $\mathcal{G}$

Physical States σ

$E(\sigma)$ = Energy of σ.

k = Boltzmann's const.

T = temperature.

$$Z_{\mathcal{G}} = \sum_{\sigma} e^{-\frac{1}{kT}E(\sigma)}$$

It is assumed that $e^{-\frac{1}{kT}E(\sigma)} \sim$ probability of the state σ at temperature T.

Thus $p(\sigma) = e^{-\frac{1}{kT}E(\sigma)} / Z_{\mathcal{G}}$ (70)

and one can derive average physical quantities from $Z_{\mathcal{G}}$, e.g.

$$\frac{dZ_{\mathcal{G}}}{dT} = \sum_{\sigma} \frac{1}{kT^2} E(\sigma) e^{-1/kT}$$

$$\Rightarrow \frac{kT^2\, dZ_{\mathcal{G}}/dT}{Z_{\mathcal{G}}} = \langle E \rangle$$

where $\langle E \rangle$ is the average energy for the system at temperature T.

Potts Model

An idealized physical system based on an arbitrary graph G and "spins" $\{1, 2, \ldots, q\}$ (q an integer > 0) that can label nodes of G. A state $\sigma : \text{Nodes}(G) \longrightarrow \{1, 2, \ldots, q\}$.

$E(\sigma) = \#$ nodes $\langle i, j \rangle$ s.t. $i \bullet\!\!-\!\!\bullet j$ in G and $\sigma(i) = \sigma(j)$.

Thus for Potts,

$$Z_{G} = \sum_{\sigma} e^{-\frac{1}{kT}\sum_{\langle i,j\rangle} \delta(\sigma(i),\sigma(j))}$$

where $\langle i,j \rangle$ is an edge of G with end nodes i, j and

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{else.} \end{cases}$$

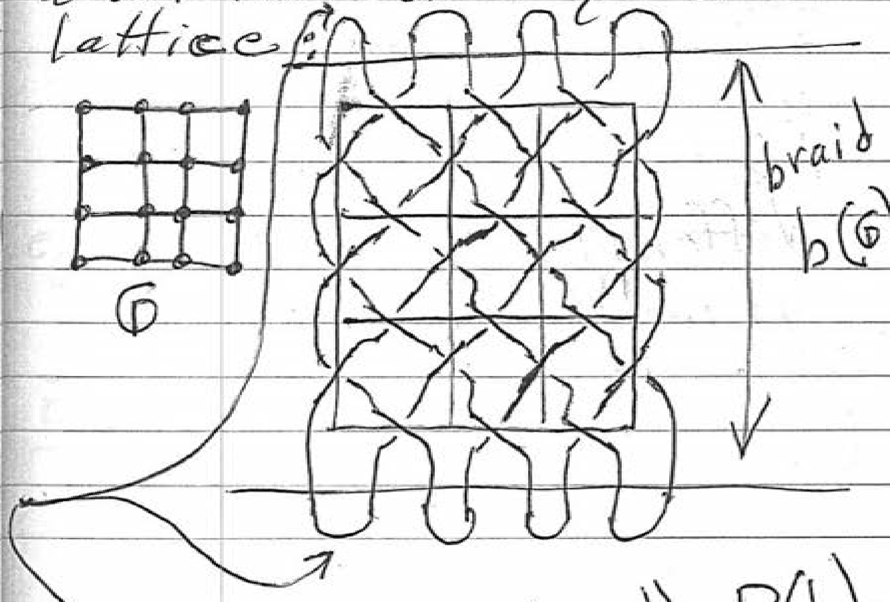
Theorem (Temperley).

For the Potts Model Z_{G} is identical with the dichromatic polynomia where $\nu = e^{-\left(\frac{1}{kT}\right)} - 1$

q = highest spin as above.

Proof. Exercise! //

From this we know that the Potts model partition function can be expressed as a bracket ($\{K\}$) evaluation and that, from this, we can expand the Potts partition function in terms of the Temperley-Lieb algebra (done diagrammatically earlier in these notes). Think about this for a square lattice:



The "Platt Closure" P(b)

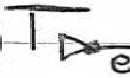
$$Z_{G} = q^{N/2}\{P(b)\}$$ (73)

and $\{P(b)\}$ can be expanded in terms of the Temperley-Lieb algebra.

This gives a complete treatment of the work of Temperley and Lieb who discovered the underlying algebra of the Potts model by quite different means at the graphtheoretic level.

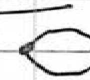
Tutte Polynomial (74)

The standard axiomatics for the Tutte polynomial $T_G(x,y)$ are:

1) T  $= T$  $+ T$ 

if e is neither an isthmus nor a loop.

2) If G consists entirely of isthmus and loops as edges, then $T_G = x^i y^l$ where
i = # isthmus edges in G
l = # loop edges in G.

e.g. T  $= T$  $+ T$ 

$= x + y$

Fact. $Z_G(q,v) = q v^{N-1} T_G(x,y)$

for $x = 1 + q v^{-1}$
$y = 1 + v$.
N = # nodes of G.

e.g. For $G =$ ,

$$Z_G(q,v) = qv(x+y)$$

$$= qv(1+qv^{-1}+1+v)$$

$$= 2qv + q^2 + qv^2$$

as we calculated before.

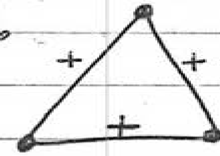
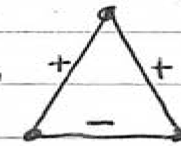
The Tutte polynomial and the dichromatic polynomial mutually determine each other. One has

$$T_G(x,y) = \frac{1}{(x-1)(y-1)^V} Z_G\big((x-1)(y-1), y-1\big)$$

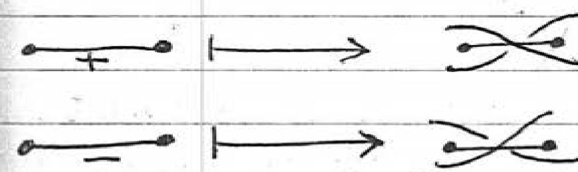
We will not discuss spanning tree expansion here. See L. Kauffman, "New Invariants in Knot Theory", Amer. Math. Monthly (1988), Vol. 95, Issue 3 (March 1988), 195–242.

Signed Tutte Polynomial

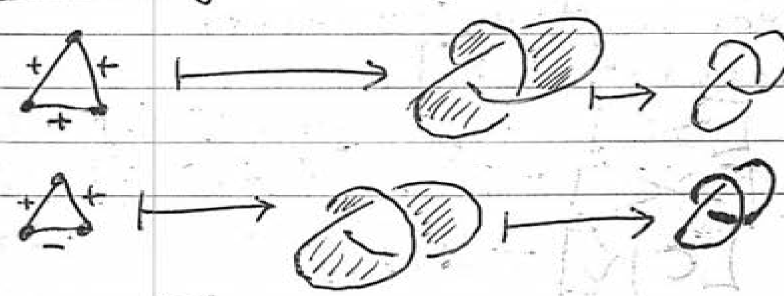
We now define a signed graph Tutte polynomial. G is now a graph with signed edges.

e.g.  or  etc.

$K(G)$ is obtained via the medial construction using



In this way, signed planar graphs represent all (unoriented) link diagrams.



etc.

Accordingly, we define (77)

1) $T_{\triangleright\!\!-\!\!+\!\!-\!\!\triangleleft} = B\,T_{\triangleright\ \triangleleft} + A\,T_{\bowtie}$

$T_{\triangleright\!\!-\!\!-\!\!-\!\!\triangleleft} = A\,T_{\triangleright\ \triangleleft} + B\,T_{\bowtie}$

For $\overset{+}{\circ\!\!-\!\!\circ}$ & $\overset{-}{\circ\!\!-\!\!\circ}$ neither isthmus nor loop.

2) If G consists in only loops and isthmus, then

$$T_G = X^{i_+ + l_-}\, Y^{i_- + l_+}$$

where $i_\pm$ = # of $\pm$ isthmus.
$l_\pm$ = # of $\pm$ loops.

With this signed Tutte polynomial, we can directly recover the 3-variable bracket [K]:

$[\times] = A[\asymp] + B[)\,(]$

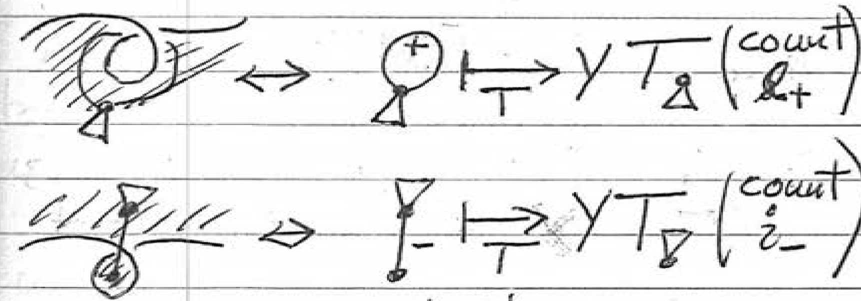
$[\times] = B[\asymp] + A[)\,(]$

$[OK] = d[K]$

(78) To see this, note that for [K] we have

$[\,\circ\!\sim\,] = A[\,\overset{\sim}{\circ}\,] + B[\,\smile\,]$
$= (Ad + B)[\,\sim\,]$

$[\,\sim\!\circ\,] = A[\,\smile\,] + B[\,\overset{\sim}{\circ}\,]$
$= (A + Bd)[\,\sim\,]$.



∴ We take

$$\boxed{\begin{aligned} Y &= Ad + B \\ X &= A + Bd \end{aligned}}$$

This implies that if G is a connected signed graph and $K(G)$ the corresponding link diagram, then

$$[K(G)] = d\,T_G(X, Y, A, B)$$ (79)

A, B, d $\qquad X = A + Bd$

$\qquad\qquad Y = Ad + B$

When we specialize to the topological bracket $\langle K \rangle$, then we take $B = A^{-1}$

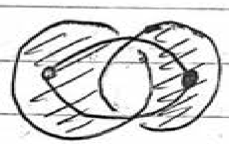
$d = -A^2 - A^{-2}$

and $\langle K \rangle = d^{-1}[K]$ so that $\langle O \rangle = 1$.

Then $Y = A(-A^2 - A^{-2}) + A^{-1} = -A^3$

and $X = -A^{-3}$

e.g. $K(G) =$  , $G =$ 



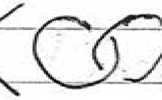
(80)

$T_{\;}$  $= A\,T_{\;}$  $+ B\,T_{\;}$ 

$= AY + BX$

$= A(-A^3) + A^{-1}(-A^{-3})$

$= -A^4 - A^{-4}$

$= \langle$  $\rangle$ ✓

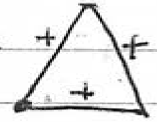
At the level of the topological bracket $\langle K \rangle$, we have

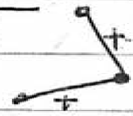
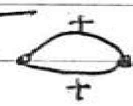
$$T_G = \langle K \rangle (A)$$

when $B = A^{-1}$

$X = -A^{-3}$, $Y = -A^3$.

This is the most direct way to relate a contraction-deletion algorithm for signed graphs to the bracket state sum.

Example. $G =$  (81)

$T_G = B\,T$  $+ A\,T$ 

$= BX^2 + A\left[B\,T_{\bullet\overset{}{\underset{+}{-}}\bullet} + A\,T_{O_+}\right]$

$= BX^2 + ABX + A^2Y$

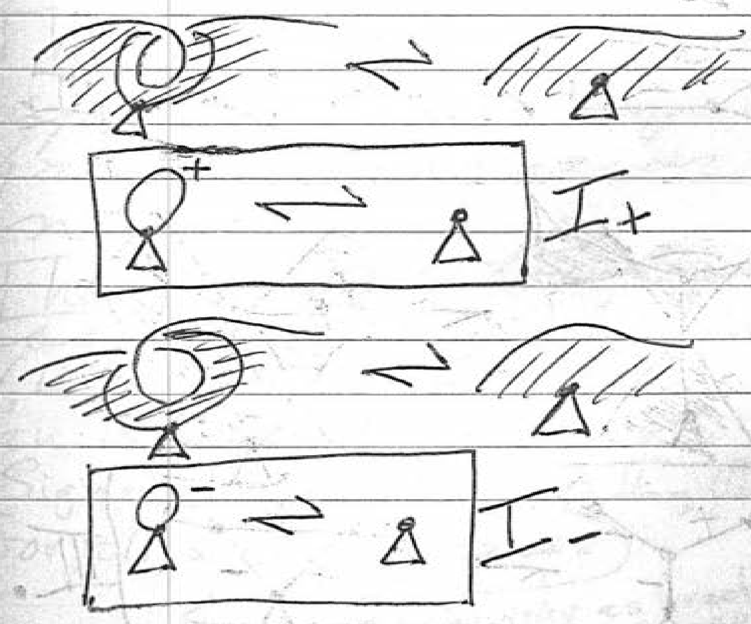
$= A^{-1}(-A^{-3})^2 + (-A^{-3}) + A^2(-A^3)$

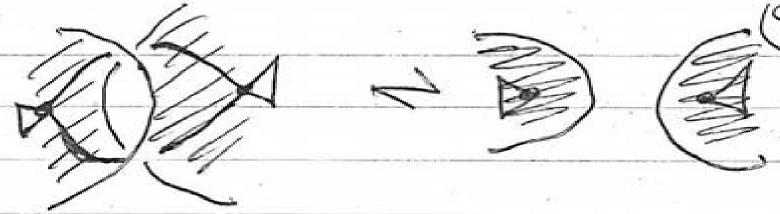
$= +A^{-7} - A^{-3} - A^5$

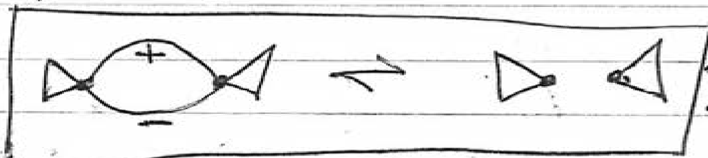
$= \langle$  $\rangle$ (as we have computed before)

We have translated the bracket polynomial into a signed Tutte polynomial for graphs. But note that the signed Tutte polynomial $T_G(A,B,X,Y)$ is well-defined for all finite signed graphs (82) G. This means that we have actually achieved a wide generalization of the bracket polynomial.

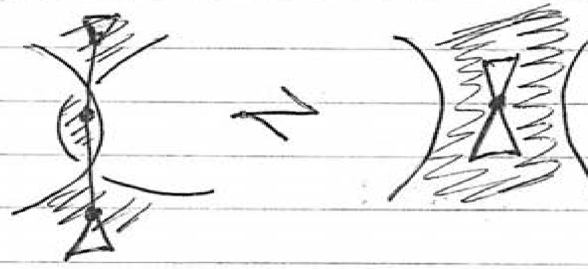
In order to put this into proper context, we now translate the Reidemeister moves into moves on signed graphs:

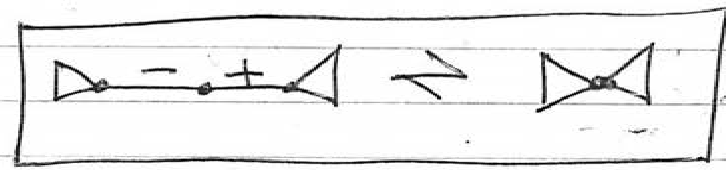




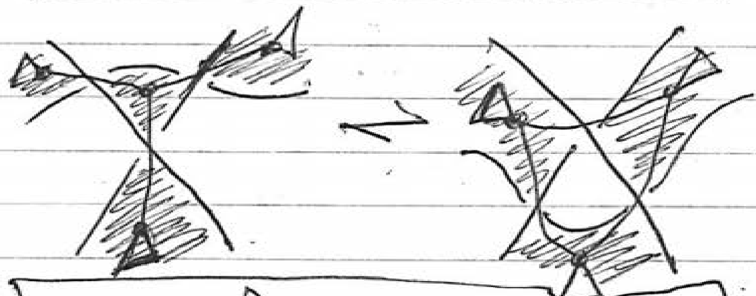


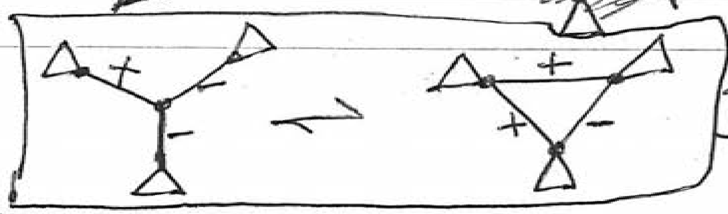
II_p





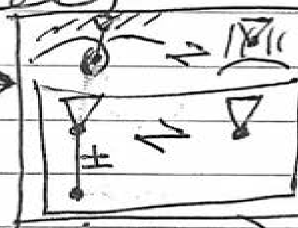
II_s





III.

Thus we see that the Reidemeister moves translate to the move on graphs $I_\pm$ and $I'_\pm$ (shown below)
(loops) (edges)



II_p (parallel) II_s (series)

and III (star-triangle).

These moves can be applied to any signed graphs, planar or not.

Theorem. The topological signed Tutte polynomial ($B=A^{-1}$, $X=A^{-3}$, $Y=-A^{3}$) is an invariant of the Signed R-Moves: ~~I, I'~~, II, III for all signed graphs.

(Same behaviour as bracket on curls for these)

Lets clarify and review.

1) $T_{G \sqcup H} \overset{def}{=} T_G T_H$
  ↑ disjoint union

2) $T_{\;+\;} = B\,T_{\;} + A\,T_{\;}$

$T_{\;-\;} = A\,T_{\;} + B\,T_{\;}$

where •—±—• is neither loop nor isthmus.

3) $T_G = X^{i_+ + l_-}\,Y^{i_- + l_+}$

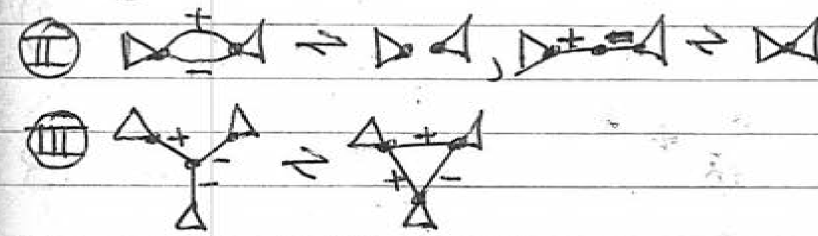
$i_\pm$ = # of ± isthmus
$l_\pm$ = # of ± loops.
when G composed of only loops & isthmus.

4) Let $\tilde{T}_G$ denote the topological signed Tutte where

$B = A^{-1}$
$X = -A^{-3}$
$Y = -A^{3}$

5) When G = Signed Medial inverse from checkerboard for classical knot diagram K, then $\tilde{T}_G = \langle K \rangle$.

6) For <u>any</u> <u>signed</u> <u>graph</u> G, $\tilde{T}_G$ is invariant under



and

$\tilde{T}_{\;\bigcirc^{+}} = Y\,\tilde{T}_{\;} = -A^{3}\,\tilde{T}_{\;}$

$\tilde{T}_{\;\bigcirc^{-}} = X\,\tilde{T}_{\;} = -A^{-3}\,\tilde{T}_{\;}$

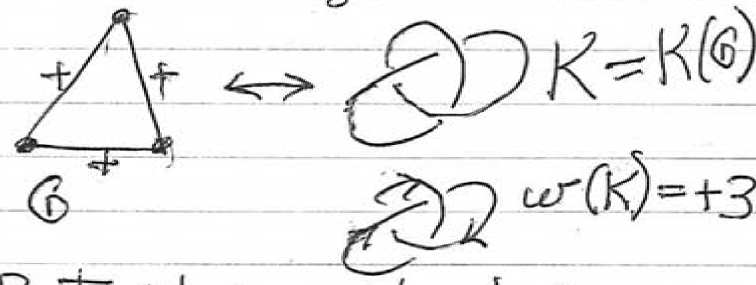
$\tilde{T}_{\;+\;} = X\,\tilde{T}_{\;} = -A^{-3}\,\tilde{T}_{\;}$

$\tilde{T}_{\;-\;} = Y\,\tilde{T}_{\;} = -A^{3}\,\tilde{T}_{\;}$

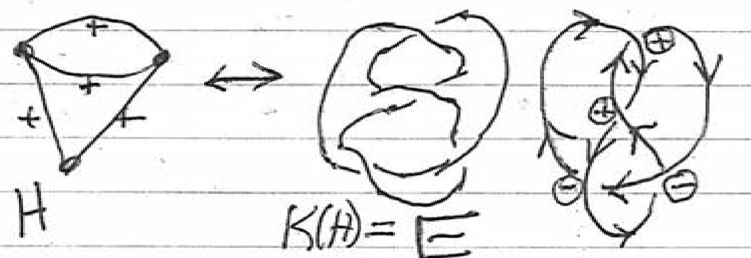
These last equations are obvious. We discuss invariance under II and III now. We call the equiv relation on signed graphs gen by II, III above **graph regular isotopy**

Remark:

In working with the bracket on knot and (oriented) link diagrams, we can normalize the bracket by using the writhe. e.g.

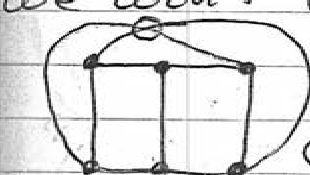


But it is not obvious what is the writhe of a planar graph! We have to translate it into a knot diagram.



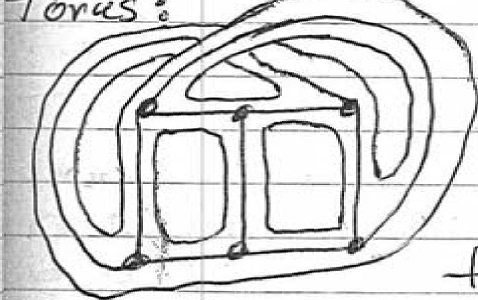
$w(E) = 0$
(using encircled oriented crossing signs)

In the case of a non-planar graph, there is no canonical knot or link diagram associated with the graph. For example, consider the graph K_{33} (all + signs, so we won't mark them)



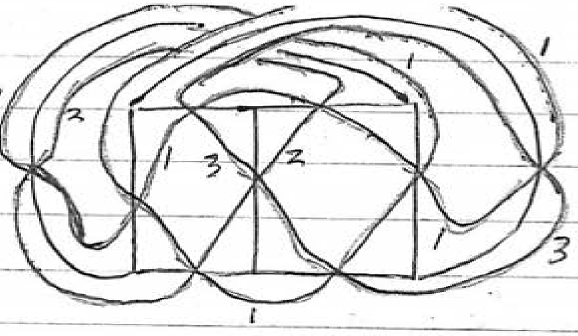
$G = K_{33}$.

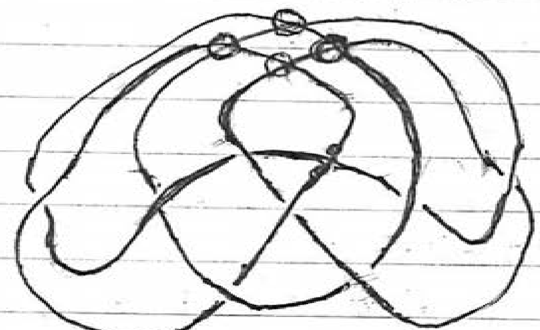
With this choice of planar immersion, we associate an embedding of K_{33} on a torus:



(Add disks to the boundary components to get the torus.)

On the torus you can make a medial construction and obtain a link of 3 components.

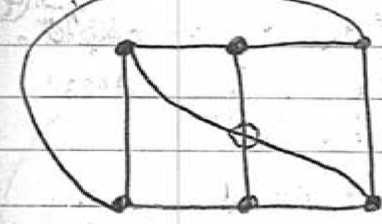




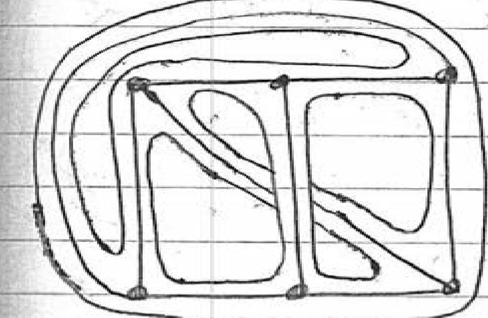
L = "K(K_{33})"

Here we represent the link L as a 3-comp virtual link.

If we choose a *different* immersion of K_{33} ~~in the plane~~ then we can get another torus embedding of K_{33} and a *different* medial link.



K'_{33} (a second immersion of K_{33})

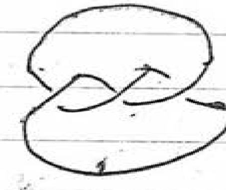


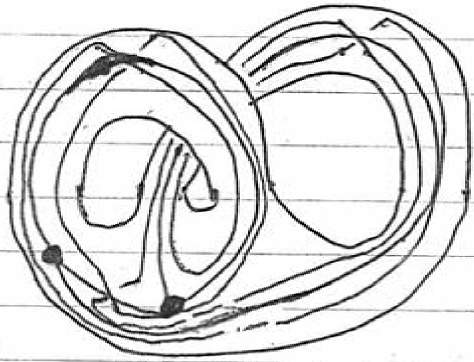
Close with 3 cells + get torus.

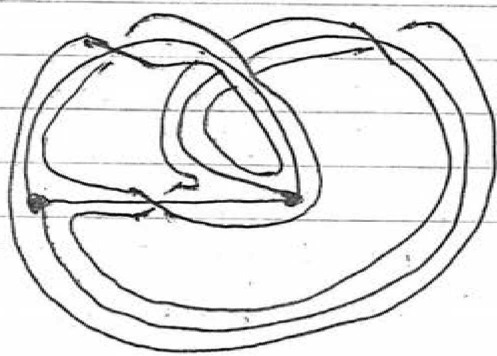
Exercise. Show that the resulting medial link has *two* components.

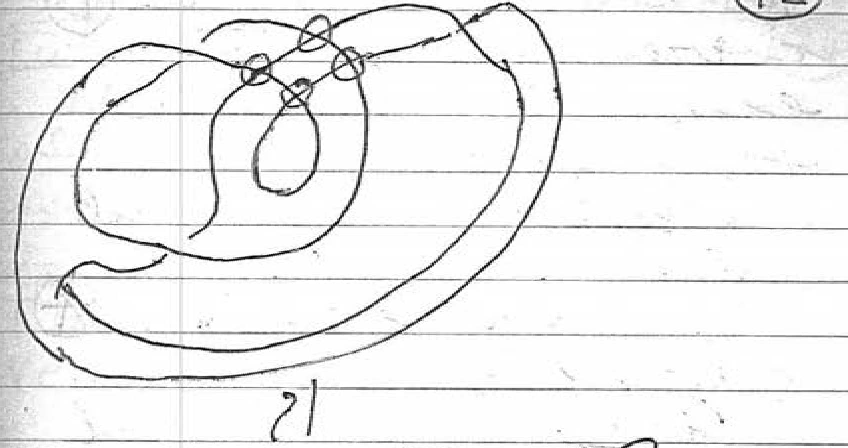
This means that the Tutte polynomial (bracket) invariant for signed graphs can sometimes give the same evaluation for distinct associated virtual links. Clearly, there is much to be investigated here.

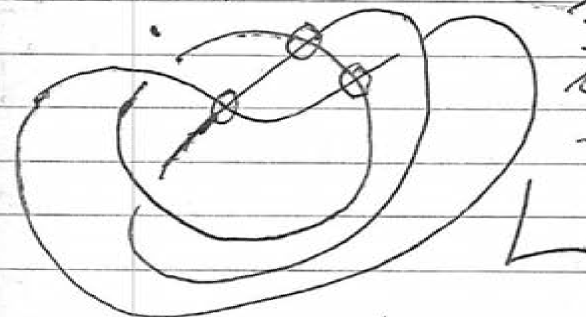
Here is another example.
If we change a planar immersion to a non-planar immersion, the associated medial link will change.

a)   1 component









3 components

L

Note that L is a non-trivial virtual link since each pair of components

 (e.g.) is virtually linked.

We conclude that corresponding to each signed graph there is a rich collection of virtual links sharing the same graphical (*) regular isotopy class and signed Tutte polynomial. (93)

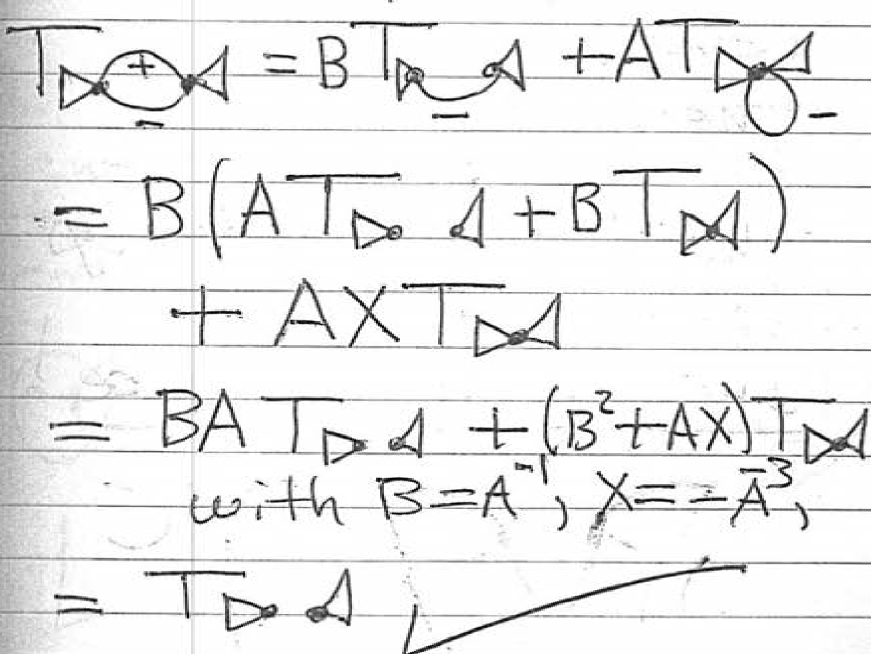
After these notes, we will prepare a paper on this subject with many examples.

Note that one should be most interested in those immersions that correspond to minimal genus surface embeddings for the graphs.

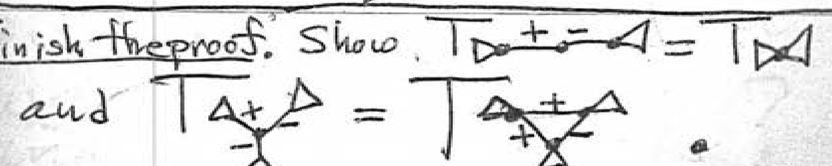
(*) See Appendix 2.

Invariance of the topological Tutte polynomial. (94)

We illustrate with one of the 2-moves, and we assume that no isthmus appears in the calculation. The rest is left to the reader:

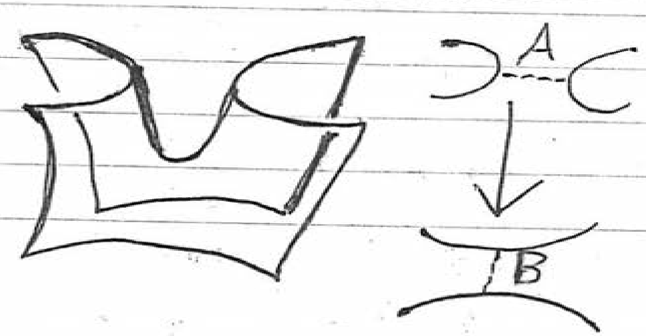


Finish the proof. Show

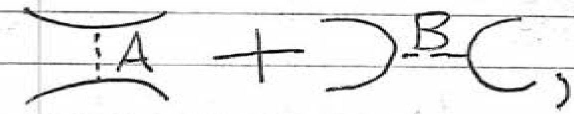


Khovanov Homology

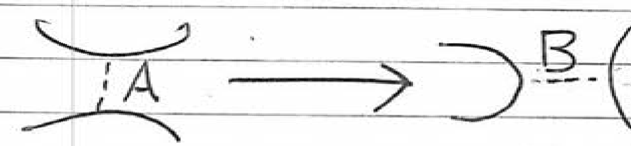
We discussed this theory in the short course, but I shall not write notes on it here. We should remark that this theory is based on the idea that the two smoothings fit together into one saddle surface!



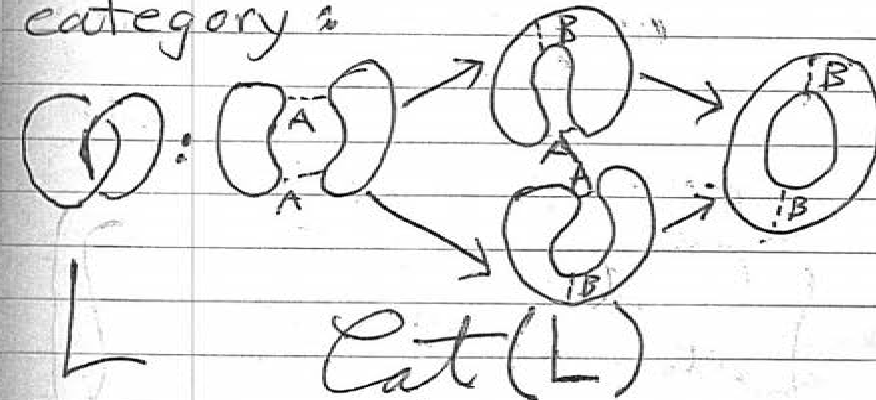
Instead of thinking of



we think of an arrow from the A-smoothing to the B-smoothing:



and we configure all the states as a category:



and we ask, how can we measure Cat(L) and find out about L?

It turns out that there are homological ways to measure $Cat(L)$ with respect to certain algebras associated with the state loops.

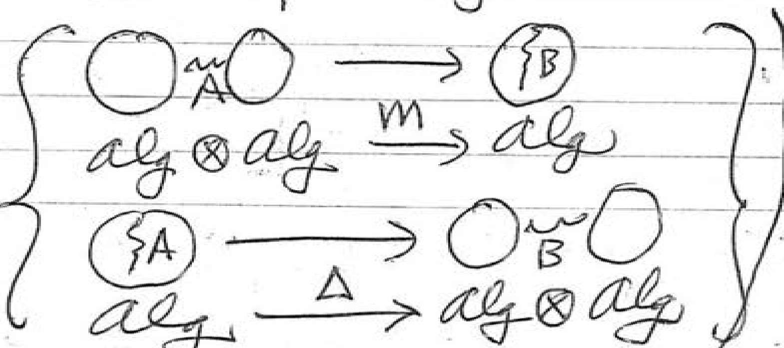
(Khovanov used

$$Algebra(O) \simeq \frac{\mathbb{Z}[x]}{(x^2)}$$

with a comultiplication

$$\Delta(x) = x \otimes x$$
$$\Delta(1) = 1 \otimes x + x \otimes 1$$

corresponding to



Then one can measure a cohomology of the category with respect to the algebra.

$$H^*(Cat(L); Alg)$$

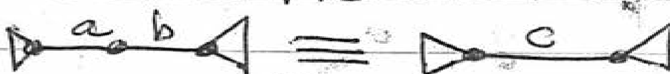
and it turns out to generalize the Jones polynomial and leads to many remarkable results.

But we shall stop here.

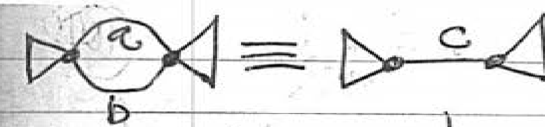
Acknowledgment. Many thanks to Prof. Xian'an Jin and all the participants in the 5 day, 3 hours/day course for which these notes are a partial record. And thanks to the School of Mathematical Sciences, Xiamen University for their support.
June 20 – 24, 2016

Appendix 1. Electricity

Recall the following formulas for conductance (conductance is the reciprocal of resistance in electrical networks).

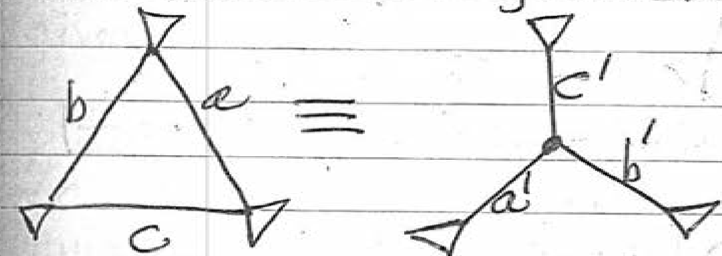


$$c = \frac{1}{\frac{1}{a}+\frac{1}{b}} = \frac{ab}{a+b}$$



$$c = a + b$$

and less well-known is the star-triangle relation

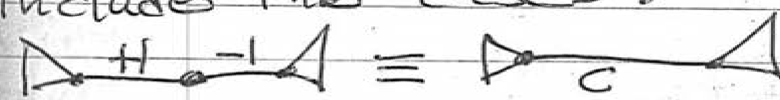


$$x' = D/x$$

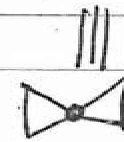
where $D = ab + ac + bc$.

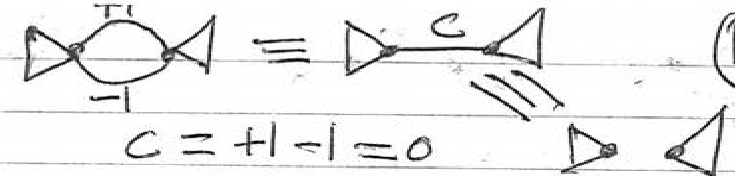
Networks in which such replacements are made have equivalent conductances.

Nota Bene. These rules include the cases:

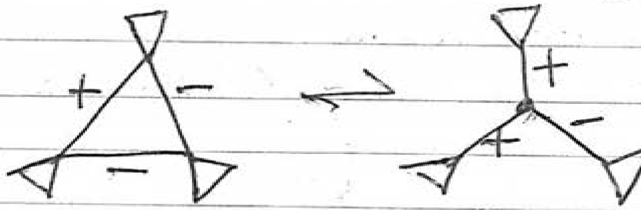


$$c = \frac{(+1)(-1)}{(+1)+(-1)} = \infty$$





$c = +1 - 1 = 0$



$D = (+1)(-1) + (+1)(-1) + (-1)(-1) = -1$

$x' = -1/x$

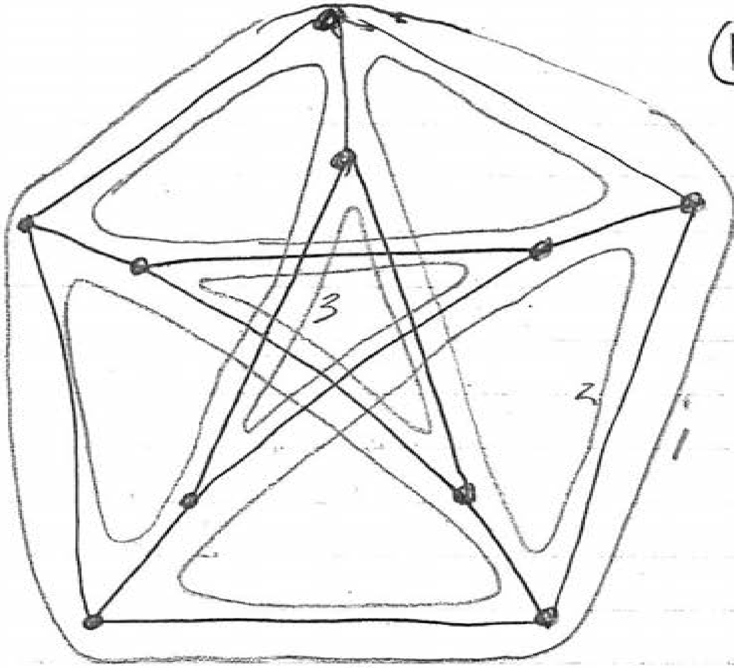
$\Rightarrow$

$+' = -$
$-' = +$

Thus we can use electrical conductance as a link invariant. See papers by Goldman & Kauffman. Again these invariants apply to arbitrary graphs and the graphical versions of the Reidemeister moves.

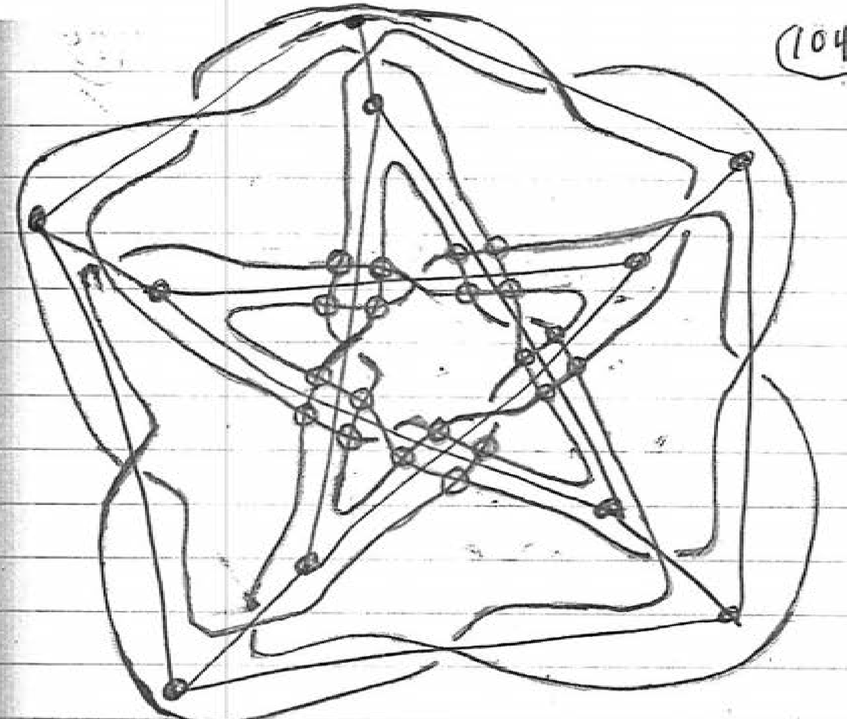
Appendix 2. Remarks on the medial construction.

In the section on the topological Tutte polynomial we have pointed out that given a specific immersion of a signed graph in the plane, we can associate to this graph a cellular embedding in an orientable surface and a medial construction of a knot or link diagram in this surface. This link can be regarded as a virtual link, and the signed Tutte polynomial of the graph is related to the bracket polynomial of this virtual link. There is a multiplicity of virtual links corresponding to different immersions of a given graph. We end these notes with an example for the Petersen graph.



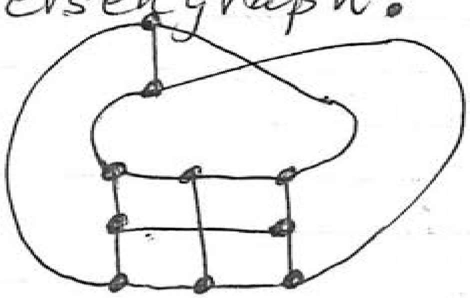
$$v - e + f = 2 - 2g$$
$$10 - 15 + 3 = 2 - 2g$$
$$-2 = 2 - 2g$$
$$\boxed{g = 2}$$

This immersion of the Petersen graph P leads to an embedding of P in a surface of genus 2.



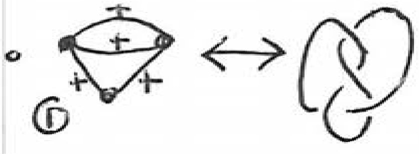
Here we have given P all + signs. The resulting medial is a single component knot; virtual and alternating.

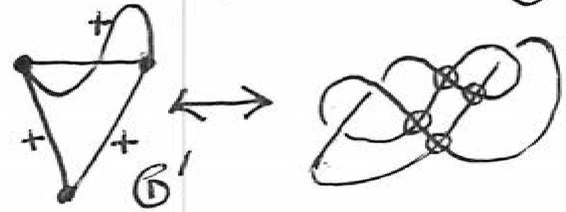
Exercise. Use this immersion of the Petersen graph:



Show that the corresponding surface has genus 2 and that the corresponding link has 2-components.

Question. Call a virtual knot K graphical if it occurs as a medial construction from some signed graph $\mathbb{G}$. Does there exist a graphical non-trivial virtual knot with unit Jones polynomial?

Exercise. $\mathbb{G}$  E



$\mathbb{G}'$ E'

Verify that $\langle E \rangle = T_{\mathbb{G}}$ but $\langle E' \rangle \neq T_{\mathbb{G}'} = T_{\mathbb{G}}$.

The reason for the difference is that the contraction-deletion algorithm (in the cases of non-planar immersions) can bottom out on loop evaluations like

that are not just products of loop values.

The relationship of the general signed Tutte polynomial and these virtual links will be investigated!