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STATISTICAL MECHANICS AND THE JONES POLYNOMIAL

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ABSTRACT. This paper studies a relationship between the formalism of knot theory and certain models in statistical mechanics. It is shown how the partition function for the Potts model may be computed from an associated link diagram, and how this provides a common algorithmic model with the Jones polynomial. Certain features of both Jones polynomial and the Potts model can be treated in common, such as the appearance of the Temperley-Lieb algebra for braid diagrams and the geometry of the ice-model for piecewise linear diagrams.

I. INTRODUCTION

In his paper [3] on the Jones polynomial, Vaughan Jones described a connection between his new polynomial invariant of links and the Potts model in statistical mechanics. Both the Potts model and the Jones polynomial involve traces defined on a von Neumann algebra A[n]. For the knot theory, Jones implicated this algebra by constructing a representation to it with domain the Artin braid group B[n]. In statistical mechanics the same algebra is used to calculate the partition function for the Potts model (also by a trace), and the algebra is known to the physicists as the Temperly-Lieb Algebra [1].

The purpose of this paper is to exhibit a direct diagrammatic connection between these two subjects. In particular, I show that the partition function of the Potts model for a planar lattice can be calculated from a link diagram that is canonically associated with the lattice. (A link diagram is a schematic planar picture of a knot or link embedded in three-dimensional space.) The algorithm for computing the partition function from the link diagram is a special case of a three-variable polynomial defined on such diagrams that I call the bracket polynomial (denoted [L] for a diagram L. See [5], [6].). The Jones polynomial, up to a normalization, is also a special case of the bracket polynomial. In this way, the Jones polynomial and the Potts partition function are both aspects of a single algorithm defined on diagrams of knots and links.

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The actual relation of the partition function of the Potts model with topological properties of knots and links is an open question. Remarkably, the Potts partition function for a planar lattice G is computed by bracket for an alternating link K(G). (An alternating link has a weaving pattern so that a given thread is seen to pass alternately under and over successive strands.) If K(G) is reduced (see section 2), then the long-standing conjectures of Tait and Little imply that [K(G)] is an invariant of the topological type of K(G). Thus, if this conjecture is true, then the Potts partition function of the lattice G is actually a topological invariant of the associated link K(G). This leads to many questions regarding the meaning of this connection between physics and topology.

The paper is organized as follows. Section II constructs the general bracket, and shows how it specializes to the Jones polynomial and to the dichromatic polynomial for planar graphs. Section III gives the background for the Potts model, shows that the Potts partition function is a dichromatic polynomial, and relates these results to the discussion of the dichromate in section II. Section IV discusses the diagram monoid of states for the bracket, and its relation to computing the dichromatic polynomial and the Potts partition function. This formalism is related to the Temperly-Lieb algebra via a diagrammatic tensor formalism due to Roger Penrose [10]. Section V returns to the Potts model, and shows how our viewpoint illuminates the discussion of the critical point for the ferromagnetic case. The link K(G) and its mirror image $K(G)^*$ correspond to the graph G and its dual G^*. This clarifies the structure of an argument [16] locating the conjectured critical point for the model.

Finally, in section VI, I show how a translation of the 6-vertex ice-model [1] of statistical mechanics into our knot-theoretic formalism gives rise to a different state-expansion for the bracket in terms of arrow coverings and local angular data for piecewise-linear diagrams. This gives a new formalism for the Jones polynomial and lets us raise further questions about the topology and the physics.

II. THE GENERAL BRACKET

I first define a three-variable polynomial $[K](A,B,d) \in Z[A,B,d]$ (Z denotes the integers) defined for unoriented link diagram K. The polynomial [K] will be referred to as the general bracket polynomial or the square bracket. Only by specializing its variables does the bracket become a topological invariant, but it is well-defined on link diagrams as combinatorial entities.

First recall a few basic facts about link diagrams and their relationship with planar graphs: A universe (or link shadow) is a planar graph with four edges locally incident to each of its vertices. See Figure 1. A link diagram is a universe endowed with extra structure at each vertex, indicating a crossing. Again see Figure 1. A link diagram can be seen as a schematic drawing of a knot or link and it can be regarded as a special species of planar graph.

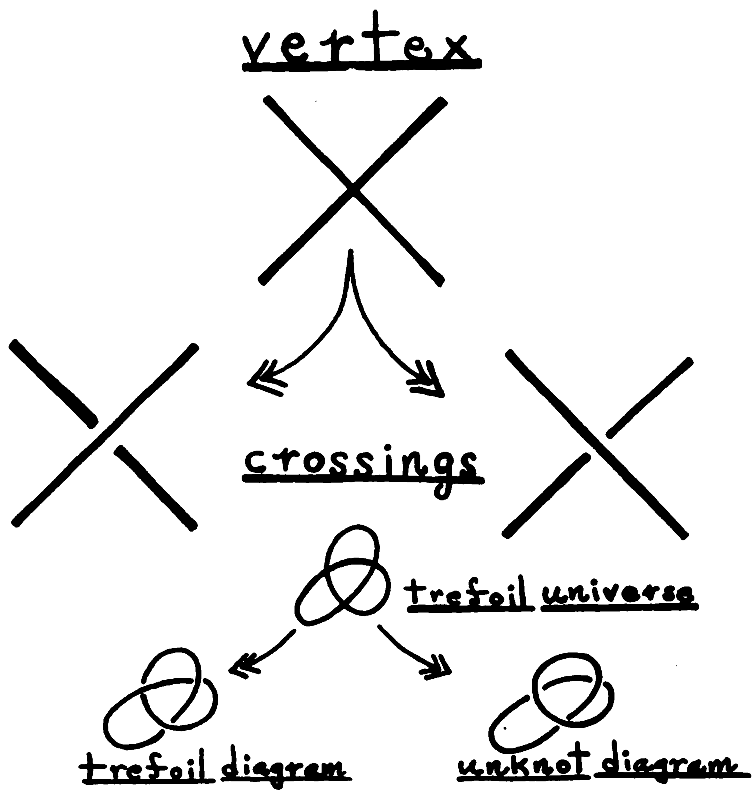
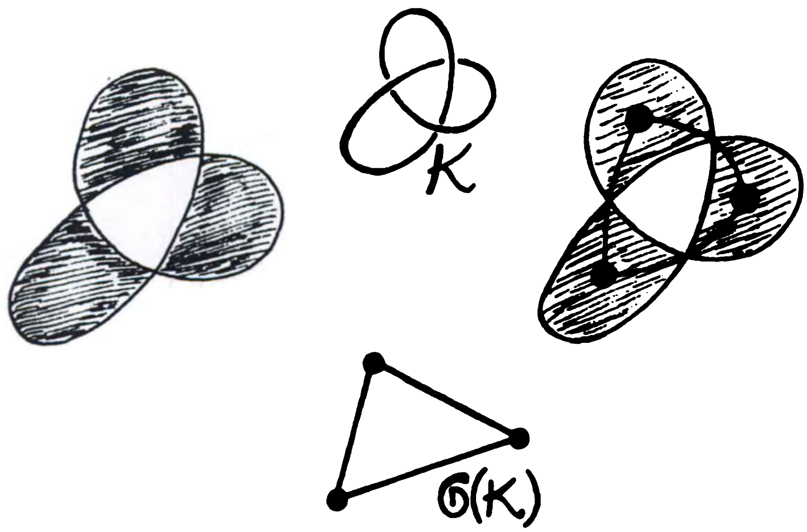


Figure 1

There is another planar graph associated with a link diagram K: This graph, G(K), is obtained by first checkerboard shading the diagram K as in Figure 2 (let the unbounded region be unshaded). The vertices of G(K) are in one-to-one correspondence with the shaded regions of K. Two vertices are joined by an edge if the two regions touch at a crossing in the knot diagram.

The graph G(K) depends only upon the underlying universe for K. If we want a complete graphical translation of the link diagram, this can be done by assigning signs (+1 or -1) to the edges of G(K) to indicate the unoriented crossing types (See [5].).

The association $U \twoheadrightarrow G(U)$ of a planar universe to a planar graph is invertible. The inverse process takes a planar graph, G, and produces a universe, M(G). M(G) is sometimes called the medial graph for G. The medial graph is produced by placing a crossing on each edge of G, and then connecting M(G) as shown in Figure 3.



Checkerboard Shading and Associated Graph

Figure 2

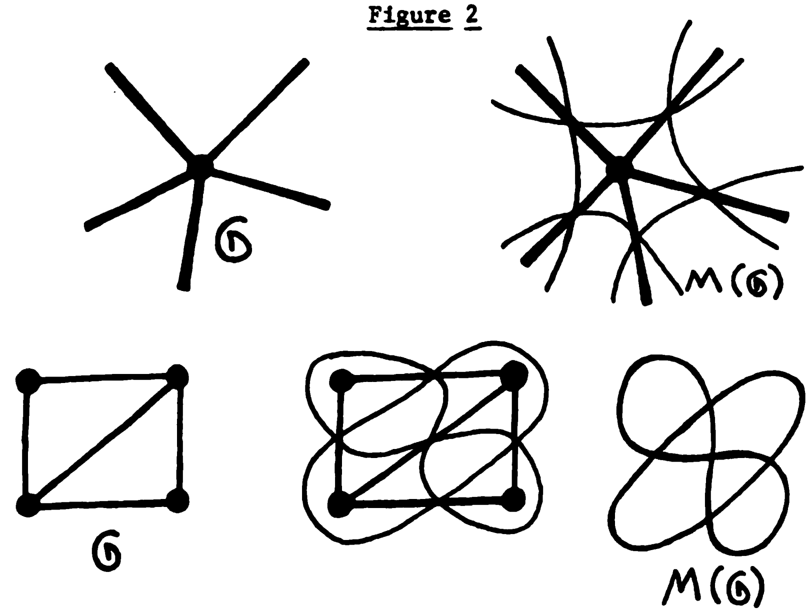


Figure 3

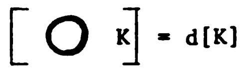
Note that in producing the medial graph M(G) from the planar graph G, we extend each crossing segment (of the crossings placed at each edge of G) toward the corresponding vertex, connecting it to a crossing segment from the next edge that is incident to this vertex - in clockwise or counterclockwise order. By this construction, <u>planar graphs and universes (shadows of link diagrams) are in one-to-one correspondence</u>.

This is the basic connection between link diagrams and graph theory.

With the correspondence of graph and medial graph in hand, we are prepared to define the general bracket, and to begin translations between the language of link diagrams and the language of graph theory.

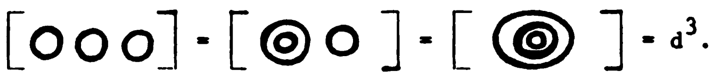
DEFINITION 2.1. Let K be a link diagram. The three-variable bracket polynomial is defined by the axioms:

1. 

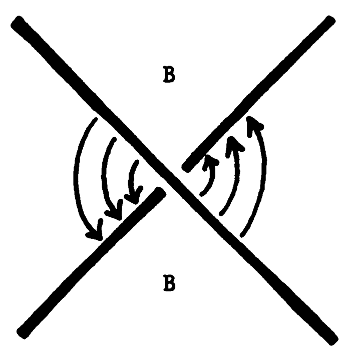
2. 

3. 

The small diagrams stand for parts of otherwise identical larger diagrams; the circle next to the letter K in 2. denotes the disjoint union of K with a Jordan curve. Thus



Some words of explanation about this definition are in order. First, the assignment of A and B to the two modes of splicing the crossing in equation 1 is determined as follows. The form of the crossing distinguishes a pair of local regions by the rule: rotate the overcrossing line counterclockwise until it coincides with the undercrossing line.



The two regions swept out by this rotation are labelled A. The other two are labelled B. When splitting the vertex fuses the two A-regions, I say that this splitting opens an A-channel. This is the part labelled A in formula 1. The other split opens the B-channel. See Figure 4.

The bracket is then well-defined recursively. Note that the order of applications of formula 1. does not affect the final result, since the choice of A or B is entirely local. Note also that switching a crossing reverses the roles of A and B. Thus

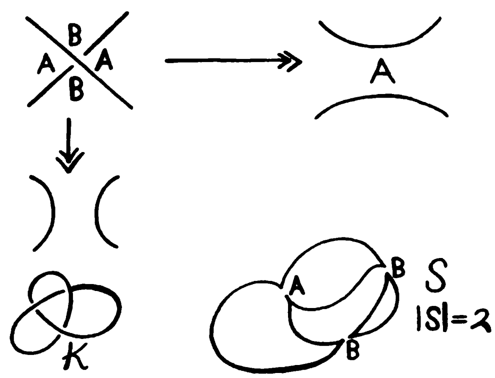


Figure 4

Here are a few sample bracket calculations:

(i) 

$= Bd + Ad^2$

(ii) 

$= Bd^2 + Ad$

(iii) 

$= A(Bd + Ad^2) + B(Bd^2 + Ad)$

$= A^2d^2 + B^2d^2 + 2ABd$

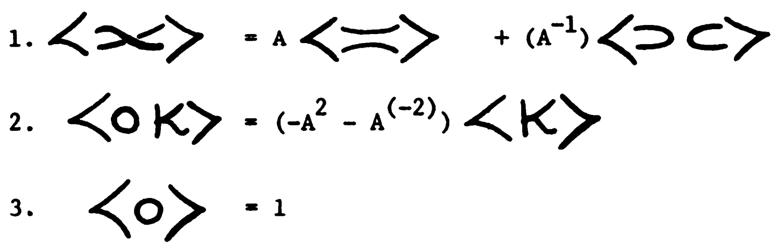
In general, we can give an explicit formula for the bracket by considering the full recursion. I define a state S of K as the configuration of simple closed curves in the plane obtained by splitting each crossing in one of the two possible ways. There are 2^V states where V denotes the number of crossings in the diagram K. See Figure 4 for an example of a state for the trefoil diagram. The formula for [K] will sum over the contributions of each state.

Let S be a state of K. Let $i_K(S)$ denote the number of A-channels in S, and let $j_K(S)$ denote the number of B-channels. Let $|S|$ denote the number of circuits in S. Then the general bracket is given by the formula:

$$[K] = \sum_S A^{i_K(S)} B^{j_K(S)} d^{|S|}.$$

EXAMPLE 1. (The Jones polynomial)

Let $B = A^{(-1)}$ and let $d = -A^2 - A^{(-2)}$. For this specialization, let $\langle K\rangle = (d^{-1})[K]$. (See [4], [5], [6].) Then $\langle K\rangle$ satisfies the axioms:



This special bracket is invariant under the Reidemeister moves of type II and type III. Under type I moves $\langle K\rangle$ is multiplied by $-A^3$ or its inverse.

For K an oriented link, let w(K) denote the writhe of K, where this number is the sum of the signs of crossings. Crossing signs are +1 or -1 as shown below.

Now define a normalized polynomial by the formula

$$f_K = ((-A)^{(-3w(K))})\langle K\rangle.$$

In [5] I show that f_K is an ambient isotopy invariant of the oriented link K. Furthermore, if $V_K(t)$ denotes the original Jones polynomial [3], then

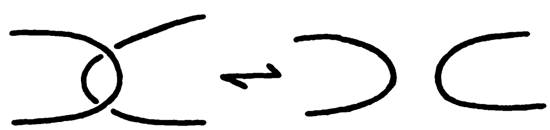
$$f_K(t^{(-1/4)}) = V_K(t).$$

This gives a model for the Jones polynomial as a (normalized) specialization of the general bracket.

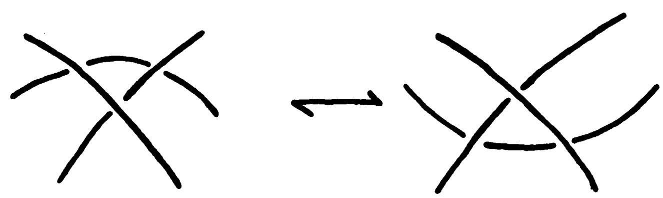
The Jones polynomial $V_K(t)$ is an invariant of ambient isotopy for oriented knots and links, and it is the first polynomial invariant capable of detecting chirality for many knots. A knot is said to be <u>chiral</u> if it is not ambient isotopic to its mirror image (obtained by switching all crossings). The bracket provides an elementary route to the Jones polynomial. For the reader interested in pursuing this direction, I record here the three types of Reidemeister moves:

I.

II.



III.

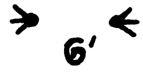


Ambient isotopy is generated diagramatically by these three types of local move. Each small diagram represents a corresponding situation in a larger diagram.

The existence of combinatorial invariants such as the Jones polynomial poses extraordinary problems for classical knot theory. For example, it is not known as of this writing whether $V_K(t)$ detects knottedness in all cases. In other words, does $V_K(t) = 1$ imply that a knot K is unknotted (ambient isotopic to an unknotted circle)?

<u>EXAMPLE 2</u>. (Chromatic Polynomial)

Let G be a planar graph. Let $C[G](q)$ denote the number of ways to properly color G with q colors. A proper coloring of G is an assignment of colors to the vertices of G so that two vertices sharing an edge are colored differently. The coloring polynomial satisfies the following graphical axioms:

1. C  = C  - C 

2. $C_{\bullet G} = qC_G$

The first axiom states that if three graphs G, G', G" are related so that G' is obtained from G by deleting an edge, and G" is obtained from G by collapsing the same edge, then

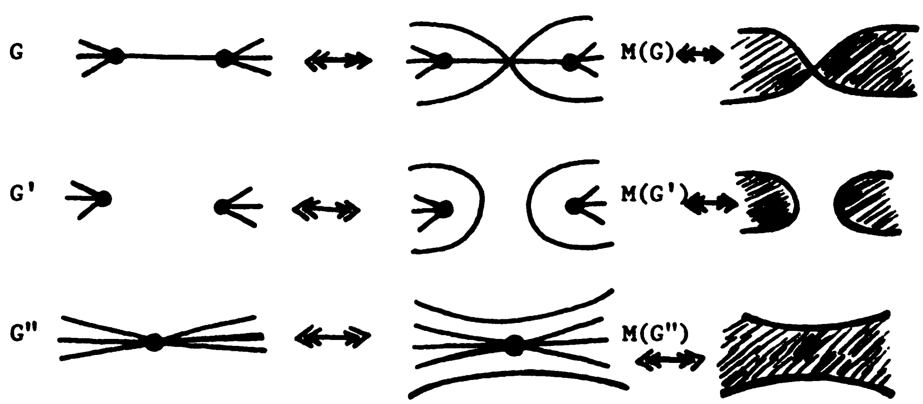
$$C[G] = C[G'] - C[G''].$$

The proof is due to Whitney [15] in his 1932 paper, "A logical identity in mathematics". The logical identity is

<u>[Different] = [All] - [Same]</u>.

The second axiom states that the number of proper colorings of a graph that is augmented by a disjoint vertex is q multiplied by the chromatic number of the original graph.

By using the medial graph M(G) and the checkerboard construction, we can begin a translation of the chromatic polynomial in the direction of the general bracket.

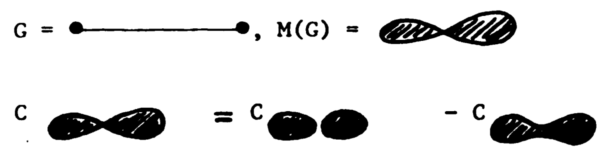


We rewrite the axioms for the chromatic polynomial in terms of shaded universes as follows:

1. C[] = C[] - C[]

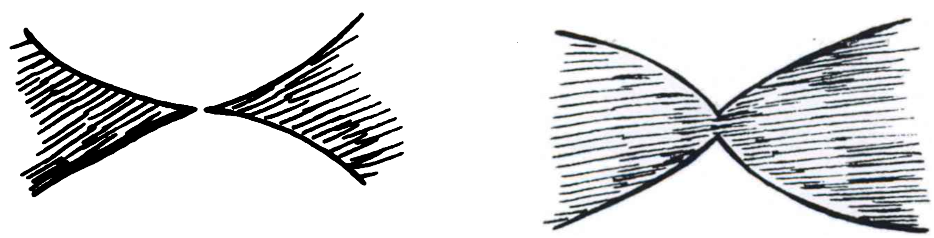
2. C[B ∪ M] = qC[M] where B is any shaded component that is free of crossings.

For example,



$$= q^2 - q = q(q-1).$$

In general, for a shaded universe M(G), we can expand C[M(G)] via the same states as the bracket expansion, except that we keep track of the number of shaded regions, and it is necessary to classify split vertices as <u>internal</u> or <u>external</u>:



<u>external</u> <u>internal</u>

The open channel for an external vertex is unshaded, while the channel for an internal vertex is shaded.

Let M be a shaded universe, and S(M) its collection of shaded states. Let I(S) denote the number of internal vertices in the state S. Let $\|S\|$ denote the number of <u>shaded components</u> in the state S. Then our axioms imply that

$$C[M] = \sum_{S(M)} (-1)^{I(S)} q^{\|S\|} .$$

This gives a specific algorithm for computing chromatic polynomials. The algorithm at this level is of independent interest. I have been investigating its behaviour as a quasi-physical system in collaboration with Mario Rasetti, Corrado Agnes and Amelia Sparavigna of the Politecnico di Torino, Torino, Italy. We compute C[G] by listing the states, starting with the state with maximal number of shaded components (all external vertices). The algorithm moves through the "space of states" changing only one vertex at a time (via a gray code enumeration). We watch the partial sums for C[G]. On a log-log plot of partial sum versus number of iterations (in the listing of states) the picture suggests a remarkable linearity of relationship for the chromatic numbers of subgraphs. These numbers arise after every 2^n steps. In other words, it seems to be possible to estimate the chromatic number rather long before the computation is completed! See Figure 5 for an example of one of our plots. This algorithm, and our joint results will be reported in more detail elsewhere.

I called this algorithm a quasi-physical system because it performs an artificial ergodic path through the space of states of the system. Thus it is a very idealized model of the way a physical system may move from state to state. Obviously, there is much experimental-mathematical and theoretical work to be done here. As we shall see in the next section, the chromatic polynomial (and its generalization to the dichromatic polynomial) is a central feature in certain models in statistical mechanics.

EXAMPLE 3. (More translation between link diagrams and graphs).

We now translate the chromatic polynomial into a direct special case of the bracket. The following lemma is the key. It gives the relation between number of shaded components and number of circuits in a state.

LEMMA 2.2. Let $M = M(G)$ be the shaded universe corresponding to a planar graph G. Let G have N vertices. Let S be a state of M with $\|S\|$ shaded components, $|S|$ circuits, and $I(S)$ internal vertices. Then $\|S\| = (1/2)(N - I(S) + |S|)$.

PROOF: Associate to each state S a graph $G(S)$: The vertices of $G(S)$ are the vertices of G (our given graph G), one for each shaded region of $M = M(G)$. Two vertices of $G(S)$ are connected by an edge exactly when this edge corresponds to an interior vertex of S. Thus $G(S)$ has $I(S)$ edges. I assert that $G(S)$ has $|S| - \|S\| + 1$ faces. (Count faces of $G(S)$ by counting circuits of S, but discard circuits forming outer boundaries of shaded components of S; add 1 to count the unbounded face.) By construction, $G(S)$ has $\|S\|$ components. According to Euler

$$(\text{vertices}) - (\text{edges}) + (\text{faces}) = (\text{components}) + 1$$

for a planar graph. Hence

$$N - I(S) + (|S| - \|S\| + 1) = \|S\| + 1.$$

This completes the proof.

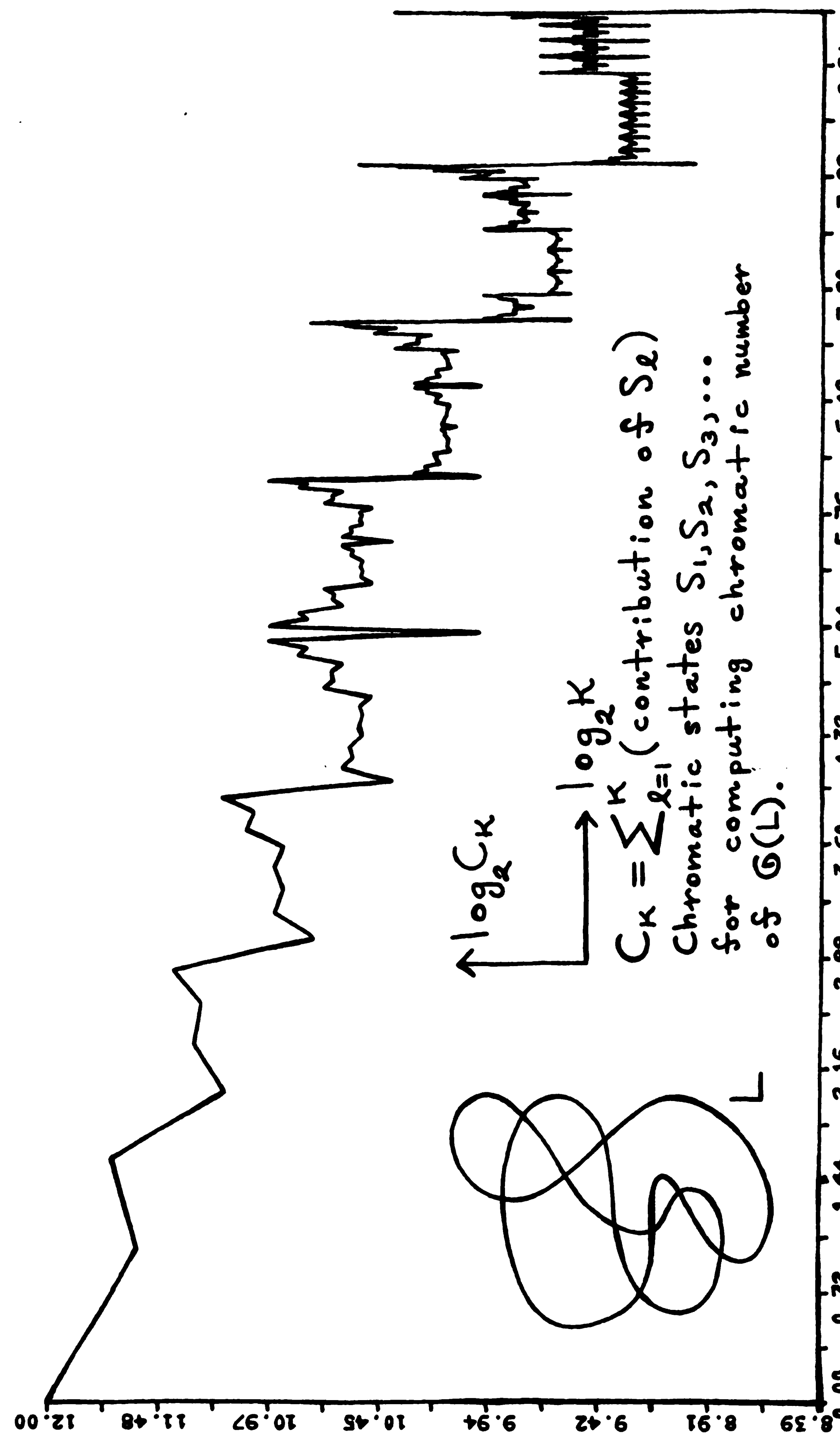
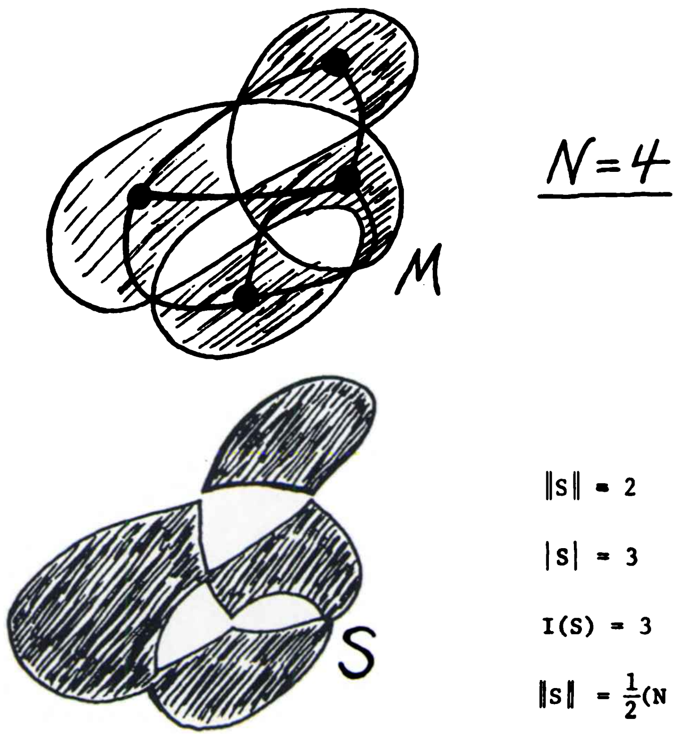

$\log_2 C_K$
$\log_2 K$
$C_K = \sum_{\ell=1}^{K}$ (contribution of S_ℓ)
Chromatic states $S_1, S_2, S_3, \ldots$
for computing chromatic number
of $G(L)$.
L
12.00
11.48
10.97
10.45
9.94
9.42
8.91
8.39
0.00
0.72
1.44
2.16
2.88
3.60
4.32
5.04
5.76
6.48
7.20
7.92
8.64

Figure 5

Examine Figure 6 for an example of the Lemma.



$\|S\| = 2$

$|S| = 3$

$I(S) = 3$

$\|S\| = \frac{1}{2}(N - I(S) + |S|)$

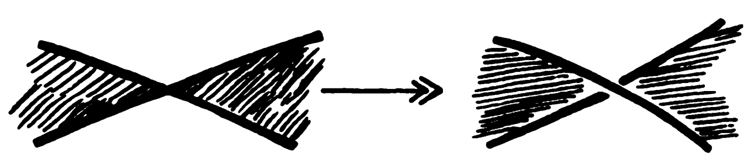
Figure 6

We are about to prove that the chromatic polynomial for a planar graph G can be computed by a bracket applied to an alternating link K(G) associated with G. Two diagrammatic facts are relevant to the construction of K(G):

Fact 1. Every connected universe is the shadow of exactly two alternating link diagrams. (In an alternating diagram the weave passes alternately under and over as one moves along a strand.)

Fact 2. One of the diagrams alluded to in Fact 1 can be obtained as follows:

1. Shade the universe U to form a shaded universe M. (The unbounded region is unshaded in our convention.)
2. Replace each shaded crossing in M by a diagrammatic crossing of type-A, as indicated below.



Call the resulting link K(G). See Figure 7 for an illustration of this process.

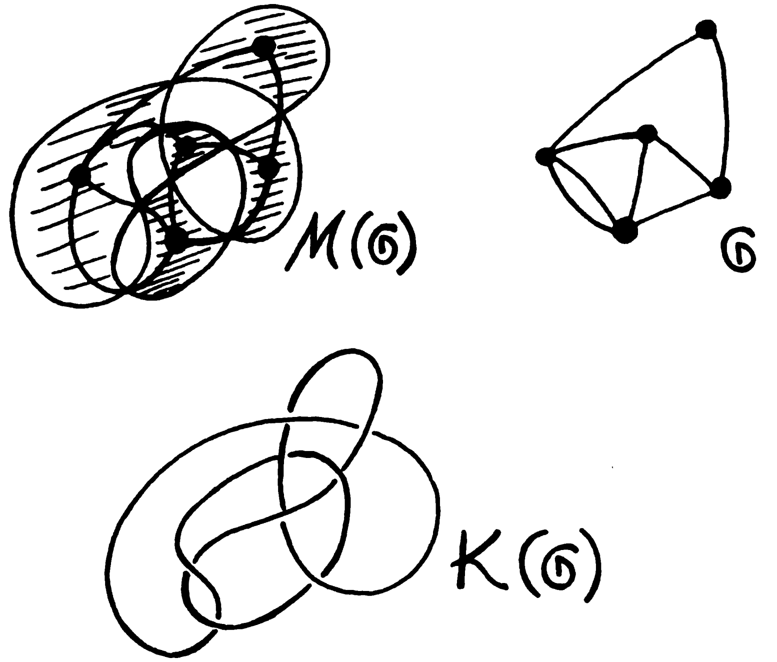


Figure 7

I shall make the translation for a generalization of the chromatic polynomial -- the dichromatic polynomial, Z[G](q,v). The dichromatic polynomial has axioms:

1. Z  = Z  + vZ 

2. $Z_{\bullet G} = qZ_G$

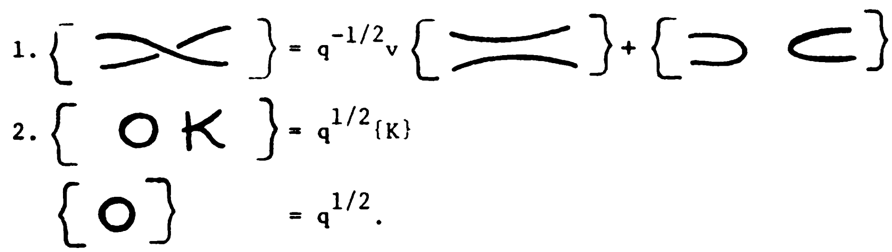
Thus $Z[G](q,-1) = C[G](q)$, giving the chromatic polynomial as a special case when the variable v in the dichromatic polynomial is equal to minus one.

EXAMPLE:

Z  $= Z_{\bullet\bullet} + vZ_{\bullet} = q^2 + vq$

Z  $= Z_{\bullet} + vZ_{\bullet} = q(1+v)$

PROPOSITION 2.3. Let G be a planar graph with N vertices. Let K(G) be the alternating link diagram associated with G by the process described above. For any diagram K let {K} be the special bracket defined by $\{K\} = [K](q^{-1/2}v,1,q^{1/2})$. That is, K satisfies the recursion



Then the dichromatic polynomial is given by the special bracket applied to K(G) via the formula

$$Z_G = q^{N/2}\{K(G)\}.$$

PROOF: Just as in our discussion of the chromatic polynomial we find the dichromatic polynomial is given by a summation over shaded states of the universe M(G) by the formula

$$Z_G = Z_{M(G)} = \sum_{S(M)} v^{I(S)} q^{\|S\|}.$$

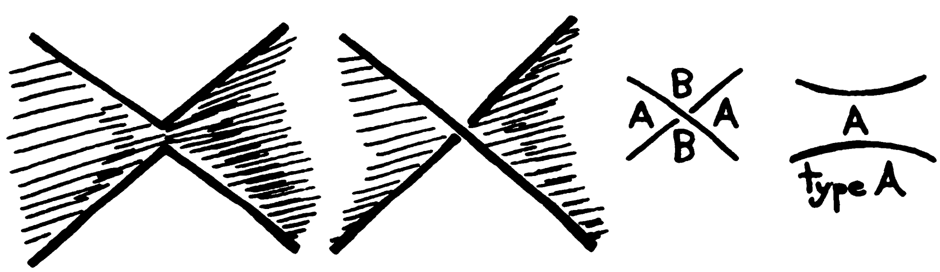
Now use the formula from Lemma 2.2.

$$Z_M = \sum_{S(M)} v^{I(S)} q^{1/2(N-I(S) + |S|)}$$

(*)
$$Z_M = q^{N/2} \sum_{S(M)} (q^{-1/2}v)^{I(S)} (q^{1/2})^{|S|}.$$

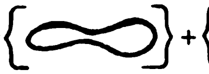
This gives a formula for the dichromatic polynomial in terms of internal vertices and circuits in the states S.

Note that internal vertices in states of M(G) are in one-to-one correspondence with type-A splits in the states of K(G).



The theorem follows from this remark, formula (*), and the form of the state expansion of the general bracket.

EXAMPLE.  G  M(G)  K(G)

$\{K(G)\} = \{$  $\} = q^{-1/2}v\{$  $\} + \{$  $\} = q^{-1/2}vq^{1/2} + q$

$\therefore \{K(G)\} = v + q$

$q^{N/2}\{K(G)\} = qv + q^2.$

SUMMARY. We have defined a general bracket polynomial on link diagrams and have shown how it specializes to the Jones polynomial for links, and to the dichromatic polynomial for planar graphs via the use of the alternating link diagram associated to a planar graph G.

III. THE POTTS MODEL AND THE DICHROMATIC POLYNOMIAL.

Here we discuss a model from statistical mechanics. The framework consists in a lattice G, taken to be any planar graph. (It is also of interest to consider lattices in three dimensional space.) The states of the physical system associated with G consist in assignments of "spins" to the vertices of G. The spins are assumed to be available in q discrete values, where q is a positive integer. Thus we may take the neutral term color in place of spin, and consider a state to be an assignment of colors to the vertices of G (not necessarily a proper coloring). The colors may correspond to spins of particles located at these vertices, or with other localizable and discrete physical states.

Hence we shall speak of states σ of a graph G where $\sigma = \{\sigma_i\}$ is an assignment σ_i to each vertex i of G where σ_i has q possible values. These states should not be confused with the states associated with a link diagram of section 2. If necessary, I shall refer to the latter as diagram states and the former as graph states. A graph state is simply any assignment of q colors to the vertices of a graph G. A diagram state is a mode of splitting the crossings of a link diagram.

To each state σ of a graph G there is an associated energy, $E(\sigma)$, and for the ensemble of all the system's states there is the partition function

$$Z = \sum_{\sigma} e^{-\frac{1}{kT}E(\sigma)}$$

where T denotes temperature, and k is a constant (Boltzman's constant). From the partition function can be deduced many physical properties of the system. For example, the probability $p(E)$ of the system being in a state of energy E is given by the formula

$$p(E) = e^{-\frac{1}{kT}E}/Z.$$

When the partition function itself is written in exponential form

$$Z = e^{-F/kT},$$

then F is the so-called <u>free energy</u> of the system. The <u>average energy</u> U is the expectation value

$$U = \sum_{\sigma} E(\sigma)e^{-E(\sigma)/kT}/Z$$

and one can show that

$$U = F + ST$$

where T denotes temperature, as above, and $S = -\partial F/\partial T$ is the <u>entropy</u> of the system.

It should be mentioned that the intent of this type of model is to create a mathematical situation that embodies the characteristics of a system of interacting particles and its changes under changes of temperature. Thus, on physical grounds, one would expect the system to exhibit <u>phase transition</u>. This means, for example, that as the temperature is started at a high value and then lowered, the distributions of spins will go from very random (at high temperature) to clumping into domains of alignment (at low temperature). The corresponding transition in the distribution of probable states should then be exhibited as a discontinuity or sharp change in the partition function (for very large lattices) at certain critical temperatures.

Historically, very simple rules for the energy of a state have been considered. The <u>Potts model</u> [1] assigns energy by the formula

$$E(\sigma) = \varepsilon \sum_{<i,j>} \delta(\sigma_i,\sigma_j)$$

where i and j are vertices of G, ε is ± 1, $<i,j>$ denotes an edge connecting i and j, and δ is the Kronecker delta:

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

In the case $q = 2$, the Potts model is equivalent to the <u>Ising Model</u>. The Ising model was solved exactly by Onsager in the 1940's. It is the first case of an exactly solved model, and was shown to have a phase transition for the limit of a square $N \times N$ planar lattice as N goes to infinity. No exact expression is known for the limit of the Potts partition function on this lattice for general q.

The epsilon ($\varepsilon = \pm 1$) in this expression for the energy divides the Potts models into two cases: anti-ferromagnetic ($\varepsilon = +1$) and ferromagnetic ($\varepsilon = -1$). Consider the ferromagnetic case. Here it is higher energy for neighboring spins to be different. Thus at low energies (low temperatures) one expects neighboring spins to be aligned as happens in magnetic domains in iron or other magnetic materials. In the anti-ferromagnetic case the low temperature domains will correspond to regions of the graph that are properly colored (neighboring spins different). Physically distinct, these two cases have the same mathematical formalism.

We shall now show that the partition function for the Potts model for G planar is a special case of the general bracket. In fact, I will give a proof of the well-known [12] fact that the Potts partition function is a dichromatic polynomial. Then we are in a position to apply our Proposition 2.3, expressing the dichromatic polynomial as a bracket.

PROPOSITION 3.1. Let G be a planar graph. Let Z be the partition function for the q-state Potts model on G. Let

$$v = e^{-\varepsilon/kT} - 1.$$

Then the partition function is the dichromatic polynomial in q and v.

$$Z = Z[G](q,v)$$

PROOF: By the definition of the partition function, we have the following sequence of equalities

$$Z = \sum_{\sigma} e^{-\frac{1}{kT}E(\sigma)} = \sum_{\sigma} e^{-\frac{\varepsilon}{kT}\sum_{\langle i,j\rangle}\delta(\sigma_i,\sigma_j)}$$

$$= \sum_{\sigma}\prod_{\langle i,j\rangle} e^{-\frac{\varepsilon}{kT}\delta(\sigma_i,\sigma_j)}$$

$$= \sum_{\sigma}\prod_{\langle i,j\rangle}(1 + v\delta(\sigma_i,\sigma_j)),\ v = e^{-\frac{\varepsilon}{kT}} - 1.$$

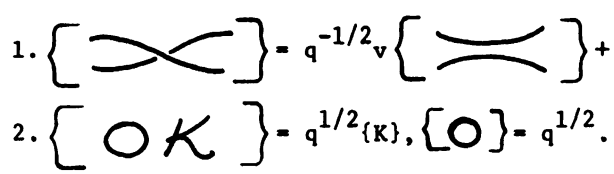
It is easy to see that this last formula is the dichromatic polynomial in q and v. Just verify the recursive axioms (section II) directly. This completes the proof.

Since we know how to translate the dichromatic polynomial into a bracket polynomial, we can finally state

COROLLARY 3.2. Let $Z[G]$ be the q-state Potts partition function for a planar graph G. Let $v = e^{-\frac{\varepsilon}{kT}} - 1$ where k is Boltzman's constant and T is the temperature. Then

$$Z[G] = q^{N/2}\{K(G)\}$$

where N is the number of vertices of G, and $K(G)$ is the alternating link associated with G (as in section II). The symbol $\{K\}$ denotes the special bracket defined by



PROOF: Apply 2.3 to 3.1.

In [9] we shall discuss specific results of this translation for the Potts model. It is remarkable that both the Jones polynomial and the Potts model fit into exactly the same combinatorial framework.

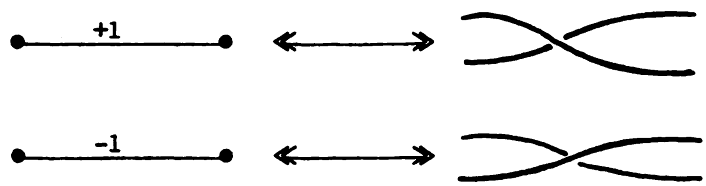
One may wonder whether there is a direct relation between the topology of the link $K(G)$ and the Potts model. This may be the case. It follows from the Tait flyping conjecture that $[K(G)](A,B,d)$ (the general bracket) is a topological invariant of $K(G)$ whenever G is a connected planar graph without isthmus. The Tait conjecture states that topological equivalences of such alternating knots are generated by special moves (flypes) that each preserve the alternating structure. These moves preserve the general bracket, and hence they preserve the Potts partition function.

This means that (modulo the Tait conjecture) we can take a planar lattice G, form the alternating link $K(G)$, transform this link topologically to any other alternating link K' (in reduced form), take the planar graph G' of K'. Then G and G' will have the same partition function. In [9] we shall explore examples of this process.

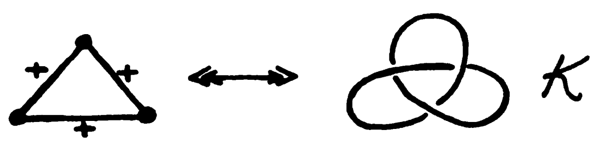
The full Tait conjecture is not yet proved, but some of its corollaries have been verified (minimality and invariance of the number of crossings and of the writhe for reduced alternating diagrams). These verifications depend crucially on the Jones polynomial, the bracket model, and on a two-variable generalization of the Jones polynomial due to the present author (see [5], [6], [10], [13], [14]).

Much remains to be explored in this conduit between physics and topology.

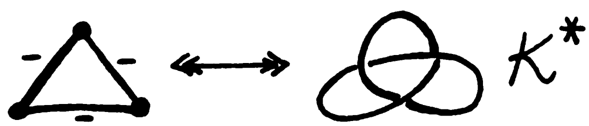
REMARK: It is interesting to reformulate the general bracket as a generalization of the Potts model. To do this, we can regard any link diagram as corresponding to the medial graph of a signed graph G (G embedded in the plane). By a signed graph G I mean that each edge of G is assigned ± 1. This encodes the crossing type for the medial graph according to the convention



Thus

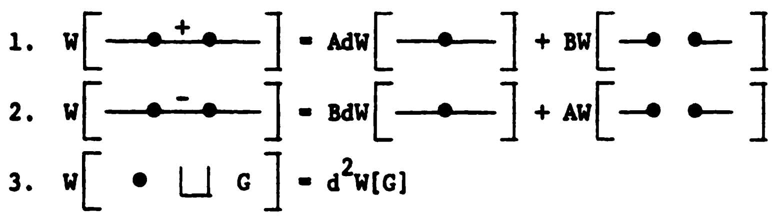


while



Changing all the signs creates the mirror image.

Letting $K(G)$ denote the knot or link diagram corresponding to a signed graph G, we then find (as in Proposition 2.3) that $[K] = d^{-N}W[G]$ where N is the number of vertices in G, and W satisfies recursions:



This generalized contraction-deletion algorithm can then be translated into a corresponding energy model for signed graphs (with d^2 spins per site, and the interaction energy dependent on the sign of the bond). This model will be investigated elsewhere.

IV. BRACKET, BRAID DIAGRAMS AND THE TEMPERLEY-LIEB ALGEBRA.

The n-strand braid group $B[n]$ is generated by elementary braids

$$\sigma_1, \bar{\sigma}_1, \sigma_2, \bar{\sigma}_2, \ldots, \sigma_{n-1}, \bar{\sigma}_{n-1}$$

as shown in Figure 8.

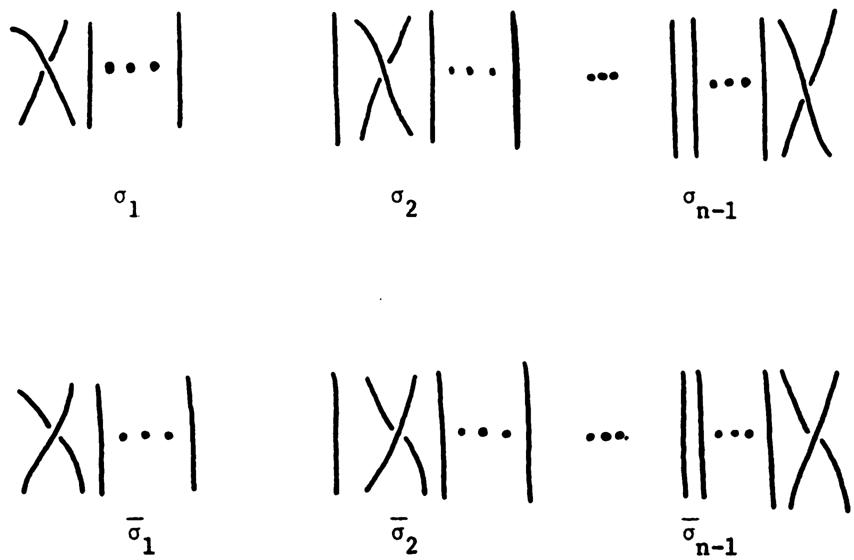
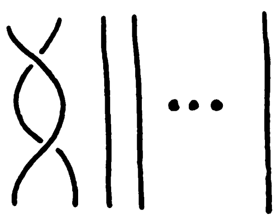


Figure 8

A braid diagram is any specific diagram obtained by taking a word in these generators. Thus $\sigma_1\bar{\sigma}_1$ is the diagram



taken in this form (even though it is isotopic to the identity braid).

If b is a braid diagram, let $\bar{b}$ denote its closure, obtained by connecting top to bottom strands as shown in Figure 9. Let P(b) denote the plat closure for braids with an even number of strands as also shown in Figure 9.

As explained in [5], [6], [8], we can represent any state in the expansion for the general bracket for a braid diagram b as a product of generators $h_1, h_2, \ldots, h_{n-1}$ of local splittings. These generators satisfy diagrammatic relations as shown in Figure 10, and form a multiplicative monoid. We call the diagram algebra the free additive algebra over this monoid, with coefficients in the polynomial ring in the variables A, B. The evaluation of the bracket is invariant under the relations in the diagram algebra.

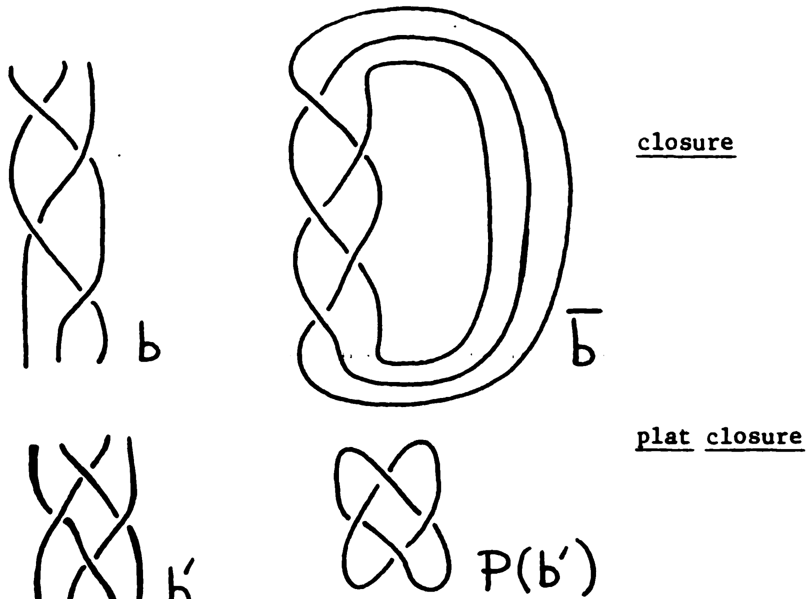
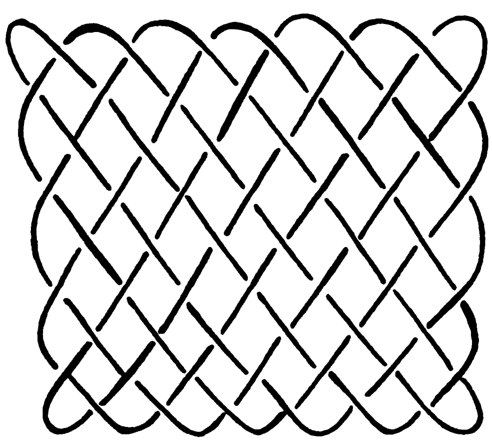
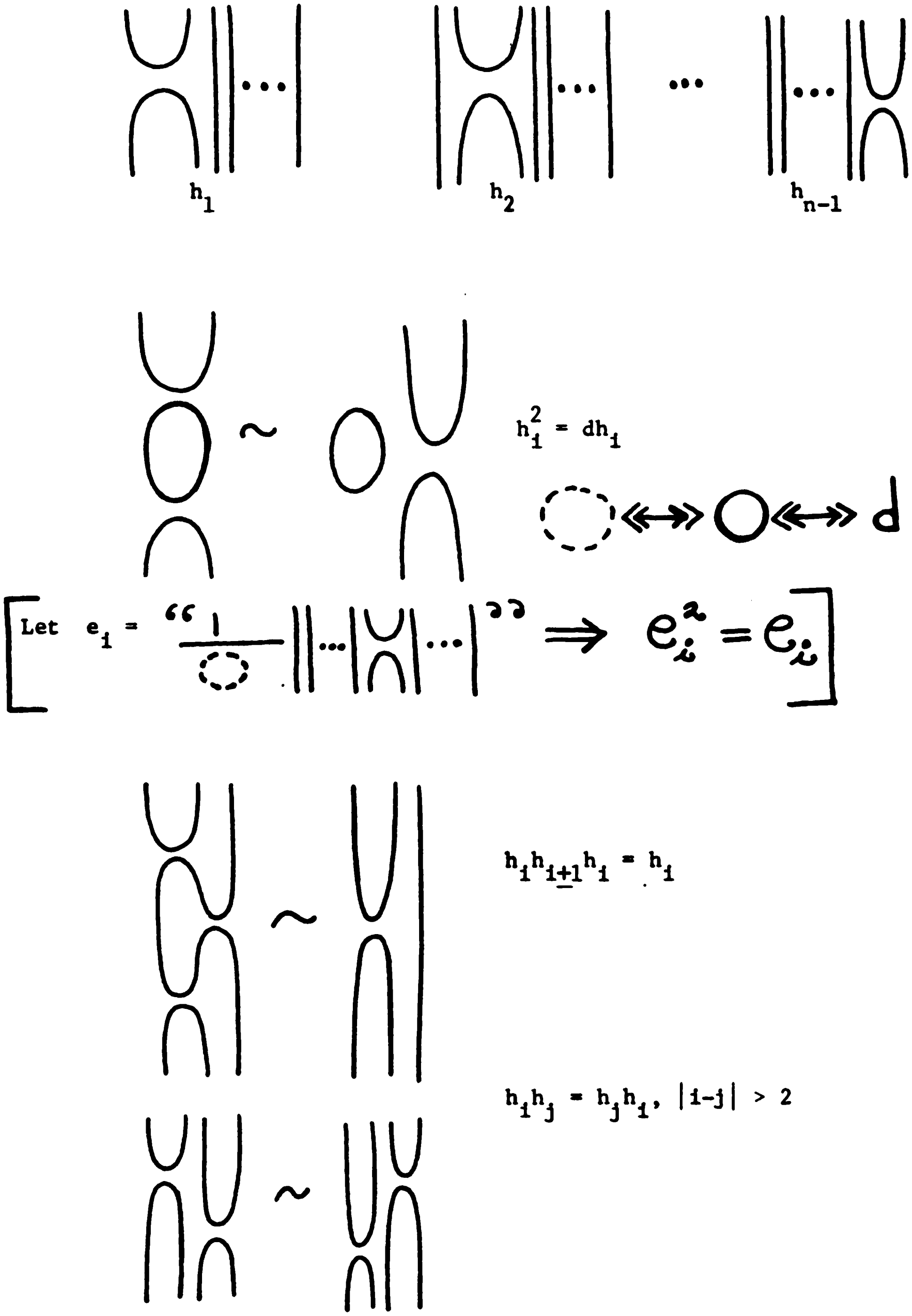


Figure 9

Since, in the diagram algebra, $h_i^2 = dh_i$, it may seem appropriate to regard d as a special element of the algebra. Since d commutes with all elements it has the same status as the polynomial generators A and B. Thus it is sometimes useful to regard the ring Z[A,B,d] as a coefficient ring for the algebra. Note however, that as we have defined it, the diagram algebra is the free additive algebra over Z[A,B] that is generated multiplicatively by the diagram monoid.

In the expression of the relations for Figure 10 the letter d stands for the closed loop. By using these relations we can expand the general bracket, needing only to know the value of the bracket on a product of the monoid generators. This value depends upon the type of closure, since the value of the bracket on a disjoint collection of Jordan curves is d raised to the number of curves. The case of plat closure is particularly interesting.





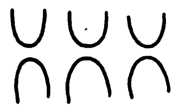
Generators and Relations for Diagram Monoid

Figure 10

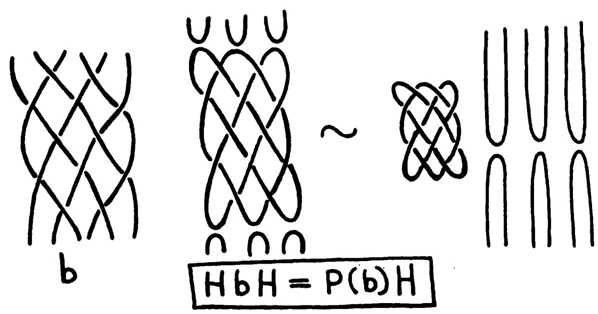
As Figure 11 shows we have, for braids b in B[2n], the diagrammatic formula

$$HbH = P(b)H$$

where H denotes the product $h_1h_3h_5 \ldots h_{2n-1} = H$.

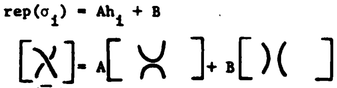


$H = h_1h_3h_5 \cdots h_{2n-1}$
(here $n = 3$)



Hrep(b)H = [b]H
where

rep(σ_i) = $Ah_i + B$



rep($\bar{\sigma}_i$) = $Bh_i + A$

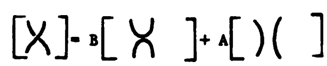


Figure 11

A consequence of this is the following description for the general bracket on the plat closure P(b):

1. The braid diagram b is given as a word in

$$\sigma_1, \bar{\sigma}_1, \ldots, \sigma_{n-1}\, \bar{\sigma}_{n-1}.$$

Replace each instance of σ_i by $Ah_i + B$, and each instance of $\bar{\sigma}_i$ by $Bh_i + A$.

Call the resulting element of the diagram algebra rep(b).

2. Then $H\,rep(b)H = [P(b)]H$ is a valid identity in the diagram algebra. By using the relations in the algebra systematically, this formula provides an algorithm for computing $[P(B)]$.

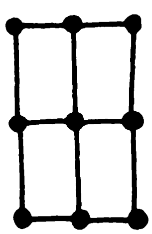
EXAMPLE 1. $b = \sigma_1\sigma_1 =$  , $P(b) =$  , $rep(b) = (Ah_1 + B)^2$

$$H\,rep(b)H = h_1(Ah_1 + B)^2 h_1 = h_1(A^2h_1^2 + 2ABh_1 + B^2)h_1$$
$$= (A^2d^3 + 2ABd^2 + B^2d)h_1.$$

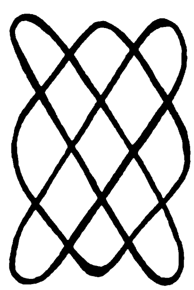
$[P(b)] = \Big[$  $\Big] = (Ad + B)^2 d$

$\therefore\ H\,rep(b)H = [P(b)]H$

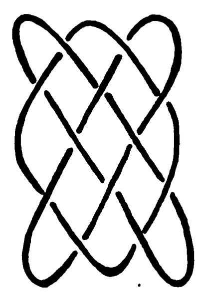
EXAMPLE 2.



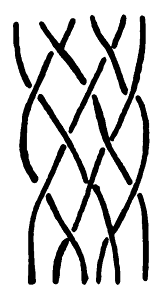
G



M(G)



K(G)



b(G)
"French Sinnet"

$$K(G) = P(b(G))$$

$$b(G) = \sigma_2\sigma_4\bar{\sigma}_1\bar{\sigma}_3\bar{\sigma}_5\sigma_2\sigma_4\bar{\sigma}_1\bar{\sigma}_3\bar{\sigma}_5\sigma_2\sigma_4,\ H = h_1h_3h_5$$

For $\{K\}$: $rep(\sigma_i) = q^{-1/2}v\,h_i + 1$

$$rep(\bar{\sigma}_i) = h_i + q^{-1/2}v$$

$$\left\{\begin{array}{l} h_i^2 = q^{1/2}h_i \\ h_ih_{i\pm1}h_i = h_i \\ h_ih_j = h_jh_i,\ |i-j| > 1 \end{array}\right\}$$

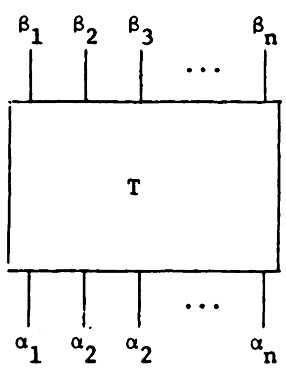
$$Z_G = q^{N/2}\{K(G)\},\ \{K(G)\}H = H\,rep(b(G))H$$

The formalism of this diagram algebra is identical to the Temperley-Lieb Algebra [1] used in evaluating the Potts model. As the example above indicates, the plat closure of the "French Sinnet" braid is the alternating link associated with a rectangular lattice in the plane. The calculation we have indicated for the dichromatic polynomial in this example is exactly paralleled in Baxter [1] in his chapter on the Potts model - but without mention of braids. The entire formalism was invented in the lattice context.

Finally, it is worth indicating that the diagram algebra is actually a "skeleton" of a specific matrix representation for the algebra. Following Penrose [11], et al, let

denote a Kronecker delta δ_{α}^{β}.

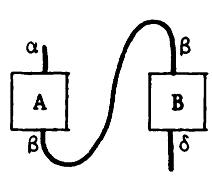
In general, let a diagram in the form



$$= T^{\beta_1\beta_2 \cdots \beta_n}_{\alpha_1\alpha_2 \cdots \alpha_n}$$

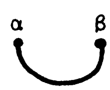
denote a matrix or tensor element. Depict contraction of indices (summation over a repeated index) by connecting corresponding lines. Thus

$$A^{\alpha}_{\beta}B^{\beta}_{\delta} =$$



Let juxtaposition of these forms correspond to tensor product.

We may let

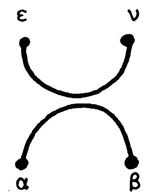


$$= M^{\alpha\beta}$$

and

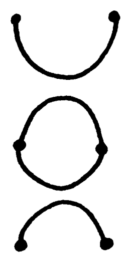
$= M_{\alpha\beta}$

Then

$h \longleftrightarrow$ 

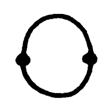
$= M_{\alpha\beta}M^{\varepsilon\nu}$

and

$h^2 \longleftrightarrow$  $\longleftrightarrow M_{\alpha\beta}M^{\varepsilon\nu}M_{\varepsilon\nu}M^{\lambda\mu} = d(M_{\alpha\beta}M^{\lambda\mu})$

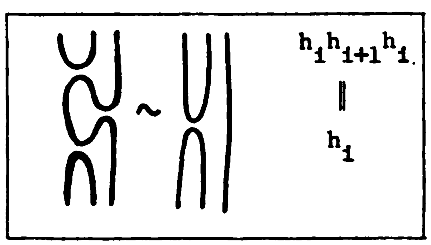
$= dh$

where

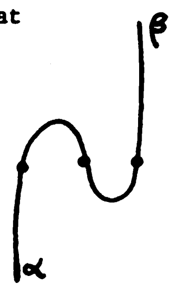
$d = M^{\varepsilon\nu}M_{\varepsilon\nu} \longleftrightarrow$  .

In order to have the relations

$$h_i h_{i\pm 1} h_i = h_i$$

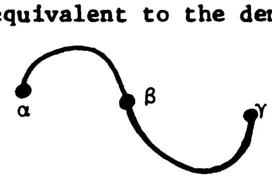


we need that



$= \delta_\alpha^\beta =$

which is equivalent to the demand that

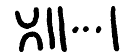


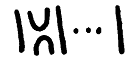
$\longleftrightarrow M_{\alpha\beta}M^{\beta\gamma} = \delta_\alpha^\gamma$

If $M_{\alpha\beta} = M^{\alpha\beta}$, then this requirement is equivalent to the demand that $MM = I$. Thus we can take M to be a 2×2 Pauli matrix such as

$$M = \begin{pmatrix} 0 & \sqrt{-1}\,A \\ -\sqrt{-1}\,A^{-1} & 0 \end{pmatrix}$$

and set

$$h_1 = (M \otimes M) \otimes I \otimes I \otimes \ldots \otimes I \longleftrightarrow$$ 

$$h_2 = I \otimes (M \otimes M) \otimes I \otimes \ldots \otimes I \longleftrightarrow$$ 

$$\vdots$$

$$h_{n-1} = I \otimes I \otimes I \otimes \ldots \otimes (M \otimes M) \longleftrightarrow$$ 

This particular representation is specifically the algebra of Temperly and Lieb. In this form note that

$$M \otimes M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \lambda = -A^2$$

$$\begin{pmatrix} \lambda & 1 \\ 1 & \lambda^{-1} \end{pmatrix}^2 = (\lambda + \lambda^{-1}) \begin{pmatrix} \lambda & 1 \\ 1 & \lambda \end{pmatrix} = (-A^2 - A^{-2}) \begin{pmatrix} \lambda & 1 \\ 1 & \lambda \end{pmatrix}$$

and recall that

$$\langle O K \rangle = (-A^2 - A^{-2})\langle K \rangle .$$

Here $\underline{\text{rep}}: B_n \to$ (Temperley-Lieb) is a representation of the braid group. $\sigma_i \to Ah_i + A^{-1}$, $\sigma_i \to A^{-1}h_i + A$. Thus the formalism of the topological bracket $\langle K \rangle$ (see section II, Example 1) lives in this algebra. It is a nice exercise to see how to compute the topological bracket through this representation. The result is equivalent to Vaughan Jones' original construction of his polynomial for braids.

For the topological bracket we define $\langle b \rangle = \langle \bar{b} \rangle$ where $\bar{b}$ denotes the closure of the braid b, as in Figure 9. Then for an element H in the diagram monoid, $\langle H \rangle = d^{|H| - 1}$ where H denotes the number of disjoint closed curves in $\bar{H}$, and $d = -A^2 - A^{-2}$.

In order to detect this evaluation in the matrix algebra, define

$$\eta = (\eta_a^b) = \begin{pmatrix} -A^2 & 0 \\ 0 & -A^{-2} \end{pmatrix}$$

and $\eta_n = \eta \otimes \ldots \otimes \eta$ (n-copies of η).

Let M_n denote the matrix algebra we have constructed, representing the n-strand diagram algebra. Let $tr: M_n \to Z[A,A^{-1}]$ be the standard matrix trace. Then for any $H \in M_n$, we have the following trace formula for the bracket:

$$\langle H \rangle = d^{-1} tr(\eta H).$$

This is our version of Vaughan Jones' trace formula for his polynomial. The bracket "is" the trace.

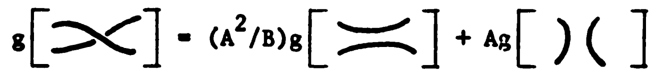
V. THE FERROMAGNETIC CRITICAL POINT

First an exercise about the general bracket. Let K be any link diagram, and let K^* be its mirror image obtained by reversing all the crossings of K. Suppose that K has c(K) crossings.

LEMMA 5.1. Under the above assumptions,

$$[K^*](A,B) = (A/B)^{c(K)}[K](B^2/A,B).$$

PROOF: Let $g[K] = (A/B)^{c(K)}[K]$. Then it is easy to check that



from the expansion formula for the general bracket. It then follows by induction that

$$g[K](A,B) = [K](A^2/B,A).$$

Now use $[K^*](A,B) = [K](A,B)$.

Specialize this lemma to the bracket $\{K^*\}$ where $A = q^{-1/2}v^*$, $B = 1$ and we find

$$\{K^*\}(q^{-1/2}v^*,1) = (q^{-1/2}v)^{c(K)}\{K\}(q^{1/2}v^{-1*},1).$$

In the case where K is the alternating link diagram associated with a rectangular lattice, then K^* is the diagram associated with its planar dual. If $\{K\} = \{K\}(q^{-1/2}v,1)$ then we see that we should make the identification

$$q^{1/2}v^{*-1} = q^{-1/2}v$$

in order to compare these models. (v and v^* being the modified temperature variables in each case). Thus

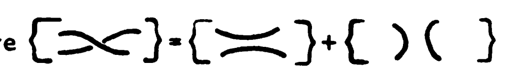
$$vv^* = q.$$

A heuristic argument [16] then suggests that unless the critical point occurs at $v = v^*$ there will, by duality, be at least <u>two</u> critical points. (And this does not make sense physically.) Therefore, we expect the criticality to be at

$$v^2 = q.$$

Here $e^{\frac{1}{kT}} - 1 = v$ (ferromagnetic case), so that

$$T_c = T_{critical} = \frac{1}{k\,\ell n(1 + \sqrt{q})}\ .$$

Note that at this temperature . Hence, at this critcality the recursive expansion for the partition function is particularly simple, and it is independent of crossing types, depending only on the medial graph.

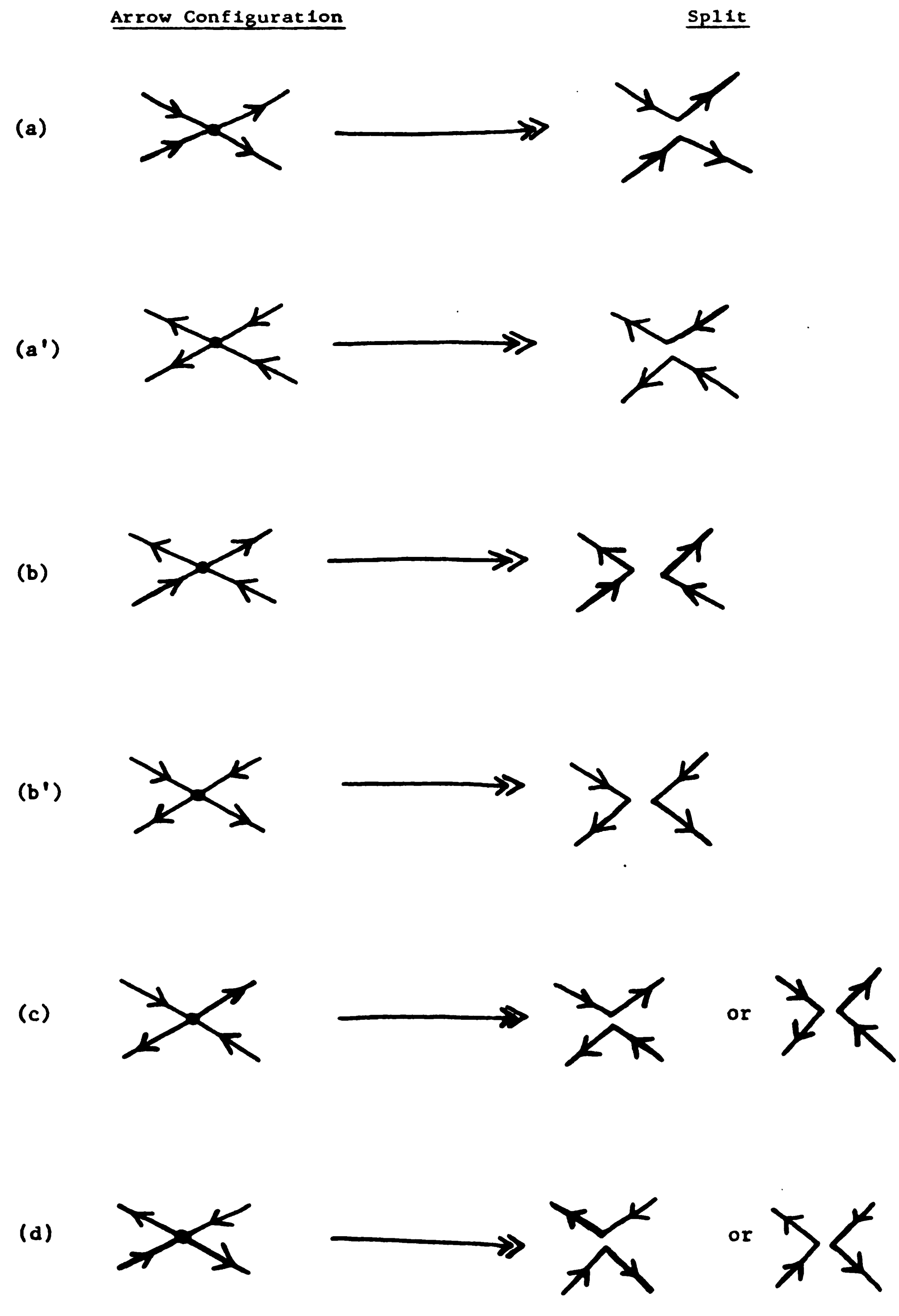
It would be very interesting to see this ferromagnetic critical point verified by an exact calculation of the Potts model.

VI. ICE AND ANGLES.

In this section I explain a method for calculating the general bracket on piecewise-linear link diagrams. The method depends on "arrow-coverings", a state-notion derived from the ice model [1] in statistical mechanics. In the ice-model (6-vertex model) the underlying graph M is 4-valent, hence it is the shadow of a link diagram. An <u>arrow-covering</u> of M is a choice of orientation for each M so that <u>at each 4-valent vertex two arrows point into the vertex and two arrows point out of it</u>.

We shall also allow 2-valent vertices, but here the arrow covering must give a consistent orientation across the vertex (one in and one out).

Figure 12 illustrates the possible arrow configuration at a vertex (four-valent).



Arrow Configurations and Their Corresponding Splittings

Figure 12

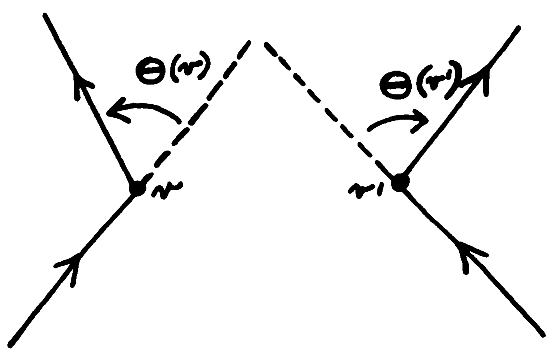
Each arrow configuration gives rise to one or two (see cases (c) and (d) of Figure 12) modes of splitting the diagram at the vertex. We insist that the splits be oriented so that the split diagrams inherit arrow coverings.

Let K be an unoriented link diagram, and let $\mathcal{S}$ denote the collection of link diagrams states S, obtained by splitting vertices as in section 1. Let $\bar{\mathcal{S}}$ denote the collection of oriented states $\bar{S}$ where each simple closed curve in $\bar{S}$ has been assigned an orientation. It is then easy to see that each $\bar{S}$ in $\bar{\mathcal{S}}$ closes to a unique arrow covering $\bar{A} \in \mathcal{A}$ where $\mathcal{A}$ is the set of arrow coverings of U (U is the shadow of K).

Since any simple closed curve μ in a diagram state S will appear with two orientations in the states $\mathcal{S}$ we shall write

$$d = Z^{2\pi} + Z^{-2\pi}$$

where Z is a new variable. Then each 2-valent vertex will contribute a power of Z, Z^{θ}, according to the rule



Counterclockwise rotations give positive angles; clockwise rotations give negative angles.

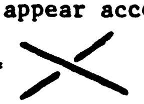
For a piecewise linear oriented simple closed curve the product $Z^{\theta_1}Z^{\theta_2} \ldots Z^{\theta_n}$ over its vertices will equal $Z^{2\pi}$ or $Z^{-2\pi}$ according as the curve is positively or negatively oriented.

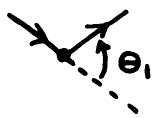
By taking into account both orientations on a simple closed curve, and summing these two products we retrieve $d = Z^{2\pi} + Z^{-2\pi}$.

Now let $\bar{A} \in \mathcal{A}$ be an arrow covering of a piecewise-linear (i.e., the graph is composed of straight line segments) link shadow M. Let M be the shadow of the link K. For each vertex v of M, define K(v) as follows:

(i) If v is 2-valent then $K_A(v) = Z^{\theta(v)}$ where $\theta(v)$ is the angle corresponding to this vertex and arrow covering.

(ii) If v is 4-valent, let v', v'' the two local vertices obtained by splitting the diagram according to the arrow covering. Let $K_A(v)$ = the sum over allowed splittings (there may be two) of $A^{i(v)}B^{j(v)}Z^{\theta(v') + \theta(v'')}$ where A or B appear according to the usual rules of section 1.

For example: if K =  locally then K_A () $= AZ^{\theta_1 + \theta_2}$ where



.

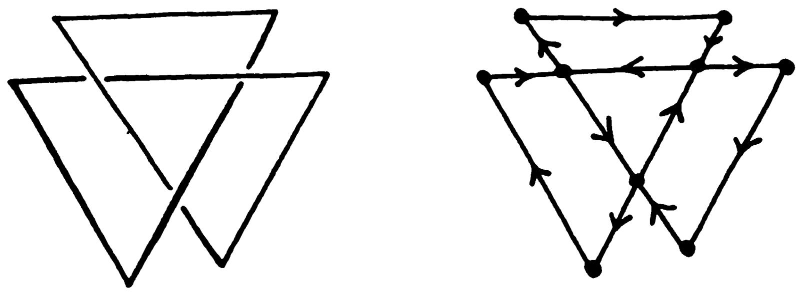
Each contribution $K_A(v)$ is strictly local, once the arrow covering has been chosen. We then have

<u>THEOREM</u> <u>6.1</u>. Let K be a link diagram represented piecewise linearly in the plane. Let M be the shadow of K, and $\mathcal{A}$ the collection of arrow coverings of M. Then the general bracket is given as a sum over all arrow coverings of the products of the ice-angle contributions from each vertex of M:

$$[K] = \sum_{A} \prod_{v} K_A(v).$$

This theorem shows that the bracket is a generalized ice-model. In particular, the Jones polynomial is given by this formula, for we can write the topological bracket in the form above, taking $Z = (\sqrt{-1}\,A)^{1/\pi}$. This reformulation of the bracket calculation in terms of strictly local data may pave the way towards new (infinitesimal or differential geometric) interpretations of the new invariants of knots and links.

In this section we have carried an idea from statistical mechanics into the knot-theoretic context.



K M

K piecewise linear.

An arrow covering of the shadow M.

Figure 13

As pointed out by Baxter [1], in the case of the Potts model this change of variables gives $q^{1/2} = Z^{2\pi} + Z^{-2\pi} = 2\cosh(\lambda)$ when $Z = e^{\lambda/2\pi}$. For q taking values in the Beraha numbers

$$q = 4\cos^2(\pi/n),\ n = 2,3,4,\ \ldots$$

we then have the λ-values

$$\lambda = i\pi/2,\ i\pi/3,\ i\pi/4,\ \ldots$$

It is possible that this reformulation of the bracket will shed geometric light on the mysterious appearances of the Beraha numbers in chromatic problems ($q = 4$, $v = -1$), the Jones algebra [3] and other recent work [2] about the meaning of these models at these special values.

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