

# From Tangle Fractions to DNA

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## Abstract

This paper draws a line from the elements of tangle fractions to the tangle model of DNA recombination. In the process, we sketch the classification of rational tangles, unoriented and oriented rational knots and the application of these subjects to DNA recombination.

## 1 Introduction

Rational knots and links are a class of alternating links of one or two unknotted components, and they are the easiest knots to make (also for Nature!). The first twenty five knots, except for  $8_5$ , are rational. Furthermore all knots and links up to ten crossings are either rational or are obtained from rational knots by insertion operations on certain simple graphs. Rational knots are also known in the literature as four-plats, Viergeflechte and 2-bridge knots. The lens spaces arise as 2-fold branched coverings along rational knots.

A rational tangle is the result of consecutive twists on neighbouring endpoints of two trivial arcs, see Definition 1. Rational knots are obtained by taking numerator closures of rational tangles (see Figure 19), which form a basis for their classification. Rational knots and rational tangles are of fundamental importance in the study of DNA recombination. Rational knots and links were first considered in [40] and [2]. Treatments of various aspects of rational knots and rational tangles can be found in [3],[7], [46], [6], [42], [16], [27], [31], [34]. A rational tangle is associated in a canonical manner with a unique, reduced rational number or  $\infty$ , called *the fraction* of the tangle. Rational tangles are classified by their fractions by means of the following theorem:

**Theorem 1 (Conway, 1970)** *Two rational tangles are isotopic if and only if they have the same fraction.*

John H. Conway [7] introduced the notion of tangle and defined the fraction of a rational tangle using the continued fraction form of the tangle and the Alexander polynomial of knots. Via the Alexander polynomial, the fraction

is defined for the larger class of all 2-tangles. In this paper we are interested in different definitions of the fraction, and we give a self-contained exposition of the construction of the invariant fraction for arbitrary 2-tangles from the bracket polynomial [19]. The tangle fraction is a key ingredient in both the classification of rational knots and in the applications of knot theory to DNA. Proofs of Theorem 1 can be found in [33], [6] p.196, [16] and [25].

More than one rational tangle can yield the same or isotopic rational knots and the equivalence relation between the rational tangles is reflected in an arithmetic equivalence of their corresponding fractions. This is marked by a theorem due originally to Schubert [45] and reformulated by Conway [7] in terms of rational tangles.

**Theorem 2 (Schubert, 1956)** *Suppose that rational tangles with fractions  $\frac{p}{q}$  and  $\frac{p'}{q'}$  are given ( $p$  and  $q$  are relatively prime. Similarly for  $p'$  and  $q'$ .) If  $K(\frac{p}{q})$  and  $K(\frac{p'}{q'})$  denote the corresponding rational knots obtained by taking numerator closures of these tangles, then  $K(\frac{p}{q})$  and  $K(\frac{p'}{q'})$  are topologically equivalent if and only if*

1.  $p = p'$  and
2. either  $q \equiv q' \pmod{p}$  or  $qq' \equiv 1 \pmod{p}$ .

This classic theorem [45] was originally proved by using an observation of Seifert that the 2-fold branched covering spaces of  $S^3$  along  $K(\frac{p}{q})$  and  $K(\frac{p'}{q'})$  are lens spaces, and invoking the results of Reidemeister [41] on the classification of lens spaces. Another proof using covering spaces has been given by Burde in [5]. Schubert also extended this theorem to the case of oriented rational knots and links described as 2-bridge links:

**Theorem 3 (Schubert, 1956)** *Suppose that orientation-compatible rational tangles with fractions  $\frac{p}{q}$  and  $\frac{p'}{q'}$  are given with  $q$  and  $q'$  odd. ( $p$  and  $q$  are relatively prime. Similarly for  $p'$  and  $q'$ .) If  $K(\frac{p}{q})$  and  $K(\frac{p'}{q'})$  denote the corresponding rational knots obtained by taking numerator closures of these tangles, then  $K(\frac{p}{q})$  and  $K(\frac{p'}{q'})$  are topologically equivalent if and only if*

1.  $p = p'$  and
2. either  $q \equiv q' \pmod{2p}$  or  $qq' \equiv 1 \pmod{2p}$ .

In [26] we give the first combinatorial proofs of Theorem 2 and Theorem 3. In this paper we sketch the proofs in [25] and [26] of the above three theorems and we give the key examples that are behind all of our proofs. We also give some applications of Theorems 2 and 3 using our methods.

The paper is organized as follows. In Section 2 we introduce 2-tangles and rational tangles, Reidemeister moves, isotopies and operations. We give the definition of flyping, and state the (now proved) Tait flyping conjecture. The Tait conjecture is used implicitly in our classification work. In Section 3 we introduce the continued fraction expression for rational tangles and its properties. We use the continued fraction expression for rational tangles to define their fractions. Then rational tangle diagrams are shown to be isotopic to alternating diagrams. The alternating form is used to obtain a canonical form for rational tangles, and we obtain a proof of Theorem 1.

Section 4 discusses alternate definitions of the tangle fraction. We begin with a self-contained exposition of the bracket polynomial for knots, links and tangles. Using the bracket polynomial we define a fraction  $F(T)$  for arbitrary 2-tangles and show that it has a list of properties that are sufficient to prove that for  $T$  rational,  $F(T)$  is identical to the continued fraction value of  $T$ , as defined in Section 3. The next part of Section 4 gives a different definition of the fraction of a rational tangle, based on coloring the tangle arcs with integers. This definition is restricted to rational tangles and those tangles that are obtained from them by tangle-arithmetic operations, but it is truly elementary, depending just on a little algebra and the properties of the Reidemeister moves. Finally, we sketch yet another definition of the fraction for 2-tangles that shows it to be the value of the conductance of an electrical network associated with the tangle.

Section 5 contains a description of our approach to the proof of Theorem 2, the classification of unoriented rational knots and links. The key to this approach is enumerating the different rational tangles whose numerator closure is a given unoriented rational knot or link, and confirming that the corresponding fractions of these tangles satisfy the arithmetic relations of the Theorem. Section 6 sketches the classification of rational knots and links that are isotopic to their mirror images. Such links are all closures of palindromic continued fraction forms of even length. Section 7 describes our proof of Theorem 3, the classification of oriented rational knots. The statement of Theorem 3 differs from the statement of Theorem 2 in the use of integers modulo  $2p$  rather than  $p$ . We see how this difference arises in relation to matching orientations on tangles. This section also includes an explanation of the fact that fractions with even numerators correspond to rational links of two components, while fractions with odd numerators correspond to single component rational knots (the denominators are odd in both cases). Section 8 discusses strongly invertible rational knots and links. These correspond to palindromic continued fractions of odd length.

Section 9 is an introduction to the tangle model for DNA recombination. The classification of the rational knots and links, and the use of the tangle

fractions is the basic topology behind the tangle model for DNA recombination. We indicate how problems in this model are reduced to properties of rational knots, links and tangles, and we show how a finite number of observations of successive DNA recombination can pinpoint the recombination mechanism.

## 2 2-Tangles and Rational Tangles

Throughout this paper we will be working with *2-tangles*. The theory of tangles was discovered by John Conway [7] in his work on enumerating and classifying knots. A 2-tangle is an embedding of two arcs (homeomorphic to the interval  $[0,1]$ ) and circles into a three-dimensional ball  $B^3$  standardly embedded in Euclidean three-space  $S^3$ , such that the endpoints of the arcs go to a specific set of four points on the surface of the ball, so that the circles and the interiors of the arcs are embedded in the interior of the ball. The left-hand side of Figure 1 illustrates a 2-tangle. Finally, a 2-tangle is *oriented* if we assign orientations to each arc and each circle. Without loss of generality, the four endpoints of a 2-tangle can be arranged on a great circle on the boundary of the ball. One can then define a *diagram* of a 2-tangle to be a regular projection of the tangle on the plane of this great circle. In illustrations we may replace this circle by a box.

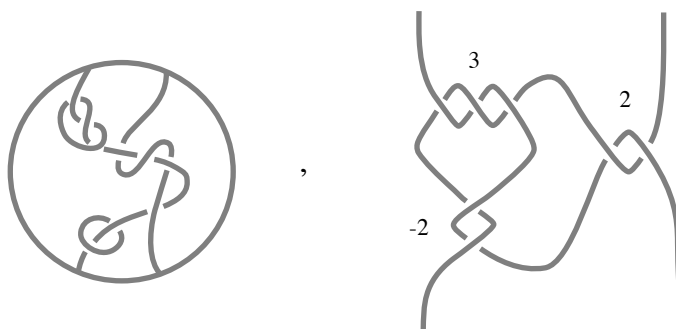


Figure 1 - A 2-tangle and a rational tangle

The simplest possible 2-tangles comprise two unlinked arcs either horizontal or vertical. These are the *trivial tangles*, denoted  $[0]$  and  $[\infty]$  tangles respectively, see Figure 2.



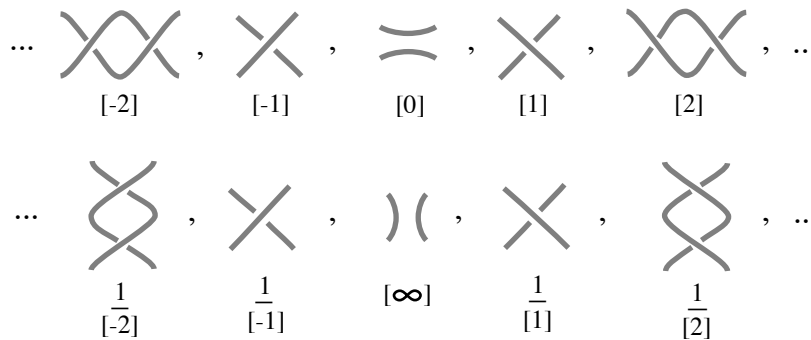
**Figure 2 - The trivial tangles  $[0]$  and  $[\infty]$**

**Definition 1** A 2-tangle is *rational* if it can be obtained by applying a finite number of consecutive twists of neighbouring endpoints to the elementary tangles  $[0]$  or  $[\infty]$ .

The simplest rational tangles are the  $[0]$ , the  $[\infty]$ , the  $[+1]$  and the  $[-1]$  tangles, as illustrated in Figure 3, while the next simplest ones are:

- (i) The *integer tangles*, denoted by  $[n]$ , made of  $n$  horizontal twists,  $n \in \mathbb{Z}$ .
- (ii) The *vertical tangles*, denoted by  $\frac{1}{[n]}$ , made of  $n$  vertical twists,  $n \in \mathbb{Z}$ . These are the inverses of the integer tangles, see Figure 3. This terminology will be clear soon.

Examples of rational tangles are illustrated in the right-hand side of Figure 1 as well as in Figures 8 and 17 below.



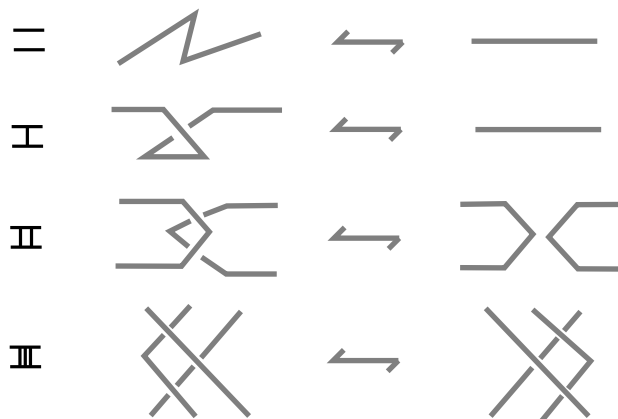
**Figure 3 - The elementary rational tangles**

We study tangles up to *isotopy*. Two 2-tangles,  $T, S$ , in  $B^3$  are said to be *isotopic*, denoted by  $T \sim S$ , if they have identical configurations of their four endpoints in the boundary  $S^2$  of the three-ball, and there is an ambient isotopy

of  $(B^3, T)$  to  $(B^3, S)$  that is the identity on the boundary  $(S^2, \partial T) = (S^2, \partial S)$ . An ambient isotopy can be imagined as a continuous deformation of  $B^3$  fixing the four endpoints on the boundary sphere, and bringing one tangle to the other without causing any self-intersections.

In terms of diagrams, Reidemeister [39] proved that the local moves on diagrams illustrated in Figure 4 capture combinatorially the notion of ambient isotopy of knots, links and tangles in three-dimensional space. That is, if two diagrams represent knots, links or tangles that are isotopic, then the one diagram can be obtained from the other by a sequence of *Reidemeister moves*. In the case of tangles *the endpoints of the tangle remain fixed* and all the moves occur inside the tangle box.

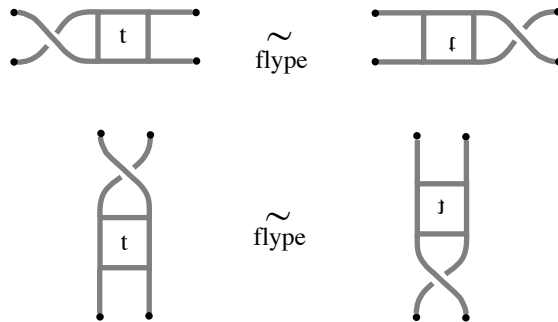
Two oriented 2-tangles are said to be *oriented isotopic* if there is an isotopy between them that preserves the orientations of the corresponding arcs and the corresponding circles. The diagrams of two oriented isotopic tangles differ by a sequence of *oriented Reidemeister moves*, i.e. Reidemeister moves with orientations on the little arcs that remain consistent during the moves.



**Figure 4 - The Reidemeister moves**

From now on we will be thinking in terms of tangle diagrams. Also, we will be referring to both knots and links whenever we say ‘knots’.

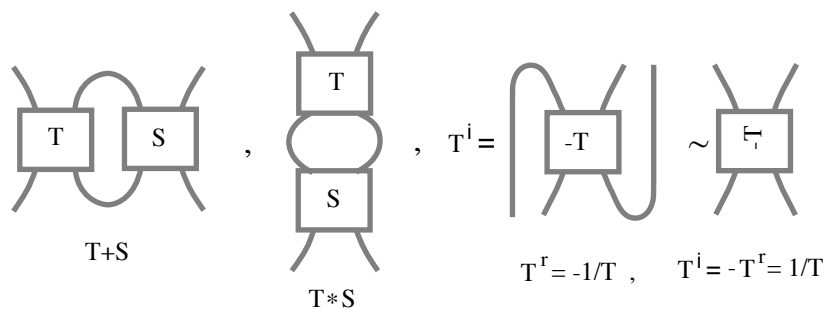
A *flype* is an isotopy move applied on a 2-subtangle of a larger tangle or knot as shown in Figure 5. A flype preserves the alternating structure of a diagram. Even more, flypes are the only isotopy moves needed in the statement of the celebrated Tait Conjecture for alternating knots, stating that *two alternating knots are isotopic if and only if any two corresponding diagrams on  $S^2$  are related by a finite sequence of flypes*. This was posed by P.G. Tait, [49] in 1898 and was proved by W. Menasco and M. Thistlethwaite, [32] in 1993.



**Figure 5 - The flype moves**

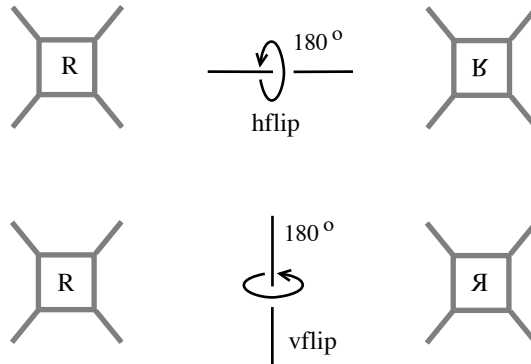
The class of 2-tangles is closed under the operations of *addition* (+) and *multiplication* (\*) as illustrated in Figure 6. Addition is accomplished by placing the tangles side-by-side and attaching the *NE* strand of the left tangle to the *NW* strand of the right tangle, while attaching the *SE* strand of the left tangle to the *SW* strand of the right tangle. The product is accomplished by placing one tangle underneath the other and attaching the upper strands of the lower tangle to the lower strands of the upper tangle.

The *mirror image* of a tangle  $T$  is denoted by  $-T$  and it is obtained by switching all the crossings in  $T$ . Another operation is *rotation* accomplished by turning the tangle counter-clockwise by  $90^\circ$  in the plane. The rotation of  $T$  is denoted by  $T^r$ . The *inverse* of a tangle  $T$ , denoted by  $1/T$ , is defined to be  $-T^r$ . See Figure 6. In general, the inversion or rotation of a 2-tangle is an order 4 operation. Remarkably, for rational tangles the inversion (rotation) is an order 2 operation. It is for this reason that we denote the inverse of a 2-tangle  $T$  by  $1/T$  or  $T^{-1}$ , and hence the rotate of the tangle  $T$  can be denoted by  $-1/T = -T^{-1}$ .



**Figure 6 - Addition, product and inversion of 2-tangles**

We describe now another operation applied on 2-tangles, which turns out to be an isotopy on rational tangles. We say that  $R^{hflip}$  is the *horizontal flip* of the tangle  $R$  if  $R^{hflip}$  is obtained from  $R$  by a  $180^\circ$  rotation around a horizontal axis on the plane of  $R$ . Moreover,  $R^{vflip}$  is the *vertical flip* of the 2-tangle  $R$  if  $R^{vflip}$  is obtained from  $R$  by a  $180^\circ$  rotation around a vertical axis on the plane of  $R$ . See Figure 7 for illustrations. Note that a flip switches the endpoints of the tangle and, in general, a flipped tangle is not isotopic to the original one. *It is a property of rational tangles that  $T \sim T^{hflip}$  and  $T \sim T^{vflip}$  for any rational tangle  $T$ .* This is obvious for the tangles  $[n]$  and  $\frac{1}{[n]}$ . The general proof crucially uses flypes, see [25].

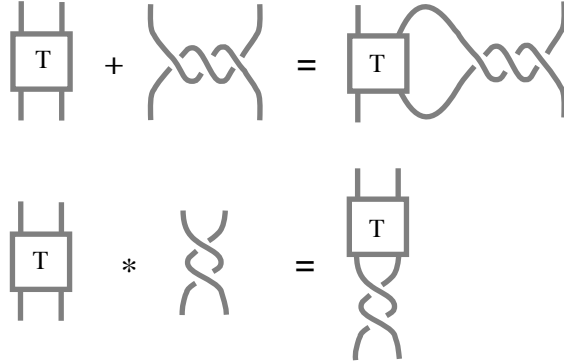


**Figure 7 - The horizontal and the vertical flip**

The above isotopies composed consecutively yield  $T \sim (T^{-1})^{-1} = (T^r)^r$  for any rational tangle  $T$ . This says that inversion (rotation) is an operation of order 2 for rational tangles, so we can rotate the mirror image of  $T$  by  $90^\circ$  either counterclockwise or clockwise to obtain  $T^{-1}$ .

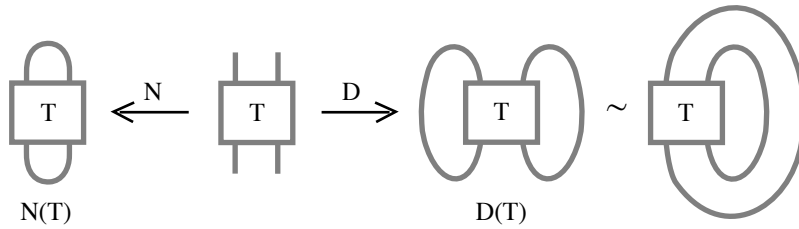
Note that the twists generating the rational tangles could take place between the right, left, top or bottom endpoints of a previously created rational tangle. Using flypes and flips inductively on subtangles one can always bring the twists to the right or bottom of the rational tangle. We shall then say that the rational tangle is in *standard form*. Thus a rational tangle in standard form is created by consecutive additions of the tangles  $[\pm 1]$  *only on the right* and multiplications by the tangles  $[\pm 1]$  *only at the bottom*, starting from the tangles  $[0]$  or  $[\infty]$ . For example, Figure 1 illustrates the tangle  $(([3] * \frac{1}{[-2]}) + [2])$ , while Figure 17 illustrates the tangle  $(([3] * \frac{1}{[2]}) + [2])$  in standard form. Figure 8 illustrates addition on the right and multiplication on the bottom by elementary tangles.





**Figure 8 - Creating new rational tangles**

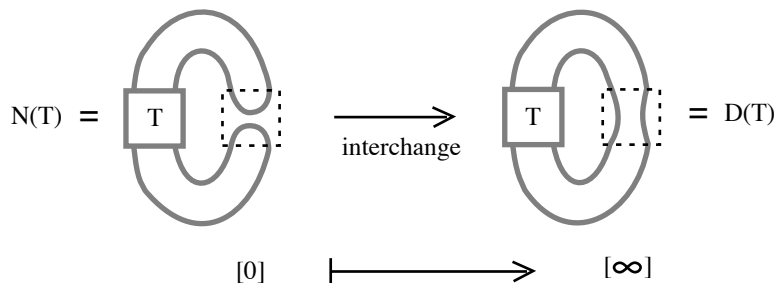
We also have the following *closing* operations, which yield two different knots: the *Numerator* of a 2-tangle  $T$ , denoted by  $N(T)$ , obtained by joining with simple arcs the two upper endpoints and the two lower endpoints of  $T$ , and the *Denominator* of a 2-tangle  $T$ , obtained by joining with simple arcs each pair of the corresponding top and bottom endpoints of  $T$ , denoted by  $D(T)$ . We have  $N(T) = D(T^r)$  and  $D(T) = N(T^r)$ . We note that every knot or link can be regarded as the numerator closure of a 2-tangle.



**Figure 9 - The numerator and denominator of a 2-tangle**

We obtain  $D(T)$  from  $N(T)$  by a  $[0] - [\infty]$  interchange, as shown in Figure 10. This ‘transmutation’ of the numerator to the denominator is a precursor to the tangle model of a recombination event in DNA, see Section 9. The  $[0] - [\infty]$  interchange can be described algebraically by the equations:

$$N(T) = N(T + [0]) \longrightarrow N(T + [\infty]) = D(T).$$



**Figure 10 - The  $[0] - [\infty]$  interchange**

We will concentrate on the class of *rational knots and links* arising from closing the rational tangles. Even though the sum/product of rational tangles is in general not rational, the numerator (denominator) closure of the sum/product of two rational tangles is still a rational knot. It may happen that two rational tangles are not isotopic but have isotopic numerators. This is the basic idea behind the classification of rational knots, see Section 5.

### 3 Continued Fractions and the Classification of Rational Tangles

In this section we assign to a rational tangle a fraction, and we explore the analogy between rational tangles and continued fractions. This analogy culminates in a common canonical form, which is used to deduce the classification of rational tangles.

We first observe that multiplication of a rational tangle  $T$  by  $\frac{1}{[n]}$  may be obtained as addition of  $[n]$  to the inverse  $\frac{1}{T}$  followed by inversion. Indeed, we have:

**Lemma 1** *The following tangle equation holds for any rational tangle  $T$ .*

$$T * \frac{1}{[n]} = \frac{1}{[n] + \frac{1}{T}}.$$

*Thus any rational tangle can be built by a series of the following operations: Addition of  $[\pm 1]$  and Inversion.*

*Proof.* Observe that a  $90^\circ$  clockwise rotation of  $T * \frac{1}{[n]}$  produces  $-[n] - \frac{1}{T}$ . Hence, from the above  $(T * \frac{1}{[n]})^r = -[n] - \frac{1}{T}$ , and thus  $(T * \frac{1}{[n]})^{-1} = [n] + \frac{1}{T}$ . So, taking inversions on both sides yields the tangle equation of the statement.  $\square$

**Definition 2** A *continued fraction in integer tangles* is an algebraic description of a rational tangle via a continued fraction built from the tangles  $[a_1], [a_2], \dots, [a_n]$  with all numerators equal to 1, namely an expression of the type:

$$[[a_1], [a_2], \dots, [a_n]] := [a_1] + \frac{1}{[a_2] + \dots + \frac{1}{[a_{n-1}] + \frac{1}{[a_n]}}$$

for  $a_2, \dots, a_n \in \mathbb{Z} - \{0\}$  and  $n$  even or odd. We allow that the term  $a_1$  may be zero, and in this case the tangle  $[0]$  may be omitted. A rational tangle described via a continued fraction in integer tangles is said to be in *continued fraction form*. The *length* of the continued fraction is arbitrary – in the previous formula illustrated with length  $n$  – whether the first summand is the tangle  $[0]$  or not.

It follows from Lemma 3.2 that inductively *every rational tangle can be written in continued fraction form*. Lemma 3.2 makes it easy to write out the continued fraction form of a given rational tangle, since horizontal twists are integer additions, and multiplications by vertical twists are the reciprocals of integer additions. For example, Figure 1 illustrates the rational tangle  $[2] + \frac{1}{[-2] + \frac{1}{[3]}}$ , Figure 17 illustrates the rational tangle  $[2] + \frac{1}{[2] + \frac{1}{[3]}}$ . Note that

$$([c] * \frac{1}{[b]}) + [a] \text{ has the continued fraction form } [a] + \frac{1}{[b] + \frac{1}{[c]}} = [[a], [b], [c]].$$

For  $T = [[a_1], [a_2], \dots, [a_n]]$  the following statements are now straightforward.

1.  $T + [\pm 1] = [[a_1 \pm 1], [a_2], \dots, [a_n]],$
2.  $\frac{1}{T} = [[0], [a_1], [a_2], \dots, [a_n]],$
3.  $-T = [[-a_1], [-a_2], \dots, [-a_n]].$

We now recall some facts about continued fractions. See for example [28], [35], [29], [50]. In this paper we shall only consider continued fractions of the type

$$[a_1, a_2, \dots, a_n] := a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}$$

for  $a_1 \in \mathbb{Z}$ ,  $a_2, \dots, a_n \in \mathbb{Z} - \{0\}$  and  $n$  even or odd. The *length* of the continued fraction is the number  $n$  whether  $a_1$  is zero or not. Note that if for  $i > 1$  all terms are positive or all terms are negative and  $a_1 \neq 0$  ( $a_1 = 0$ ), then the absolute value of the continued fraction is greater (smaller) than one. Clearly, the two simple algebraic operations *addition of +1 or -1* and *inversion* generate inductively the whole class of continued fractions starting from zero. For any rational number  $\frac{p}{q}$  the following statements are straightforward.

1. there are  $a_1 \in \mathbb{Z}$ ,  $a_2, \dots, a_n \in \mathbb{Z} - \{0\}$  such that  $\frac{p}{q} = [a_1, a_2, \dots, a_n]$ ,
2.  $\frac{p}{q} \pm 1 = [a_1 \pm 1, a_2, \dots, a_n]$ ,
3.  $\frac{q}{p} = [0, a_1, a_2, \dots, a_n]$ ,
4.  $-\frac{p}{q} = [-a_1, -a_2, \dots, -a_n]$ .

We can now define the fraction of a rational tangle.

**Definition 3** Let  $T$  be a rational tangle isotopic to the continued fraction form  $[[a_1], [a_2], \dots, [a_n]]$ . We define *the fraction  $F(T)$  of  $T$*  to be the numerical value of the continued fraction obtained by substituting integers for the integer tangles in the expression for  $T$ , i.e.

$$F(T) := a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_{n-1} + \frac{1}{a_n}}} = [a_1, a_2, \dots, a_n],$$

if  $T \neq [\infty]$ , and  $F([\infty]) := \infty = \frac{1}{0}$ , as a formal expression.

**Remark 1** This definition is good in the sense that one can show that isotopic rational tangles always differ by flypes, and that the fraction is unchanged by flypes [25].

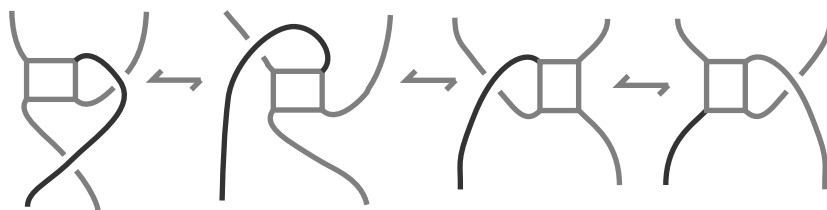
Clearly the tangle fraction has the following properties.

1.  $F(T + [\pm 1]) = F(T) \pm 1$ ,
2.  $F(\frac{1}{T}) = \frac{1}{F(T)}$ ,
3.  $F(-T) = -F(T)$ .

The main result about rational tangles (Theorem 1) is that two rational tangles are isotopic if and only if they have the same fraction. We will show that every rational tangle is isotopic to a unique alternating continued fraction form, and that this alternating form can be deduced from the fraction of the tangle. The Theorem then follows from this observation.

**Lemma 2** *Every rational tangle is isotopic to an alternating rational tangle.*

*Proof.* Indeed, if  $T$  has a non-alternating continued fraction form then the following configuration, shown in the left of Figure 11, must occur somewhere in  $T$ , corresponding to a change of sign from one term to an adjacent term in the tangle continued fraction. This configuration is isotopic to a simpler isotopic configuration as shown in that figure.



**Figure 11 - Reducing to the alternating form**

Therefore, it follows by induction on the number of crossings in the tangle that  $T$  is isotopic to an alternating rational tangle.  $\square$

Recall that a tangle is alternating if and only if it has crossings all of the same type. Thus, a rational tangle  $T = [[a_1], [a_2], \dots, [a_n]]$  is alternating if the  $a_i$ 's are all positive or all negative. For example, the tangle of Figure 17 is alternating.

A rational tangle  $T = [[a_1], [a_2], \dots, [a_n]]$  is said to be in *canonical form* if  $T$  is alternating and  $n$  is odd. The tangle of Figure 17 is in canonical form. We note that if  $T$  is alternating and  $n$  even, then we can bring  $T$  to canonical form by breaking  $a_n$  by a unit, e.g.  $[[a_1], [a_2], \dots, [a_n]] = [[a_1], [a_2], \dots, [a_n - 1], [1]]$ , if  $a_n > 0$ .

The last key observation is the following well-known fact about continued fractions.

**Lemma 3** *Every continued fraction  $[a_1, a_2, \dots, a_n]$  can be transformed to a unique canonical form  $[\beta_1, \beta_2, \dots, \beta_m]$ , where all  $\beta_i$ 's are positive or all negative integers and  $m$  is odd.*

*Proof.* It follows immediately from Euclid's algorithm. We evaluate first  $[a_1, a_2, \dots, a_n] = \frac{p}{q}$ , and using Euclid's algorithm we rewrite  $\frac{p}{q}$  in the desired form. We illustrate the proof with an example. Suppose that  $\frac{p}{q} = \frac{11}{7}$ . Then

$$\begin{aligned} \frac{11}{7} &= 1 + \frac{4}{7} = 1 + \frac{1}{\frac{7}{4}} = 1 + \frac{1}{1 + \frac{3}{4}} = 1 + \frac{1}{1 + \frac{1}{\frac{4}{3}}} \\ &= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3}}} = [1, 1, 1, 3] = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1}}}} = [1, 1, 1, 2, 1]. \end{aligned}$$

This completes the proof.  $\square$

Note that if  $T = [[a_1], [a_2], \dots, [a_n]]$  and  $S = [[b_1], [b_2], \dots, [b_m]]$  are rational tangles in canonical form with the same fraction, then it follows from this Lemma that  $[a_1, a_2, \dots, a_n]$  and  $[b_1, b_2, \dots, b_m]$  are canonical continued fraction forms for the same rational number, and hence are equal term-by-term. Thus the uniqueness of canonical forms for continued fractions implies the uniqueness of canonical forms for rational tangles. For example, let  $T = [[2], [-3], [5]]$ . Then  $F(T) = [2, -3, 5] = \frac{23}{14}$ . But  $\frac{23}{14} = [1, 1, 1, 1, 4]$ , thus  $T \sim [[1], [1], [1], [1], [4]]$ , and this last tangle is the canonical form of  $T$ .

*Proof of Theorem 1.* We have now assembled all the ingredients for the proof of Theorem 1. In one direction, suppose that rational tangles  $T$  and  $S$  are isotopic. Then each is isotopic to its canonical form  $T'$  and  $S'$  by a sequence of flypes. Hence the alternating tangles  $T'$  and  $S'$  are isotopic to one another. By the Tait conjecture, there is a sequence of flypes from  $T'$  to  $S'$ . Hence there is a sequence of flypes from  $T$  to  $S$ . One verifies that the fraction as we defined it is invariant under flypes. Hence  $T$  and  $S$  have the same fraction. In the other direction, suppose that  $T$  and  $S$  have the same fraction. Then, by the remark above, they have identical canonical forms to which they are isotopic, and therefore they are isotopic to each other. This completes the proof of the Theorem.  $\square$

## 4 Alternate Definitions of the Tangle Fraction

In the last section and in [25] the fraction of a rational tangle is defined directly from its combinatorial structure, and we verify the topological invariance of the fraction using the Tait conjecture.

In [25] we give yet another definition of the fraction for rational tangles by using coloring of the tangle arcs. There are definitions that associate a fraction  $F(T)$  (including  $0/1$  and  $1/0$ ) to any 2-tangle  $T$  whether or not it is rational. The first definition is due to John Conway in [7] using the Alexander polynomial of the knots  $N(T)$  and  $D(T)$ . In [16] an alternate definition is given that uses the bracket polynomial of the knots  $N(T)$  and  $D(T)$ , and in

[15] the fraction of a tangle is related to the conductance of an associated electrical network. In all these definitions the fraction is by definition an isotopy invariant of tangles. Below we discuss the bracket polynomial and coloring definitions of the fraction.

#### 4.1 $F(T)$ Through the Bracket Polynomial

In this section we shall discuss the structure of the the bracket state model for the Jones polynomial [19, 20] and how to construct the tangle fraction by using this technique. We first construct the bracket polynomial (state summation), which is a regular isotopy invariant (invariance under all but the Reidemeister move I). The bracket polynomial can be normalized to produce an invariant of all the Reidemeister moves. This invariant is known as the Jones polynomial [17, 18]. The Jones polynomial was originally discovered by a different method.

The *bracket polynomial*,  $\langle K \rangle = \langle K \rangle (A)$ , assigns to each unoriented link diagram  $K$  a Laurent polynomial in the variable  $A$ , such that

1. If  $K$  and  $K'$  are regularly isotopic diagrams, then  $\langle K \rangle = \langle K' \rangle$ .
2. If  $K \amalg O$  denotes the disjoint union of  $K$  with an extra unknotted and unlinked component  $O$  (also called ‘loop’ or ‘simple closed curve’ or ‘Jordan curve’), then

$$\langle K \amalg O \rangle = \delta \langle K \rangle,$$

where

$$\delta = -A^2 - A^{-2}.$$

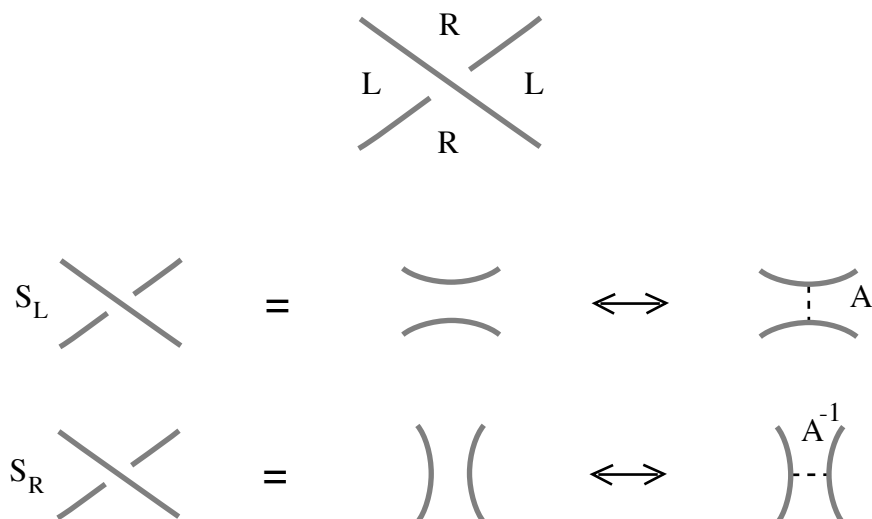
3.  $\langle K \rangle$  satisfies the following formulas

$$\begin{aligned} \langle \chi \rangle &= A \langle \text{crossing} \rangle + A^{-1} \langle \bar{\text{crossing}} \rangle \\ \langle \bar{\chi} \rangle &= A^{-1} \langle \text{crossing} \rangle + A \langle \bar{\text{crossing}} \rangle, \end{aligned}$$

where the small diagrams represent parts of larger diagrams that are identical except at the site indicated in the bracket. We take the convention that the letter chi,  $\chi$ , denotes a crossing where *the curved line is crossing over the straight segment*. The barred letter denotes the switch of this crossing, where *the curved line is undercrossing the straight segment*. The above formulas can be summarized by the single equation

$$\langle K \rangle = A \langle S_L K \rangle + A^{-1} \langle S_R K \rangle .$$

In this text formula we have used the notations  $S_L K$  and  $S_R K$  to indicate the two new diagrams created by the two smoothings of a single crossing in the diagram  $K$ . That is,  $K$ ,  $S_L K$  and  $S_R K$  differ at the site of one crossing in the diagram  $K$ . These smoothings are described as follows. Label the four regions locally incident to a crossing by the letters  $L$  and  $R$ , with  $L$  labelling the region to the left of the undercrossing arc for a traveller who approaches the overcrossing on a route along the undercrossing arc. There are two such routes, one on each side of the overcrossing line. This labels two regions with  $L$ . The remaining two are labelled  $R$ . A smoothing is of *type L* if it connects the regions labelled  $L$ , and it is of *type R* if it connects the regions labelled  $R$ , see Figure 12.



**Figure 12 - Bracket Smoothings**

It is easy to see that Properties 2 and 3 define the calculation of the bracket on arbitrary link diagrams. The choices of coefficients ( $A$  and  $A^{-1}$ ) and the value of  $\delta$  make the bracket invariant under the Reidemeister moves II and III (see [19]). Thus Property 1 is a consequence of the other two properties.

In order to obtain a closed formula for the bracket, we now describe it as a state summation. Let  $K$  be any unoriented link diagram. Define a *state*,  $S$ , of  $K$  to be a choice of smoothing for each crossing of  $K$ . There are two choices for smoothing a given crossing, and thus there are  $2^N$  states of a diagram with  $N$  crossings. In a state we label each smoothing with  $A$  or  $A^{-1}$  according to the left-right convention discussed in Property 3 (see Figure 12). The label



is called a *vertex weight* of the state. There are two evaluations related to a state. The first one is the product of the vertex weights, denoted

$$\langle K|S \rangle .$$

The second evaluation is the number of loops in the state  $S$ , denoted

$$||S||.$$

Define the *state summation*,  $\langle K \rangle$ , by the formula

$$\langle K \rangle = \sum_S \langle K|S \rangle \delta^{||S||-1}.$$

It follows from this definition that  $\langle K \rangle$  satisfies the equations

$$\langle \chi \rangle = A \langle \smile \rangle + A^{-1} \langle \frown \rangle,$$

$$\langle K \amalg O \rangle = \delta \langle K \rangle,$$

$$\langle O \rangle = 1.$$

The first equation expresses the fact that the entire set of states of a given diagram is the union, with respect to a given crossing, of those states with an  $A$ -type smoothing and those with an  $A^{-1}$ -type smoothing at that crossing. The second and the third equation are clear from the formula defining the state summation. Hence this state summation produces the bracket polynomial as we have described it at the beginning of the section.

In computing the bracket, one finds the following behaviour under Reidemeister move I:

$$\langle \gamma \rangle = -A^3 \langle \smile \rangle$$

and

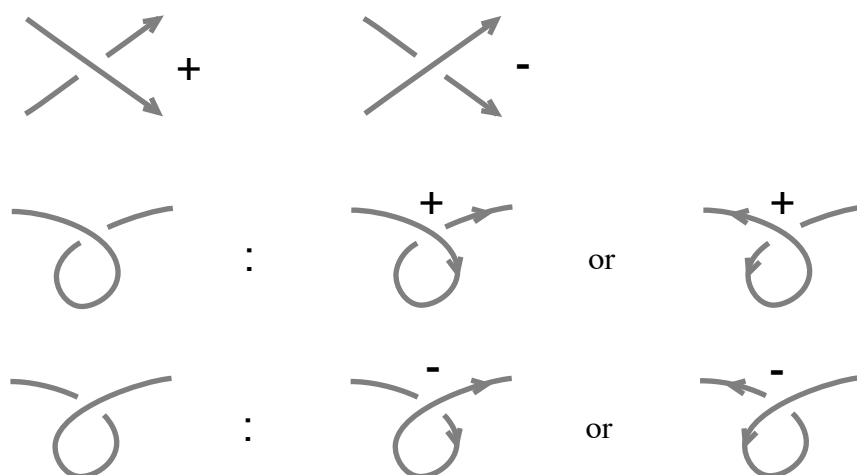
$$\langle \bar{\gamma} \rangle = -A^{-3} \langle \frown \rangle$$

where  $\gamma$  denotes a curl of positive type as indicated in Figure 13, and  $\bar{\gamma}$  indicates a curl of negative type, as also seen in this figure. The type of a curl is the sign of the crossing when we orient it locally. Our convention of signs is also given in Figure 13. Note that the type of a curl does not depend on the orientation we choose. The small arcs on the right hand side of these formulas indicate the removal of the curl from the corresponding diagram.

The bracket is invariant under regular isotopy and can be normalized to an invariant of ambient isotopy by the definition

$$f_K(A) = (-A^3)^{-w(K)} \langle K \rangle (A),$$

where we chose an orientation for  $K$ , and where  $w(K)$  is the sum of the crossing signs of the oriented link  $K$ .  $w(K)$  is called the *writhe* of  $K$ . The convention for crossing signs is shown in Figure 13.



**Figure 13 - Crossing Signs and Curls**

By a change of variables one obtains the original Jones polynomial,  $V_K(t)$ , for oriented knots and links from the normalized bracket:

$$V_K(t) = f_K(t^{-\frac{1}{4}}).$$

The bracket model for the Jones polynomial is quite useful both theoretically and in terms of practical computations. One of the neatest applications is to simply compute  $f_K(A)$  for the trefoil knot  $T$  and determine that  $f_T(A)$  is not equal to  $f_T(A^{-1}) = f_{-T}(A)$ . This shows that the trefoil is not ambient isotopic to its mirror image, a fact that is quite tricky to prove by classical methods.

For 2-tangles, we do smoothings on the tangle diagram until there are no crossings left. As a result, a *state of a 2-tangle* consists in a collection of loops in the tangle box, plus simple arcs that connect the tangle ends. The loops evaluate to powers of  $\delta$ , and what is left is either the tangle  $[0]$  or the tangle  $[\infty]$ , since  $[0]$  and  $[\infty]$  are the only ways to connect the tangle inputs

and outputs without introducing any crossings in the diagram. In analogy to knots and links, we can find a *state summation* formula for the *bracket of the tangle*, denoted  $\langle T \rangle$ , by summing over the states obtained by smoothing each crossing in the tangle. For this we define the *remainder of a state*, denoted  $R_S$ , to be either the tangle  $[0]$  or the tangle  $[\infty]$ . Then the evaluation of  $\langle T \rangle$  is given by

$$\langle T \rangle = \sum_S \langle T|S \rangle \delta^{\|S\|} \langle R_S \rangle,$$

where  $\langle T|S \rangle$  is the product of the vertex weights ( $A$  or  $A^{-1}$ ) of the state  $S$  of  $T$ . The above formula is consistent with the formula for knots obtained by taking the closure  $N(T)$  or  $D(T)$ . In fact, we have the following formula:

$$\langle N(T) \rangle = \sum_S \langle T|S \rangle \delta^{\|S\|} \langle N(R_S) \rangle.$$

Note that  $\langle N([0]) \rangle = \delta$  and  $\langle N([\infty]) \rangle = 1$ . A similar formula holds for  $\langle D(T) \rangle$ . Thus,  $\langle T \rangle$  appears as a linear combination with Laurent polynomial coefficients of  $\langle [0] \rangle$  and  $\langle [\infty] \rangle$ , i.e.  $\langle T \rangle$  takes values in the free module over  $\mathbb{Z}[A, A^{-1}]$  with basis  $\{\langle [0] \rangle, \langle [\infty] \rangle\}$ . Notice that two elements in this module are equal iff the corresponding coefficients of the basis elements coincide. Note also that  $\langle T \rangle$  is an invariant of regular isotopy with values in this module. We have just proved the following:

**Lemma 4** *Let  $T$  be any 2-tangle and let  $\langle T \rangle$  be the formal expansion of the bracket on this tangle. Then there exist elements  $n_T(A)$  and  $d_T(A)$  in  $\mathbb{Z}[A, A^{-1}]$ , such that*

$$\langle T \rangle = d_T(A) \langle [0] \rangle + n_T(A) \langle [\infty] \rangle,$$

and  $n_T(A)$  and  $d_T(A)$  are regular isotopy invariants of the tangle  $T$ .

In order to evaluate  $\langle N(T) \rangle$  in the formula above we need only apply the closure  $N$  to  $[0]$  and  $[\infty]$ . More precisely, we have:

**Lemma 5**  $\langle N(T) \rangle = d_T \delta + n_T$  and  $\langle D(T) \rangle = d_T + n_T \delta$ .

*Proof.* Since the smoothings of crossings do not interfere with the closure ( $N$  or  $D$ ), the closure will carry through linearly to the whole sum of  $\langle T \rangle$ . Thus,

$$\begin{aligned} \langle N(T) \rangle &= d_T(A) \langle N([0]) \rangle + n_T(A) \langle N([\infty]) \rangle = d_T(A) \delta + n_T(A), \\ \langle D(T) \rangle &= d_T(A) \langle D([0]) \rangle + n_T(A) \langle D([\infty]) \rangle = d_T(A) + n_T(A) \delta. \square \end{aligned}$$

We define now the *polynomial fraction*,  $frac_T(A)$ , of the 2-tangle  $T$  to be the ratio

$$frac_T(A) = \frac{n_T(A)}{d_T(A)}$$

in the ring of fractions of  $\mathbb{Z}[A, A^{-1}]$  with a formal symbol  $\infty$  adjoined.

**Lemma 6**  *$frac_T(A)$  is an invariant of ambient isotopy for 2-tangles.*

*Proof.* Since  $d_T$  and  $n_T$  are regular isotopy invariants of  $T$ , it follows that  $frac_T(A)$  is also a regular isotopy invariant of  $T$ . Suppose now  $T\gamma$  is  $T$  with a curl added. Then  $\langle T\gamma \rangle = (-A^3) \langle T \rangle$  (same remark for  $\bar{\gamma}$ ). So,  $n_{T\gamma}(A) = -A^3 n_T(A)$  and  $d_{T\gamma}(A) = -A^3 d_T(A)$ . Thus,  $n_{T\gamma}/d_{T\gamma} = n_T/d_T$ . This shows that  $frac_T$  is also invariant under the Reidemeister move I, and hence an ambient isotopy invariant.  $\square$

**Lemma 7** *Let  $T$  and  $S$  be two 2-tangles. Then, we have the following formula for the bracket of the sum of the tangles.*

$$\langle T + S \rangle = d_T d_S \langle [0] \rangle + (d_T n_S + n_T d_S + n_S \delta) \langle [\infty] \rangle .$$

Thus

$$frac_{T+S} = frac_T + frac_S + \frac{n_S \delta}{d_T d_S} .$$

*Proof.* We do first the smoothings in  $T$  leaving  $S$  intact, and then in  $S$ :

$$\begin{aligned} \langle T + S \rangle &= d_T \langle [0] + S \rangle + n_T \langle [\infty] + S \rangle \\ &= d_T \langle S \rangle + n_T \langle [\infty] + S \rangle \\ &= d_T (d_S \langle [0] \rangle + n_S \langle [\infty] \rangle) \\ &\quad + n_T (d_S \langle [\infty] + [0] \rangle + n_S \langle [\infty] + [\infty] \rangle) \\ &= d_T (d_S \langle [0] \rangle + n_S \langle [\infty] \rangle) + n_T (d_S \langle [\infty] \rangle + n_S \delta \langle [\infty] \rangle) \\ &= d_T d_S \langle [0] \rangle + (d_T n_S + n_T d_S + n_S \delta) \langle [\infty] \rangle . \end{aligned}$$

Thus,  $n_{T+S} = (d_T n_S + n_T d_S + n_S \delta)$  and  $d_{T+S} = d_T d_S$ . A straightforward calculation gives now  $frac_{T+S}$ .  $\square$

As we see from Lemma 4,  $frac_T(A)$  will be additive on tangles if

$$\delta = -A^2 - A^{-2} = 0 .$$

Moreover, from Lemma 2 we have for  $\delta = 0$ ,  $\langle N(T) \rangle = n_T$ ,  $\langle D(T) \rangle = d_T$ . This nice situation will be our main object of study in the rest of this section. Now, if we set  $A = \sqrt{i}$  where  $i^2 = -1$ , then it is

$$\delta = -A^2 - A^{-2} = -i - i^{-1} = -i + i = 0.$$

For this reason, we shall henceforth assume that  $A$  takes the value  $\sqrt{i}$ . So  $\langle K \rangle$  will denote  $\langle K \rangle (\sqrt{i})$  for any knot or link  $K$ .

We now define the 2-tangle fraction  $F(T)$  by the following formula:

$$F(T) = i \frac{n_T(\sqrt{i})}{d_T(\sqrt{i})}.$$

We will let  $n(T) = n_T(\sqrt{i})$  and  $d(T) = d_T(\sqrt{i})$ , so that

$$F(T) = i \frac{n(T)}{d(T)}.$$

**Lemma 8** *The 2-tangle fraction has the following properties.*

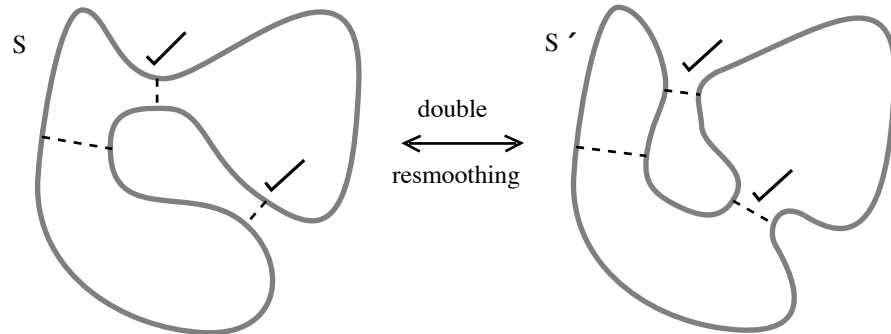
1.  $F(T) = i \langle N(T) \rangle / \langle D(T) \rangle$ , and it is a real number or  $\infty$ ,
2.  $F(T + S) = F(T) + F(S)$ ,
3.  $F([0]) = \frac{0}{1}$ ,
4.  $F([1]) = \frac{1}{1}$ ,
5.  $F([\infty]) = \frac{1}{0}$ ,
6.  $F(-T) = -F(T)$ , in particular  $F([-1]) = -\frac{1}{1}$ ,
7.  $F(1/T) = 1/F(T)$ ,
8.  $F(T^r) = -1/F(T)$ .

As a result we conclude that for a tangle obtained by arithmetic operations from integer tangles  $[n]$ , the fraction of that tangle is the same as the fraction obtained by doing the same operations to the corresponding integers. (This will be studied in detail in the next section.)

*Proof.* The formula  $F(T) = i \langle N(T) \rangle / \langle D(T) \rangle$  and Statement 2. follow from the observations above about  $\delta = 0$ . In order to show that  $F(T)$  is a real number or  $\infty$  we first consider  $\langle K \rangle := \langle K \rangle (\sqrt{i})$ , for  $K$  a knot or link, as in the hypotheses prior to the lemma. Then we apply this information to the ratio  $i \langle N(T) \rangle / \langle D(T) \rangle$ .

Let  $K$  be any knot or link. We claim that then  $\langle K \rangle = \omega p$ , where  $\omega$  is a power of  $\sqrt{i}$  and  $p$  is an integer. In fact, we will show that each non-trivial

state of  $K$  contributes  $\pm\omega$  to  $\langle K \rangle$ . In order to show this, we examine how to get from one non-trivial state to another. It is a fact that, for any two states, we can get from one to the other by resmoothing a subset of crossings. It is possible to get from any single loop state (and only single loop states of  $K$  contribute to  $\langle K \rangle$ , since  $\delta = 0$ ) to any other single loop state by a series of *double resmoothings*. In a double resmoothing we resmooth two crossings, such that one of the resmoothings disconnects the state and the other reconnects it. See Figure 14 for an illustration. Now consider the effect of a double resmoothing on the evaluation of one state. Two crossings change. If one is labelled  $A$  and the other  $A^{-1}$ , then there is no net change in the evaluation of the state. If both are  $A$ , then we go from  $A^2P$  ( $P$  is the rest of the product of state labels) to  $A^{-2}P$ . But  $A^2 = i$  and  $A^{-2} = -i$ . Thus if one state contributes  $\omega = ip$ , then the other state contributes  $-\omega = -ip$ . These remarks prove the claim.



**Figure 14 - A Double Resmoothing**

Now, a state that contributes non-trivially to  $N(T)$  must have the form of the tangle  $[\infty]$ . We will show that if  $S$  is a state of  $T$  contributing non-trivially to  $\langle N(T) \rangle$  and  $S'$  a state of  $T$  contributing non-trivially to  $\langle D(T) \rangle$ , then  $\langle S \rangle / \langle S' \rangle = \pm i$ . Here  $\langle S \rangle$  denotes the product of the vertex weights for  $S$ , and  $\langle S' \rangle$  is the product of the vertex weights for  $S'$ . If this ratio is verified for some pair of states  $S, S'$ , then it follows from the first claim that it is true for all pairs of states, and that  $\langle N(T) \rangle = \omega p$ ,  $\langle D(T) \rangle = \omega' q$ ,  $p, q \in \mathbb{Z}$  and  $\omega/\omega' = \langle S \rangle / \langle S' \rangle = \pm i$ . Hence  $\langle N(T) \rangle / \langle D(T) \rangle = \pm i p/q$ , where  $p/q$  is a rational number (or  $q = 0$ ). This will complete the proof that  $F(T)$  is real or  $\infty$ .

To see this second claim we consider specific pairs of states as in Figure 15. We have illustrated representative states  $S$  and  $S'$  of the tangle  $T$ . We obtain

$S'$  from  $S$  by resmoothing at one site that changes  $S$  from an  $[\infty]$  tangle to the  $[0]$  tangle underlying  $S'$ . Then  $\langle S \rangle / \langle S' \rangle = A^{\pm 2} = \pm i$ . If there is no such resmoothing site available, then it follows that  $D(T)$  is a disjoint union of two diagrams, and hence  $\langle D(T) \rangle = 0$  and  $F(T) = \infty$ . This does complete the proof of Statement 1.

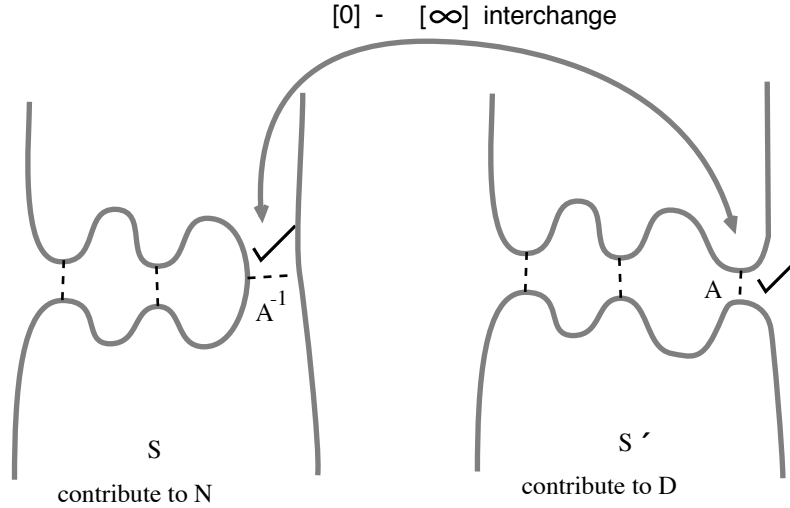


Figure 15 - Non-trivial States

At  $\delta = 0$  we also have:

$\langle N([0]) \rangle = 0$ ,  $\langle D([0]) \rangle = 1$ ,  $\langle N([\infty]) \rangle = 1$ ,  $\langle D([\infty]) \rangle = 0$ , and so, the evaluations 3. to 5. are easy. For example, note that

$$\langle [1] \rangle = A \langle [0] \rangle + A^{-1} \langle [\infty] \rangle,$$

hence

$$F([1]) = i \frac{A^{-1}}{A} = i A^{-2} = i (i^{-1}) = 1.$$

To have the fraction value 1 for the tangle  $[1]$  is the reason that in the definition of  $F(T)$  we normalized by  $i$ . Statement 6. follows from the fact that the bracket of the mirror image of a knot  $K$  is the same as the bracket of  $K$ , but with  $A$  and  $A^{-1}$  switched. For proving 7. we observe first that for any 2-tangle  $T$ ,  $d(\frac{1}{T}) = \overline{n(T)}$  and  $n(\frac{1}{T}) = \overline{d(T)}$ , where the overline denotes the complex conjugate. Complex conjugates occur because  $A^{-1} = \overline{A}$  when  $A = \sqrt{i}$ . Now, since  $F(T)$  is real, we have

$$F(\frac{1}{T}) = i \overline{d(T)} / \overline{n(T)} = -i d(T) / n(T) = 1 / (i n(T) / d(T)) = \overline{1 / F(T)} = 1 / F(T).$$

Statement 8. follows immediately from 6. and 7. This completes the proof.  $\square$

For a related approach to the well-definedness of the 2-tangle fraction, the reader should consult [30]. The double resmoothing idea originates from [23].

**Remark 2** For any knot or link  $K$  we define the *determinant* of  $K$  by the formula

$$\text{Det}(K) := | \langle K \rangle (\sqrt{i}) |$$

where  $|z|$  denotes the modulus of the complex number  $z$ . Thus we have the formula

$$|F(T)| = \frac{\text{Det}(N(T))}{\text{Det}(D(T))}$$

for any 2-tangle  $T$ .

In other approaches to the theory of knots, the determinant of the knot is actually the determinant of a certain matrix associated either to the diagram for the knot or to a surface whose boundary is the knot. See [42, 22] for more information on these connections. Conway's original definition of the fraction [7] is  $\Delta_{N(T)}(-1)/\Delta_{D(T)}(-1)$  where  $\Delta_K(-1)$  denotes the evaluation of the Alexander polynomial of a knot  $K$  at the value  $-1$ . In fact,  $|\Delta_K(-1)| = \text{Det}(K)$ , and with appropriate attention to signs, the Conway definition and our definition using the bracket polynomial coincide for all 2-tangles.

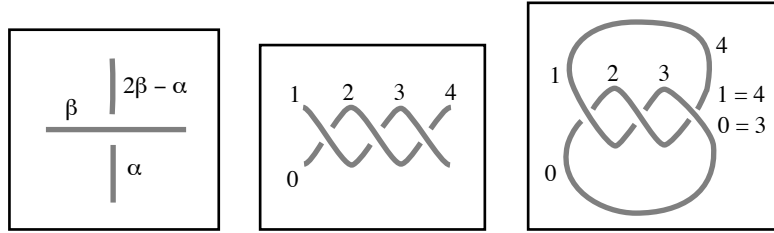
## 4.2 The Fraction through Coloring

We conclude this section by giving an alternate definition of the fraction that uses the concept of coloring of knots and tangles. We color the arcs of the knot/tangle with integers, using the basic coloring rule that if two undercrossing arcs colored  $\alpha$  and  $\gamma$  meet at an overcrossing arc colored  $\beta$ , then  $\alpha + \gamma = 2\beta$ . We often think of one of the undercrossing arc colors as determined by the other two colors. Then one writes  $\gamma = 2\beta - \alpha$ .

It is easy to verify that this coloring method is invariant under the Reidemeister moves in the following sense: Given a choice of coloring for the tangle/knot, there is a way to re-color it each time a Reidemeister move is performed, so that no change occurs to the colors on the external strands of the tangle (so that we still have a valid coloring). This means that a coloring potentially contains topological information about a knot or a tangle. In coloring a knot (and also many non-rational tangles) it is usually necessary to restrict the colors to the set of integers modulo  $N$  for some modulus  $N$ . For example, in Figure 16 it is clear that the color set  $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$  is forced for coloring a trefoil knot. When there exists a coloring of a tangle by integers,



so that it is not necessary to reduce the colors over some modulus we shall say that the tangle is *integrally colorable*.



**Figure 16 - The coloring rule, integral and modular coloring**

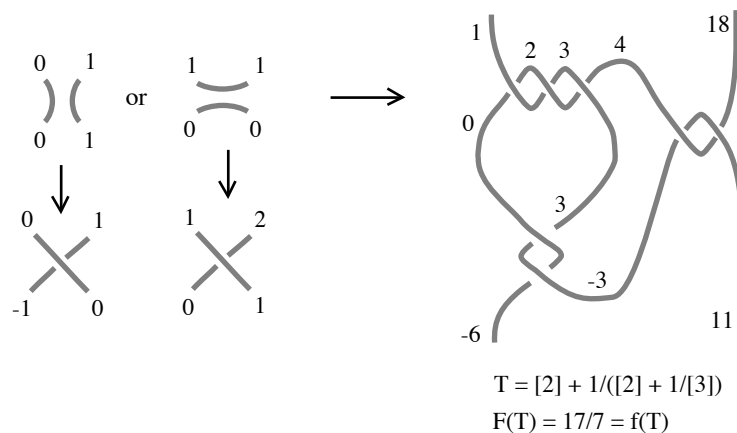
It turns out that *every rational tangle is integrally colorable*: To see this choose two ‘colors’ for the initial strands (e.g. the colors 0 and 1) and color the rational tangle as you create it by successive twisting. We call the colors on the initial strands the *starting colors*. See Figure 17 for an example. It is important that we start coloring from the initial strands, because then the coloring propagates automatically and uniquely. If one starts from somewhere else, one might get into an edge with an undetermined color. The resulting colored tangle now has colors assigned to its external strands at the northwest, northeast, southwest and southeast positions. Let  $NW(T)$ ,  $NE(T)$ ,  $SW(T)$  and  $SE(T)$  denote these respective colors of the colored tangle  $T$  and define the *color matrix of  $T$* ,  $M(T)$ , by the equation

$$M(T) = \begin{bmatrix} NW(T) & NE(T) \\ SW(T) & SE(T) \end{bmatrix}.$$

**Definition 4** To a rational tangle  $T$  with color matrix  $M(T) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  we associate the number

$$f(T) := \frac{b - a}{b - d} \in \mathcal{Q} \cup \infty.$$

It turns out that the entries  $a, b, c, d$  of a color matrix of a rational tangle satisfy the ‘diagonal sum rule’:  $a + d = b + c$ .



**Figure 17 - Coloring Rational Tangles**

**Proposition 1** *The number  $f(T)$  is a topological invariant associated with the tangle  $T$ . In fact,  $f(T)$  has the following properties:*

1.  $f(T + [\pm 1]) = f(T) \pm 1$ ,
2.  $f(-\frac{1}{T}) = -\frac{1}{f(T)}$ ,
3.  $f(-T) = -f(T)$ ,
4.  $f(\frac{1}{T}) = \frac{1}{f(T)}$ ,
5.  $f(T) = F(T)$ .

*Thus the coloring fraction is identical to the arithmetical fraction defined earlier.*

It is easy to see that  $f([0]) = \frac{0}{1}$ ,  $f([\infty]) = \frac{1}{0}$ ,  $f([\pm 1]) = \pm 1$ . Hence Statement 5 follows by induction. For proofs of all statements above as well as for a more general set-up we refer the reader to our paper [25]. This definition is quite elementary, but applies only to rational tangles and tangles generated from them by the algebraic operations of ‘+’ and ‘\*’.

In Figure 17 we have illustrated a coloring over the integers for the tangle  $[[2], [2], [3]]$  such that every edge is labelled by a different integer. This is always the case for an alternating rational tangle diagram  $T$ . For the numerator closure  $N(T)$  one obtains a coloring in a modular number system. For example in Figure 17 the coloring of  $N(T)$  will be in  $\mathbb{Z}/17\mathbb{Z}$ , and it is easy to check that the labels remain distinct in this example. For rational tangles, this is always the case when  $N(T)$  has a prime determinant, see [25] and [36]. It is part of a more general conjecture about alternating knots and links [24, 1].

### 4.3 The Fraction through Conductance

Conductance is a quantity defined in electrical networks as the inverse of resistance. For pure resistances, conductance is a positive quantity. Negative conductance corresponds to amplification, and is commonly included in the physical formalism. One defines the conductance between two vertices in a graph (with positive or negative conductance weights on the edges of the graph) as a sum of weighted trees in the graph divided by a sum of weighted trees of the same graph, but with the two vertices identified. This definition allows negative values for conductance and it agrees with the classical one. Conductance satisfies familiar laws of parallel and series connection as well as a star-triangle relation.

By associating to a given knot or link diagram the corresponding signed checkerboard graph (see [25, 15] for a definition of this well-known association of graph to link diagram), one can define [15] the conductance of a knot or link between any two regions that receive the same color in the checkerboard graph. The conductance of the link between these two regions is an isotopy invariant of the link (with motion restricted to Reidemeister moves that do not pass across the selected regions). This invariance follows from properties of series/parallel connection and the star-triangle relation. These circuit laws turn out to be images of the Reidemeister moves under the translation from knot or link diagram to checkerboard graph! For a 2-tangle we take the conductance to be the conductance of the numerator of the tangle, between the two bounded regions adjacent to the closures at the top and bottom of the tangle.

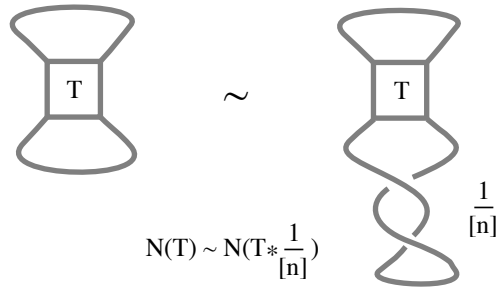
The conductance of a 2-tangle turns out to be the same as the fraction of the tangle. This provides yet another way to define and verify the isotopy invariance of the tangle fraction for any 2-tangle.

## 5 The Classification of Unoriented Rational Knots

By taking their numerators or denominators rational tangles give rise to a special class of knots, the rational knots. We have seen so far that rational tangles are directly related to finite continued fractions. We carry this insight further into the classification of rational knots (Schubert's theorems). In this section we consider unoriented knots, and by Remark 3.1 we will be using the 3-strand-braid representation for rational tangles with odd number of terms. Also, by Lemma 2, we may assume all rational knots to be alternating. Note

that we only need to take numerator closures, since the denominator closure of a tangle is simply the numerator closure of its rotate.

As already said in the introduction, it may happen that two rational tangles are non-isotopic but have isotopic numerators. The simplest instance of this phenomenon is adding  $n$  twists at the bottom of a tangle  $T$ , see Figure 18. This operation does not change the knot  $N(T)$ , i.e.  $N(T * 1/[n]) \sim N(T)$ , but it does change the tangle, since  $F(T * 1/[n]) = F(1/([n] + 1/T)) = 1/(n + 1/F(T))$ ; so, if  $F(T) = p/q$ , then  $F(T * 1/[n]) = p/(np + q)$ . Hence, if we set  $np + q = q'$  we have  $q \equiv q' \pmod{p}$ , just as Theorem 2 dictates. Note that reducing all possible bottom twists implies  $|p| > |q|$ .



**Figure 18 - Twisting the bottom of a tangle**

Another key example of the arithmetic relationship of the classification of rational knots is illustrated in Figure 19. Here we see that the ‘palindromic’ tangles

$$T = [[2], [3], [4]] = [2] + \frac{1}{[3] + \frac{1}{[4]}}$$

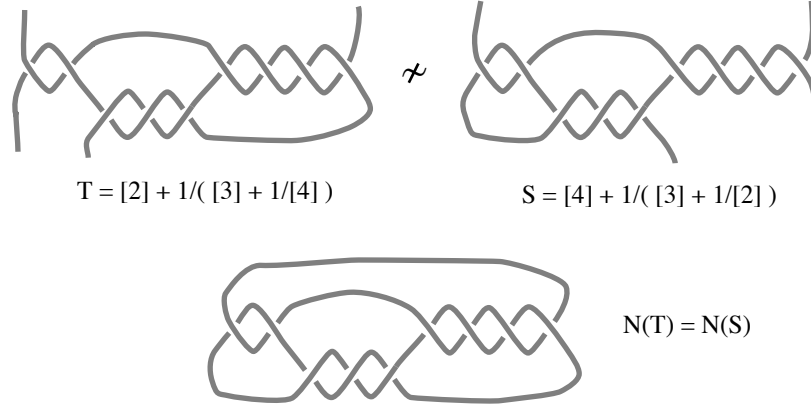
and

$$S = [[4], [3], [2]] = [4] + \frac{1}{[3] + \frac{1}{[2]}}$$

both close to the same rational knot, shown at the bottom of the figure. The two tangles are different, since they have different corresponding fractions:

$$F(T) = 2 + \frac{1}{3 + \frac{1}{4}} = \frac{30}{13} \quad \text{and} \quad F(S) = 4 + \frac{1}{3 + \frac{1}{2}} = \frac{30}{7}.$$

Note that the product of 7 and 13 is congruent to 1 modulo 30.



**Figure 19 - An Instance of the Palindrome Equivalence**

More generally, consider the following two fractions:

$$F = [a, b, c] = a + \frac{1}{b + \frac{1}{c}} \quad \text{and} \quad G = [c, b, a] = c + \frac{1}{b + \frac{1}{a}}.$$

We find that

$$F = a + c \frac{1}{cb + 1} = \frac{abc + a + c}{bc + 1} = \frac{P}{Q},$$

while

$$G = c + a \frac{1}{ab + 1} = \frac{abc + c + a}{ab + 1} = \frac{P}{Q'}.$$

Thus we found that  $F = \frac{P}{Q}$  and  $G = \frac{P}{Q'}$ , where

$$QQ' = (bc + 1)(ab + 1) = ab^2c + ab + bc + 1 = bP + 1.$$

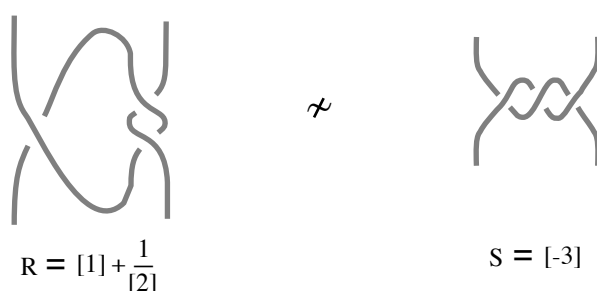
Assuming that  $a$ ,  $b$  and  $c$  are integers, we conclude that

$$QQ' \equiv 1 \pmod{P}.$$

This pattern generalizes to arbitrary continued fractions and their palindromes (obtained by reversing the order of the terms). I.e. *If  $\{a_1, a_2, \dots, a_n\}$  is a collection of  $n$  non-zero integers, and if  $A = [a_1, a_2, \dots, a_n] = \frac{P}{Q}$  and  $B = [a_n, a_{n-1}, \dots, a_1] = \frac{P'}{Q'}$ , then  $P = P'$  and  $QQ' \equiv (-1)^{n+1} \pmod{P}$ .* We will be referring to this as ‘the Palindrome Theorem’. The Palindrome Theorem is a known result about continued fractions. For example, see [46] and [26]. Note that we need  $n$  to be odd in the previous congruence. This agrees

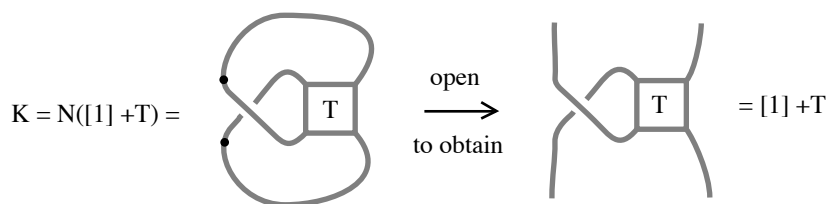
with Remark 3.1 that without loss of generality the terms in the continued fraction of a rational tangle may be assumed to be odd.

Finally, Figure 20 illustrates another basic example for the unoriented Schubert Theorem. The two tangles  $R = [1] + \frac{1}{[2]}$  and  $S = [-3]$  are non-isotopic by the Conway Theorem, since  $F(R) = 1 + 1/2 = 3/2$  while  $F(S) = -3 = 3/-1$ . But they have isotopic numerators:  $N(R) \sim N(S)$ , the left-handed trefoil. Now 2 is congruent to  $-1$  modulo 3, confirming Theorem 2.



**Figure 20 - An Example of the Special Cut**

We now analyse the above example in general. From the analysis of the bottom twists we can assume without loss of generality that a rational tangle  $R$  has fraction  $\frac{P}{Q}$ , for  $|P| > |Q|$ . Thus  $R$  can be written in the form  $R = [1] + T$  or  $R = [-1] + T$ . We consider the rational knot diagram  $K = N([1] + T)$ , see Figure 21. (We analyze  $N([-1] + T)$  in the same way.) The tangle  $[1] + T$  is said to arise as a *standard cut* on  $K$ .



**Figure 21 - A Standard Cut**

Notice that the indicated horizontal crossing of  $N([1] + T)$  could be also seen as a vertical one. So, we could also cut the diagram  $K$  at the two other marked points (see Figure 22) and still obtain a rational tangle, since  $T$  is

rational. The tangle obtained by cutting  $K$  in this second pair of points is said to arise as a *special cut* on  $K$ . Figure 22 demonstrates that the tangle of the special cut is the tangle  $[-1] - 1/T$ . So we have  $N([1] + T) \sim N([-1] - \frac{1}{T})$ . Suppose now  $F(T) = p/q$ . Then  $F([1] + T) = 1 + p/q = (p + q)/q$ , while  $F([-1] - 1/T) = -1 - q/p = (p + q)/(-p)$ , so the two rational tangles that give rise to the same knot  $K$  are not isotopic. Since  $-p \equiv q \pmod{p + q}$ , this equivalence is another example for Theorem 2. In Figure 22 if we took  $T = \frac{1}{[2]}$  then  $[-1] - 1/T = [-3]$  and we would obtain the example of Figure 20.

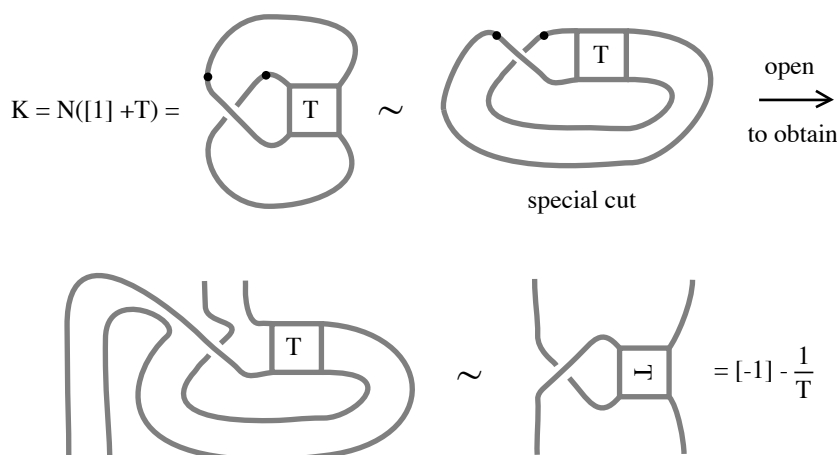
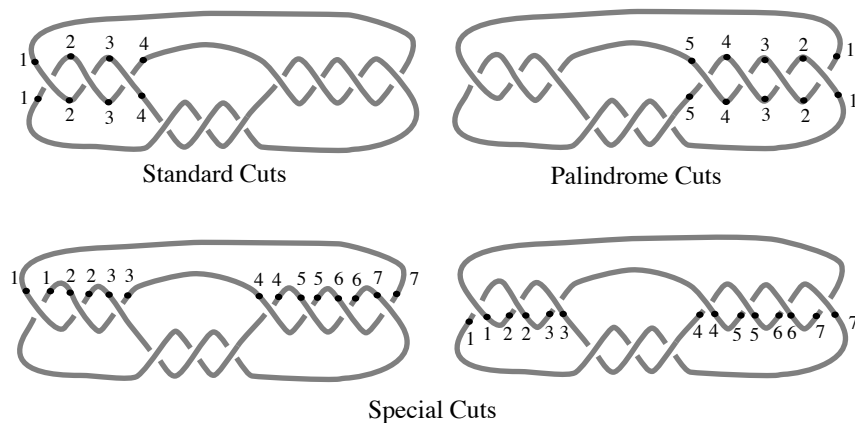


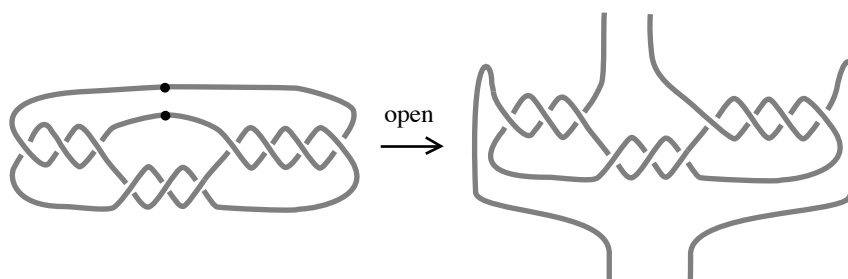
Figure 22 - A Special Cut

The proof of Theorem 2 can now proceed in two stages. First, given a rational knot diagram we look for all possible places where we could cut and open it to a rational tangle. The crux of our proof in [26] is the fact that all possible ‘rational cuts’ on a rational knot fall into one of the basic cases that we have already discussed. I.e. we have the standard cuts, the palindrome cuts and the special cuts. In Figure 23 we illustrate on a representative rational knot, all the cuts that exhibit that knot as a closure of a rational tangle. Each pair of points is marked with the same number. The arithmetics is similar to the cases that have been already verified. It is convenient to say that reduced fractions  $p/q$  and  $p'/q'$  are *arithmetically equivalent*, written  $p/q \sim p'/q'$  if  $p = p'$  and either  $qq' \equiv 1 \pmod{p}$  or  $q \equiv q' \pmod{p}$ . In this language, Schubert’s theorem states that two rational tangles close to form isotopic knots if and only if their fractions are arithmetically equivalent.



**Figure 23 - Standard, Palindrome and Special Cuts**

In Figure 24 we illustrate one example of a cut that is not allowed since it opens the knot to a non-rational tangle.



**Figure 24 - A Non-Rational Cut**

In the second stage of the proof we want to check the arithmetic equivalence for two different given knot diagrams, numerators of some rational tangles. By Lemma 2 the two knot diagrams may be assumed alternating, so by the Tait Conjecture they will differ by flypes. We analyse all possible flypes to prove that no new cases for study arise. Hence the proof becomes complete at that point. We refer the reader to our paper [26] for the details.  $\square$

**Remark 3** The original proof of the classification of unoriented rational knots by Schubert [45] proceeded by a different route than the proof we have just



sketched. Schubert used a 2-bridge representation of rational knots (representing the knots and links as diagrams in the plane with two special overcrossing arcs, called the bridges). From the 2-bridge representation, one could extract a fraction  $p/q$ , and Schubert showed by means of a canonical form, that if two such presentations are isotopic, then their fractions are arithmetically equivalent (in the sense that we have described here). On the other hand, Seifert [45] observed that the 2-fold branched covering space of a 2-bridge presentation with fraction  $p/q$  is a lens space of type  $L(p, q)$ . Lens spaces are a particularly tractable set of three manifolds, and it is known by work of Reidemeister and Franz [41, 14] that  $L(p, q)$  is homeomorphic to  $L(p', q')$  if and only if  $p/q$  and  $p'/q'$  are arithmetically equivalent. Furthermore, one knows that if knots  $K$  and  $K'$  are isotopic, then their 2-fold branched covering spaces are homeomorphic. Hence it follows that if two rational knots are isotopic, then their fractions are arithmetically equivalent (via the result of Reidemeister and Franz classifying lens spaces). In this way Schubert proved that two rational knots are isotopic if and only if their fractions are arithmetically equivalent.

## 6 Rational Knots and Their Mirror Images

In this section we give an application of Theorem 2. An unoriented knot or link  $K$  is said to be *achiral* if it is topologically equivalent to its mirror image  $-K$ . If a link is not equivalent to its mirror image then it is said to be *chiral*. One then can speak of the *chirality* of a given knot or link, meaning whether it is chiral or achiral. Chirality plays an important role in the applications of Knot Theory to Chemistry and Molecular Biology. It is interesting to use the classification of rational knots and links to determine their chirality. Indeed, we have the following well-known result (for example see [46] and also page 24, Exercise 2.1.4 in [27]):

**Theorem 4** *Let  $K = N(T)$  be an unoriented rational knot or link, presented as the numerator of a rational tangle  $T$ . Suppose that  $F(T) = p/q$  with  $p$  and  $q$  relatively prime. Then  $K$  is achiral if and only if  $q^2 \equiv -1 \pmod{p}$ . It follows that achiral rational knots and links are all numerators of rational tangles of the form  $[[a_1], [a_2], \dots, [a_k], [a_k], \dots, [a_2], [a_1]]$  for any integers  $a_1, \dots, a_k$ .*

Note that in this description we are using a representation of the tangle with an even number of terms. The leftmost twists  $[a_1]$  are horizontal, thus the rightmost starting twists  $[a_1]$  are vertical.

*Proof.* With  $-T$  the mirror image of the tangle  $T$ , we have that  $-K = N(-T)$  and  $F(-T) = p/(-q)$ . If  $K$  is topologically equivalent to  $-K$ , then  $N(T)$

and  $N(-T)$  are equivalent, and it follows from the classification theorem for rational knots that either  $q(-q) \equiv 1 \pmod{p}$  or  $q \equiv -q \pmod{p}$ . Without loss of generality we can assume that  $0 < q < p$ . Hence  $2q$  is not divisible by  $p$  and therefore it is not the case that  $q \equiv -q \pmod{p}$ . Hence  $q^2 \equiv -1 \pmod{p}$ .

Conversely, if  $q^2 \equiv -1 \pmod{p}$ , then it follows from the Palindrome Theorem (described in the previous section) [?] that *the continued fraction expansion of  $p/q$  has to be symmetric with an even number of terms*. It is then easy to see that the corresponding rational knot or link, say  $K = N(T)$ , is equivalent to its mirror image. One rotates  $K$  by  $180^\circ$  in the plane and swings an arc, as Figure 25 illustrates. This completes the proof.  $\square$

In [11] the authors find an explicit formula for the number of achiral rational knots among all rational knots with  $n$  crossings.

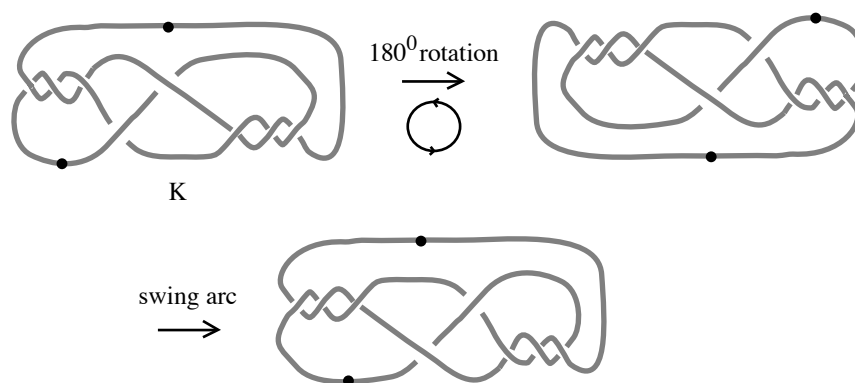


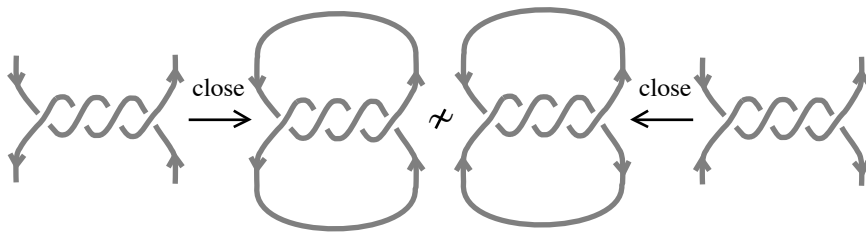
Figure 25 - An Achiral Rational Link

## 7 The Oriented Case

Oriented rational knots and links arise as numerator closures of oriented rational tangles. In order to compare oriented rational knots via rational tangles we need to examine how rational tangles can be oriented. We orient rational tangles by choosing an orientation for each strand of the tangle. Here we are only interested in orientations that yield consistently oriented knots upon taking the numerator closure. This means that the two top end arcs have to be oriented one inward and the other outward. Same for the two bottom end arcs. We shall say that two oriented rational tangles are *isotopic* if they are isotopic as unoriented tangles, by an isotopy that carries the orientation of

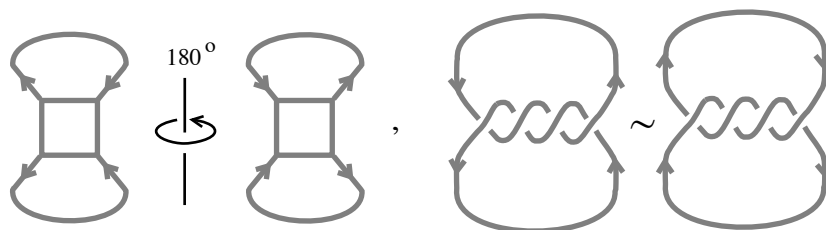
one tangle to the orientation of the other. Note that, since the end arcs of a tangle are fixed during a tangle isotopy, this means that the tangles must have identical orientations at their four end arcs  $NW$ ,  $NE$ ,  $SW$ ,  $SE$ . It follows that if we change the orientation of one or both strands of an oriented rational tangle we will always obtain a non-isotopic oriented rational tangle.

Reversing the orientation of one strand of an oriented rational tangle may or may not give rise to isotopic oriented rational knots. Figure 26 illustrates an example of non-isotopic oriented rational knots, which are isotopic as un-oriented knots.



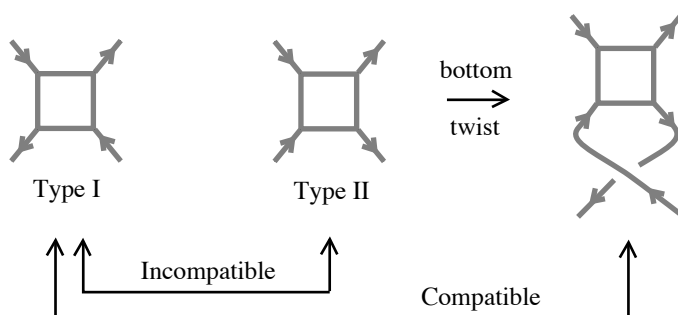
**Figure 26 - Non-isotopic Oriented Rational Links**

Reversing the orientation of both strands of an oriented rational tangle will always give rise to two isotopic oriented rational knots or links. We can see this by doing a vertical flip, as Figure 27 demonstrates. Using this observation we conclude that, as far as the study of oriented rational knots is concerned, *all oriented rational tangles may be assumed to have the same orientation for their  $NW$  and  $NE$  end arcs*. We fix this orientation to be downward for the  $NW$  end arc and upward for the  $NE$  arc, as in the examples of Figure 26 and as illustrated in Figure 28. Indeed, if the orientations are opposite of the fixed ones doing a vertical flip the knot may be considered as the numerator of the vertical flip of the original tangle. But this is unoriented isotopic to the original tangle (recall Section 2, Figure 7), whilst its orientation pattern agrees with our convention.



**Figure 27 - Isotopic Oriented Rational Knots and Links**

Thus we reduce our analysis to two basic types of orientation for the four end arcs of a rational tangle. We shall call an oriented rational tangle of *type I* if the *SW* arc is oriented upward and the *SE* arc is oriented downward, and of *type II* if the *SW* arc is oriented downward and the *SE* arc is oriented upward, see Figure 28. From the above remarks, any tangle is of type I or type II. Two tangles are said to be *compatible* if they are both of type I or both of type II and *incompatible* if they are of different types. In order to classify oriented rational knots seen as numerator closures of oriented rational tangles, we will always compare compatible rational tangles. Note that if two oriented tangles are incompatible, adding a single half twist at the bottom of one of them yields a new pair of compatible tangles, as Figure 28 illustrates. Note also that adding such a twist, although it changes the tangle, it does not change the isotopy type of the numerator closure. Thus, up to bottom twists, we are always able to compare oriented rational tangles of the same orientation type.



**Figure 28 - Compatible and Incompatible Orientations**

We shall now introduce the notion of *connectivity* and we shall relate it to orientation and the fraction of unoriented rational tangles. We shall say

that an unoriented rational tangle has *connectivity* type  $[0]$  if the *NW* end arc is connected to the *NE* end arc and the *SW* end arc is connected to the *SE* end arc. Similarly, we say that the tangle has *connectivity* type  $[+1]$  or type  $[\infty]$  if the end arc connections are the same as in the tangles  $[+1]$  and  $[\infty]$  respectively. The basic connectivity patterns of rational tangles are exemplified by the tangles  $[0]$ ,  $[\infty]$  and  $[+1]$ . We can represent them iconically by the symbols shown below.

$$[0] = \asymp$$

$$[\infty] = \succ\prec$$

$$[+1] = \chi$$

Note that connectivity type  $[0]$  yields two-component rational links, while type  $[+1]$  or  $[\infty]$  yields one-component rational links. Also, adding a bottom twist to a rational tangle of connectivity type  $[0]$  will not change the connectivity type of the tangle, while adding a bottom twist to a rational tangle of connectivity type  $[\infty]$  will switch the connectivity type to  $[+1]$  and vice versa. While the connectivity type of unoriented rational tangles may be  $[0]$ ,  $[+1]$  or  $[\infty]$ , note that an oriented rational tangle of type I will have connectivity type  $[0]$  or  $[\infty]$  and an oriented rational tangle of type II will have connectivity type  $[0]$  or  $[+1]$ .

Further, we need to keep an accounting of the connectivity of rational tangles in relation to the parity of the numerators and denominators of their fractions. We refer the reader to our paper [26] for a full account.

We adopt the following notation: *e* stands for *even* and *o* stands for *odd*. The *parity* of a fraction  $p/q$  is defined to be the ratio of the parities (*e* or *o*) of its numerator and denominator  $p$  and  $q$ . Thus the fraction  $2/3$  is of parity  $e/o$ . The tangle  $[0]$  has fraction  $0 = 0/1$ , thus parity  $e/o$ , the tangle  $[\infty]$  has fraction  $\infty = 1/0$ , thus parity  $o/e$ , and the tangle  $[+1]$  has fraction  $1 = 1/1$ , thus parity  $o/o$ . We then have the following result.

**Theorem 5** *A rational tangle  $T$  has connectivity type  $\asymp$  if and only if its fraction has parity  $e/o$ .  $T$  has connectivity type  $\succ\prec$  if and only if its fraction has parity  $o/e$ .  $T$  has connectivity type  $\chi$  if and only if its fraction has parity  $o/o$ . (Note that the formal fraction of  $[\infty]$  itself is  $1/0$ .) Thus the link  $N(T)$  has two components if and only if  $T$  has fraction  $F(T)$  of parity  $e/o$ .*

We will now proceed with sketching the proof of Theorem 3. We shall prove Schubert's oriented theorem by appealing to our previous work on the unoriented case and then analyzing how orientations and fractions are related.

Our strategy is as follows: Consider an oriented rational knot or link diagram  $K$  in the form  $N(T)$  where  $T$  is a rational tangle in continued fraction form. Then any other rational tangle that closes to this knot  $N(T)$  is available, up to bottom twists if necessary, as a cut from the given diagram. If two rational tangles close to give  $K$  as an unoriented rational knot or link, then there are orientations on these tangles, induced from  $K$  so that the oriented tangles close to give  $K$  as an oriented knot or link. The two tangles may or may not be compatible. Thus, we must analyze when, comparing with the standard cut for the rational knot or link, another cut produces a compatible or incompatible rational tangle. However, assuming the top orientations are the same, we can replace one of the two incompatible tangles by the tangle obtained by adding a twist at the bottom. *It is this possible twist difference that gives rise to the change from modulus  $p$  in the unoriented case to the modulus  $2p$  in the oriented case.* We will now perform this analysis. There are many interesting aspects to this analysis and we refer the reader to our paper [26] for these details. Schubert [45] proved his version of the oriented theorem by using the 2-bridge representation of rational knots and links, see also [6]. We give a tangle-theoretic combinatorial proof based upon the combinatorics of the unoriented case.

The simplest instance of the classification of oriented rational knots is adding an *even number of twists* at the bottom of an oriented rational tangle  $T$ , see Figure 28. We then obtain a compatible tangle  $T * 1/[2n]$ , and  $N(T * 1/[2n]) \sim N(T)$ . If now  $F(T) = p/q$ , then  $F(T * 1/[2n]) = F(1/([2n] + 1/T)) = 1/(2n + 1/F(T)) = p/(2np + q)$ . Hence, if we set  $2np + q = q'$  we have  $q \equiv q' \pmod{2p}$ , just as the oriented Schubert Theorem predicts. Note that reducing all possible bottom twists implies  $|p| > |q|$  for both tangles, if the two tangles that we compare each time are compatible or for only one, if they are incompatible.

We then have to compare the special cut and the palindrome cut with the standard cut. In the oriented case the special cut is the easier to see whilst the palindrome cut requires a more sophisticated analysis. Figure 29 illustrates the general case of the special cut. In order to understand Figure 29 it is necessary to also view Figure 22 for the details of this cut.

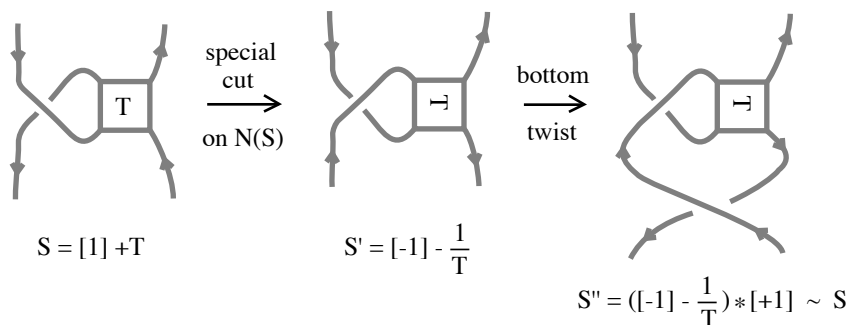


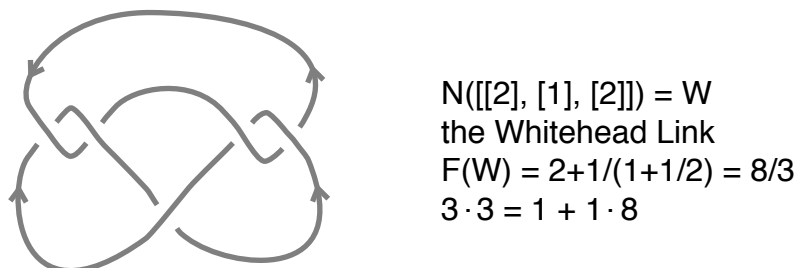
Figure 29 - The Oriented Special Cut

Recall that if  $S = [1] + T$  then the tangle of the special cut on the knot  $N([1] + T)$  is the tangle  $S' = [-1] - \frac{1}{T}$ . And if  $F(T) = p/q$  then  $F([1] + T) = \frac{p+q}{q}$  and  $F([-1] - \frac{1}{T}) = \frac{p+q}{-p}$ . Now, the point is that the orientations of the tangles  $S$  and  $S'$  are incompatible. Applying a  $[+1]$  bottom twist to  $S'$  yields  $S'' = (-1 - \frac{1}{T}) * [1]$ , and we find that  $F(S'') = \frac{p+q}{q}$ . Thus, the oriented rational tangles  $S$  and  $S''$  have the same fraction and by Theorem 1 and their compatibility they are oriented isotopic and the arithmetics of Theorem 3 is straightforward.

We are left to examine the case of the palindrome cut. For this part of the proof, we refer the reader to our paper [26].

## 8 Strongly Invertible Links

An oriented knot or link is invertible if it is oriented isotopic to the link obtained from it by reversing the orientation of each component. We have seen (see Figure 27) that rational knots and links are invertible. A link  $L$  of two components is said to be *strongly invertible* if  $L$  is ambient isotopic to itself with the orientation of only one component reversed. In Figure 30 we illustrate the link  $L = N([[2], [1], [2]])$ . This is a strongly invertible link as is apparent by a  $180^\circ$  vertical rotation. This link is well-known as the Whitehead link, a link with linking number zero. Note that since  $[[2], [1], [2]]$  has fraction equal to  $1 + 1/(1 + 1/2) = 8/3$  this link is non-trivial via the classification of rational knots and links. Note also that  $3 \cdot 3 = 1 + 1 \cdot 8$ .

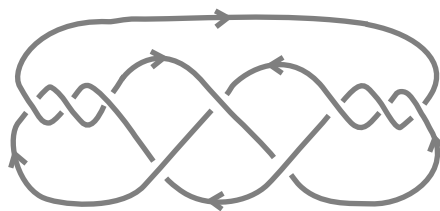


**Figure 30 - The Whitehead Link is Strongly Invertible**

In general we have the following. For our proof, see [26].

**Theorem 6** *Let  $L = N(T)$  be an oriented rational link with associated tangle fraction  $F(T) = p/q$  of parity  $e/o$ , with  $p$  and  $q$  relatively prime and  $|p| > |q|$ . Then  $L$  is strongly invertible if and only if  $q^2 = 1 + up$  with  $u$  an odd integer. It follows that strongly invertible links are all numerators of rational tangles of the form  $[[a_1], [a_2], \dots, [a_k], [\alpha], [a_k], \dots, [a_2], [a_1]]$  for any integers  $a_1, \dots, a_k, \alpha$ .*

See Figure 31 for another example of a strongly invertible link. In this case the link is  $L = N([[3], [1], [1], [1], [3]])$  with  $F(L) = 40/11$ . Note that  $11^2 = 1 + 3 \cdot 40$ , fitting the conclusion of Theorem 6.



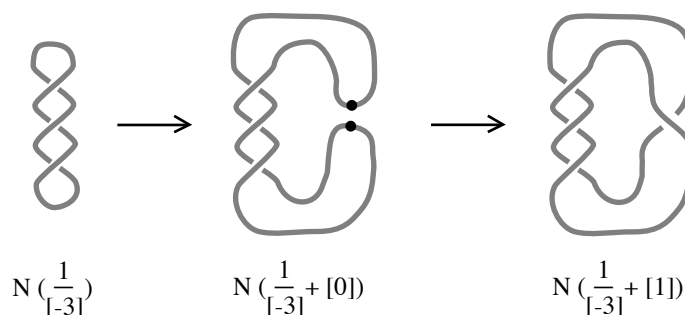
$$L = N([[3], [1], [1], [1], [3]])$$

**Figure 31 - An Example of a Strongly Invertible Link**



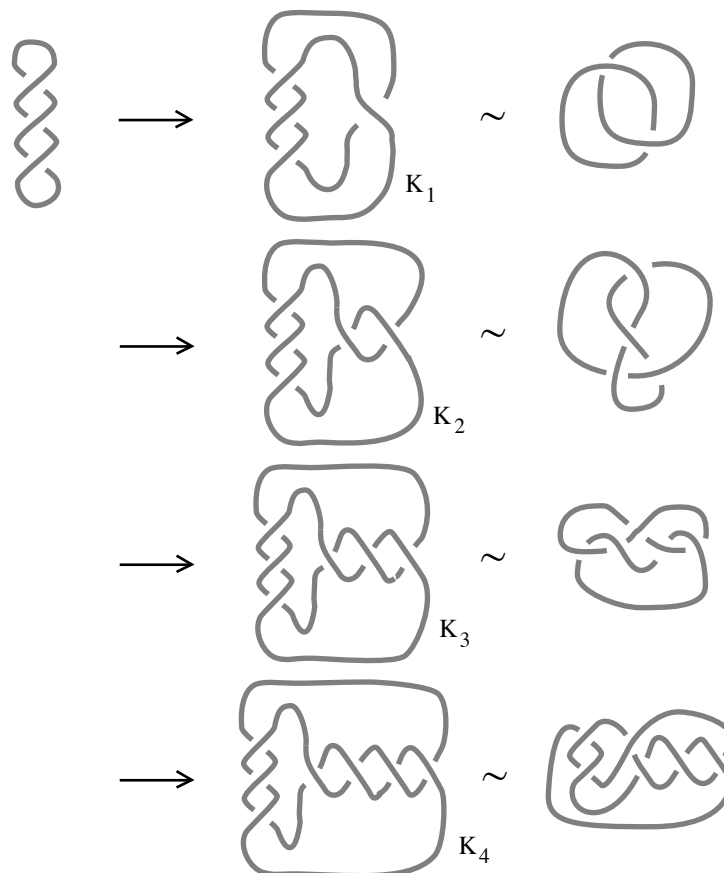
## 9 Applications to the Topology of DNA

DNA supercoils, replicates and recombines with the help of certain enzymes. *Site-specific recombination* is one of the ways nature alters the genetic code of an organism, either by moving a block of DNA to another position on the molecule or by integrating a block of alien DNA into a host genome. For a closed molecule of DNA a global picture of the recombination would be as shown in Figure 32, where double-stranded DNA is represented by a single line and the recombination sites are marked with points. This picture can be interpreted as  $N(S + [0]) \rightarrow N(S + [1])$ , for  $S = \frac{1}{[-3]}$  in this example. This operation can be repeated as in Figure 33. Note that the  $[0] - [\infty]$  interchange of Figure 10 can be seen as the first step of the process.



**Figure 32 - Global Picture of Recombination**

In this depiction of recombination, we have shown a local replacement of the tangle  $[0]$  by the tangle  $[1]$  connoting a new cross-connection of the DNA strands. In general, it is not known without corroborating evidence just what the topological geometry of the recombination replacement will be. Even in the case of a single half-twist replacement such as  $[1]$ , it is certainly not obvious beforehand that the replacement will always be  $[+1]$  and not sometimes the reverse twist of  $[-1]$ . It was at the juncture raised by this question that a combination of topological methods in biology and a tangle model using knot theory developed by C.Ernst and D.W. Summers resolved the issue in some specific cases. See [12], [47] and references therein.



**Figure 33 - Multiple Recombinations**

On the biological side, methods of protein coating developed by N. Cozzarelli, S.J. Spengler and A. Stasiak et al. In [8] it was made possible for the first time to see knotted DNA in an electron micrograph with sufficient resolution to actually identify the topological type of these knots. The protein coating technique made it possible to design an experiment involving successive DNA recombinations and to examine the topology of the products. In [8] the knotted DNA produced by such successive recombinations was consistent with the hypothesis that all recombinations were of the type of a positive half twist as in  $[+1]$ . Then D.W. Sumners and C. Ernst [12] proposed a *tangle model for successive DNA recombinations* and showed, in the case of the experiments in question, that there was no other topological possibility for the recombination mechanism than the positive half twist  $[+1]$ . This constituted a unique use of topology as a theoretical underpinning for a problem in molecular biology.

Here is a brief description of the tangle model for DNA recombination. It is assumed that the initial state of the DNA is described as the numerator closure  $N(S)$  of a *substrate tangle*  $S$ . The local geometry of the recombination is assumed to be described by the replacement of the tangle  $[0]$  with a specific tangle  $R$ . The results of the successive rounds of recombination are the knots and links

$$N(S + R) = K_1, \quad N(S + R + R) = K_2, \quad N(S + R + R + R) = K_3, \quad \dots$$

Knowing the knots  $K_1, K_2, K_3, \dots$  one would like to solve the above system of equations with the tangles  $S$  and  $R$  as unknowns.

For such experiments Ernst and Sumners [12] used the classification of rational knots in the unoriented case, as well as results of Culler, Gordon, Luecke and Shalen [9] on Dehn surgery to prove that the solutions  $S + nR$  must be *rational tangles*. These results of Culler, Gordon, Luecke and Shalen tell the topologist under what circumstances a three-manifold with cyclic fundamental group must be a lens space. By showing when the 2-fold branched covers of the DNA knots must be lens spaces, the recombination problems are reduced to the consideration of rational knots. This is a deep application of the three-manifold approach to rational knots and their generalizations.

One can then apply the theorem on the classification of rational knots to deduce (in these instances) the uniqueness of  $S$  and  $R$ . Note that, in these experiments, the substrate tangle  $S$  was also pinpointed by the sequence of knots and links that resulted from the recombination.

Here we shall solve tangle equations like the above under rationality assumptions on all tangles in question. This allows us to use only the mathematical techniques developed in this paper. We shall illustrate how a sequence of rational knots and links

$$N(S + nR) = K_n, \quad n = 0, 1, 2, 3, \dots$$

with  $S$  and  $R$  rational tangles, such that  $R = [r]$ ,  $F(S) = \frac{p}{q}$  and  $p, q, r \in \mathbb{Z}$  ( $p > 0$ ) *determines*  $\frac{p}{q}$  and  $r$  *uniquely* if we know sufficiently many  $K_n$ . We call this the “DNA Knitting Machine Analysis”.

**Theorem 7** *Let a sequence  $K_n$  of rational knots and links be defined by the equations  $K_n = N(S + nR)$  with specific integers  $p, q, r$  ( $p > 0$ ), where  $R = [r]$ ,  $F(S) = \frac{p}{q}$ . Then  $\frac{p}{q}$  and  $r$  are uniquely determined if one knows the topological type of the unoriented links  $K_0, K_1, \dots, K_N$  for any integer  $N \geq |q| - \frac{p}{qr}$ .*

*Proof.* In this proof we shall write  $N(\frac{p}{q} + nr)$  or  $N(\frac{p+qnr}{q})$  for  $N(S + nR)$ . We shall also write  $K = K'$  to mean that  $K$  and  $K'$  are isotopic links. Moreover we shall say for a pair of reduced fractions  $P/q$  and  $P/q'$  that  $q$  and  $q'$  are *arithmetically related relative to  $P$*  if either  $q \equiv q' \pmod{P}$  or  $qq' \equiv 1 \pmod{P}$ . Suppose the integers  $p, q, r$  give rise to the sequence of links  $K_0, K_1, \dots$ . Suppose there is some other triple of integers  $p', q', r'$  that give rise to the same sequence of links. We will show uniqueness of  $p, q, r$  under the conditions of the theorem. We shall say “the equality holds for  $n$ ” to mean that  $N((p+qnr)/q) = N((p'+q'r'n)/q')$ . We suppose that  $K_n = N((p+qnr)/q)$  as in the hypothesis of the theorem, and suppose that there are  $p', q', r'$  such that for some  $n$  (or a range of values of  $n$  to be specified below)  $K_n = N((p'+q'r'n)/q')$ .

If  $n = 0$  then we have  $N(p/q) = N(p'/q')$ . Hence by the classification theorem we know that  $p = p'$  and that  $q$  and  $q'$  are arithmetically related. Note that the same argument shows that if the equality holds for any two consecutive values of  $n$ , then  $p = p'$ . Hence we shall assume henceforth that  $p = p'$ . With this assumption in place, we see that if the equality holds for any  $n \neq 0$  then  $qr = q'r'$ . Hence we shall assume this as well from now on.

If  $|p + qnr|$  is sufficiently large, then the congruences for the arithmetical relation of  $q$  and  $q'$  must be *equalities over the integers*. Since  $qq' = 1$  over the integers can hold only if  $q = q' = 1$  or  $-1$  we see that it must be the case that  $q = q'$  if the equality is to hold for sufficiently large  $n$ . From this and the equation  $qr = q'r'$  it follows that  $r = r'$ . It remains to determine a bound on  $n$ . In order to be sure that  $|p + qnr|$  is sufficiently large, we need that  $|qq'| \leq |p + qnr|$ . Since  $q'r' = qr$ , we also know that  $|q'| \leq |qr|$ . Hence  $n$  is sufficiently large if  $|q^2r| \leq |p + qnr|$ .

If  $qr > 0$  then, since  $p > 0$ , we are asking that  $|q^2r| \leq p + qnr$ . Hence

$$n \geq (|q^2r| - p)/(qr) = |q| - (p/qr).$$

If  $qr < 0$  then for  $n$  large we will have  $|p + qnr| = -p - qnr$ . Thus we want to solve  $|q^2r| \leq -p - qnr$ , whence

$$n \geq (|q^2r| + p)/(-qr) = |q| - (p/qr).$$

Since these two cases exhaust the range of possibilities, this completes the proof of the theorem.  $\square$

Here is a special case of Theorem 7. See Figure 33. Suppose that we were given a sequence of knots and links  $K_n$  such that

$$K_n = N\left(\frac{1}{[-3]} + [1] + [1] + \dots + [1]\right) = N\left(\frac{1}{[-3]} + n[1]\right).$$

We have  $F\left(\frac{1}{[-3]} + n[1]\right) = (3n - 1)/3$  and we shall write  $K_n = N([(3n - 1)/3])$ . We are told that each of these rational knots is in fact the numerator closure of a rational tangle denoted

$$[p/q] + n[r]$$

for some rational number  $p/q$  and some integer  $r$ . That is, we are told that they come from a DNA knitting machine that is using rational tangle patterns. But we only know the knots and the fact that they are indeed the closures for  $p/q = -1/3$  and  $r = 1$ . By this analysis, the uniqueness is implied by the knots and links  $\{K_1, K_2, K_3, K_4\}$ . This means that a DNA knitting machine  $K_n = N(S + nR)$  that emits the four specific knots  $K_n = N([(3n - 1)/3])$  for  $n = 1, 2, 3, 4$  must be of the form  $S = 1/[-3]$  and  $R = [1]$ . It was in this way (with a finite number of observations) that the structure of recombination in  $T_{n3}$  resolvase was determined [47].

In this version of the tangle model for DNA recombination we have made a blanket assumption that the substrate tangle  $S$  and the recombination tangle  $R$  and all the tangles  $S + nR$  were rational. Actually, if we assume that  $S$  is rational and that  $S + R$  is rational, then it follows that  $R$  is an integer tangle. Thus  $S$  and  $R$  necessarily form a DNA knitting machine under these conditions. It is relatively natural to assume that  $S$  is rational on the grounds of simplicity. On the other hand it is not so obvious that the recombination tangle should be an integer. The fact that the products of the DNA recombination experiments yield rational knots and links, lends credence to the hypothesis of rational tangles and hence integral recombination tangles. But there certainly is a subtlety here, since we know that the numerator closure of the sum of two rational tangles is always a rational knot or link. In fact, it is here that some deeper topology shows that certain rational products from a generalized knitting machine of the form  $K_n = N(S + nR)$  where  $S$  and  $R$  are arbitrary tangles will force the rationality of the tangles  $S + nR$ . We refer the reader to [12], [13], [10] for the details of this approach.

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