A Retrospective Look at Ricci Flow: Lecture 1

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Abstract

This is the first talk in the short course
"A Retrospective Look at Ricci Flow" given via Tencent
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at Xiamen University from March 20 to 30, 2023.

Lecture 1: Ricci Flow on Surfaces

In this talk we discuss the Ricci flow in dimension two, called the Ricci flow on surfaces.

References:

Bennett Chow and Dan Knopf, Ricci Flow: *An Introduction*, Chapter 5, AMS 2004.

Bennett Chow and Yutze Chow, *Lectures on Differential Geometry*, Chapter 14, in preparation.

The Gauss and scalar curvatures

Let (M^2, g) be a Riemannian surface and let Rm denote its Riemann curvature tensor. The **Gauss curvature** of g is defined by

$$K(x) := \mathrm{Rm}(e_1, e_2)e_2 \cdot e_1,$$

where $\{e_1, e_2\}$ is any orthonormal frame at x on M^2.

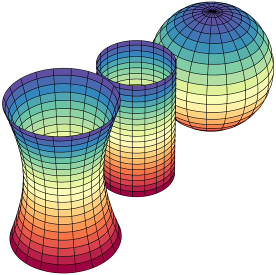
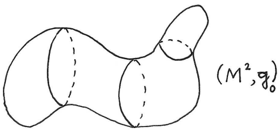


Figure: Author: Nicoguaro.

The Ricci flow on surfaces equation



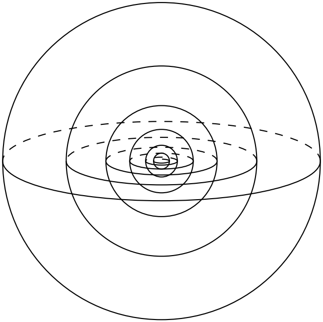
Let M^2 be a closed (i.e., compact without boundary) surface. We say that a 1-parameter family of metrics $g(t)$, $t \in I$, where I is an interval, is a solution to the **Ricci flow on surfaces** if

$$\partial_t g(x,t) = -R(x,t)g(x,t) \tag{0.1}$$

for all $x \in M^2$ and $t \in I$, where $R = 2K$ is the scalar curvature.

Shrinking round 2-spheres, I

A shrinking round 2-sphere is a solution to the Ricci flow on surfaces:



Shrinking round 2-spheres, II

Let g_0 be the Riemannian metric associated to the 2-sphere S^2 of radius $r_0 > 0$. The scalar of g_0 is

$$R_{g_0} = 2r_0^{-2}.$$

Let $g(t)$ be the Riemannian metric associated to the 2-sphere of radius $r(t)$, where

$$r(t) = \sqrt{r_0^2 - 2t}.$$

Them $g(t)$ is a solution to the Ricci flow on surfaces (exercise!) and the metric shrinks to a point as $t \to T := r_0^2/2$.

Normalized Ricci flow on surfaces

The shrinking round 2-spheres example has the defect that the Riemannian surfaces shrink to a point in a finite amount of time T. To remedy this, we consider the **normalized Ricci flow** (**NRF**) defined as follows:

$$\partial_t g(t) = (r - R(t))g(t),$$

where r denotes the average scalar curvature; i.e.,

$$r := \frac{\int_M R\, d\mu}{\int_M d\mu}.$$

This equation has the nice property that (exercise!):

$$\text{Area}(g(t)) \equiv \text{constant} = \text{Area}(g(0)).$$

Short-time existence of the Ricci flow on surfaces

Theorem (Short-time existence of NRF)

For any closed Riemannian surface (M^2, g_0), there exists $T \in (0, \infty]$ and a unique family of metrics $g(t)$, $t \in [0, T)$, that satisfy the normalized Ricci flow

$$\partial_t g(t) = (r - R(t))g(t)$$

with the initial condition $g(0) = g_0$.

Moreover, the metrics $g(t)$ are pointwise conformal to the initial metric g_0; that is, there exist functions $u(t) : M^2 \to \mathbb{R}$ such that

$$g(x, t) = \mathrm{e}^{2u(x,t)} g_0(x)$$

for all $x \in M^2$. That is, for the metrics $g(t)$ the angles between tangent vectors are independent of time.

What is the behavior of solutions?

Short-time existence means that given any Riemannian metric g_0 on a closed surface, there exists a solution $g(t)$ for short time $t \in [0, T)$. The main question now is:

What is the qualitative behavior of solutions in general?

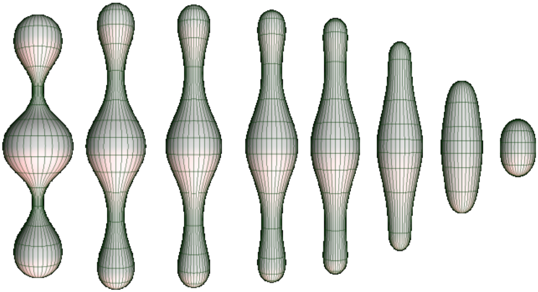


Figure: Author: CBM.

Evolution of the curvature function

Under the normalized Ricci flow on surfaces, the scalar curvature function evolves by the equation:

$$\partial_t R(t) = \Delta_{g(t)} R(t) + R(t)^2 - r\,R(t). \qquad (*)$$

Note that this is a **heat-type** (“diffusion”) **equation** with a quadratic “reaction” term. Thus, such an equation is called a **reaction-diffusion equation**.

Question: **How can we use this equation to help us understand the qualitative behavior of the solution?**

Proof of the curvature evolution formula

Lemma

If Riemannian metrics $g(t)$, $t \in [0, T)$, on M^2 satisfies $\frac{\partial}{\partial t} g(t) = v(t) g(t)$, where $v(t) : M^2 \to \mathbb{R}$ for each $t \in [0, T)$, then

$$\partial_t R(t) = -\Delta_{g(t)} v(t) - v(t) R(t).$$

In particular, taking $v(t) = r - R(t)$ yields $(*)$.

Proof. If $g(t) = \mathrm{e}^{2u(t)} g_0$, then

$$R(t) = \mathrm{e}^{-2u(t)} (R_0 - 2\Delta_0 u(t)) = \mathrm{e}^{-2u(t)} R_0 - 2\Delta_{g(t)} u(t),$$

Thus, if $\partial_t g = vg$, then $2\partial_t u(t) = v(t)$ and

$$\begin{aligned} \partial_t R(t) &= -2\partial_t u(t) \mathrm{e}^{-2u(t)} (R_0 - 2\Delta_0 u(t)) - 2\mathrm{e}^{-2u(t)} \Delta_0 (\partial_t u(t)) \\ &= -v(t) R(t) - \mathrm{e}^{-2u(t)} \Delta_0 v(t) \\ &= -v(t) R(t) - \Delta_{g(t)} v(t). \end{aligned}$$

Curvature as a supersolution to the heat equation

Since r is constant in time, we have (exercise!):

$$\partial_t(R(t) - r) = \Delta_{g(t)}(R(t) - r) + (R(t) - r)^2 + r(R(t) - r).$$

Since $(R(t) - r)^2 \geq 0$, this implies:

$$\frac{\partial}{\partial t}(R(t) - r) \geq \Delta_{g(t)}(R(t) - r) + r(R(t) - r),$$

which in turn implies:

$$\partial_t(\mathrm{e}^{-rt}(R(t) - r)) \geq \Delta_{g(t)}(\mathrm{e}^{-rt}(R(t) - r)).$$

Because of this, we say that $\mathrm{e}^{-rt}(R(t) - r)$ is a **supersolution** to the heat equation.

The parabolic maximum principle

The parabolic maximum principle is a tool to estimate supersolutions to the heat equation.

Theorem (Parabolic minimum principle)

If $w : M^2 \times [0, T) \to \mathbb{R}$, where M^2 is compact, is a supersolution to the heat equation and if $w(x, 0) \geq -C$ for all $x \in M^2$, where C is some constant, then

$$w(x, t) \geq -C \quad \text{for all } x \in M^2, \ t \in [0, T). \tag{0.2}$$

By applying the parabolic maximum principle to $\mathrm{e}^{-rt}(R(t) - r)$, we conclude that there exists a constant C such that

$$\mathrm{e}^{-rt}(R(x, t) - r) \geq -C$$

for some constant C depending only on the initial metric g_0.

A good lower bound for the scalar curvature

By the previous slide, we have that the scalar curvature function has the following lower bound:

$$R(x,t) - r \geq -C\mathrm{e}^{rt}$$

for all $x \in M^2$ and $t \in [0, T)$. This estimate is particularly effective when $r < 0$. This is because in this case we have a lower bound for $\min_{x \in M^2}(R(x,t) - r)$ that is exponentially decaying in time.

Can we prove an effective upper bound in the case where $r < 0$?

For this, the quadratic term $(R(t) - r)^2$ in

$$\partial_t (R(t) - r) = \Delta_{g(t)}(R(t) - r) + (R(t) - r)^2 + r(R(t) - r)$$

is a bad (positive) term.

The difficulty with obtaining an upper bound for R

The ODE associated to the PDE for $R(t) - r$ is obtained by dropping the Laplacian term. The result is the ODE:

$$\frac{d}{dt}\mathrm{S} = \mathrm{S}^2 + r\mathrm{S}. :$$

The solution to this ODE with initial data $\mathrm{S}(0) = \mathrm{S}_0 \neq 0$ is given by

$$\mathrm{S}(t) = \frac{r}{1 - (1 - r/\mathrm{S}_0)\mathrm{e}^{rt}}.$$

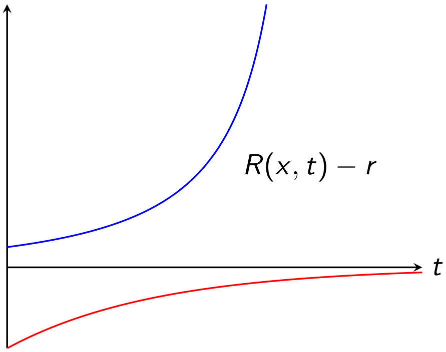
If $\mathrm{S}_0 > 0$, then unfortunately (even when $r < 0$)

$$\mathrm{S}(t) \to \infty \quad \text{as} \;\; t \to T,$$

where $T := -\frac{1}{r}\ln(1 - r/\mathrm{S}_0)$. That is, we have finite-time blow up of the solution to the ODE.

The lower and upper bounds for $R - r$

In the figure below, the lower bound for $R(x,t) - r$ is represented by the red curve. The upper bound is represented by the blue curve. This is the best we can do by looking only at the evolution equation for $R - r$.



The potential function

Let $\overline{R}(x,t) := R(x,t) - r$. By definition,

$$\int_{M^2} \overline{R}(x,t)\, d\mu(x,t) = 0.$$

By Hodge theory, there exists a function $f(t) : M^2 \to \mathbb{R}$ satisfying:

$$\Delta_{g(t)} f(t) = \overline{R}(t). \tag{0.3}$$

f is called the **potential function**. One can derive that the functions $f(t)$ satisfy the linear heat-type equation:

$$\frac{\partial f}{\partial t}(t) = \Delta_{g(t)} f(t) + r\, f(t).$$

So, by the maximum principle, we have that

$$|f(x,t)| \leq C \mathrm{e}^{rt}.$$

But what about estimating the curvature $R(x,t)$?

Curvature upper bound, I

Not only is the potential function a solution to a linear heat-type equation, its gradient squared is a subsolution to a linear heat-type equation:

$$(\partial_t - \Delta)\,\|\nabla f\|^2 = -2\|\nabla^2 f\|^2 + r\|\nabla f\|^2.$$

In particular,

$$(\partial_t - \Delta)\,\|\nabla f\|^2 \le r\|\nabla f\|^2.$$

By applying the maximum principle to this equation, we obtain the estimate:

$$\|\nabla f\|^2(x,t) \le \mathrm{e}^{rt}.$$

Next, we want to take advantage of the good (negative) term $-2\|\nabla^2 f\|^2$.

Curvature upper bound, II

Recall that $\overline{R} := R - r$ satisfies

$$(\partial_t - \Delta)\overline{R} = \overline{R}^2 + r\overline{R} = (\Delta f)^2 + r\overline{R}.$$

By combining this with

$$(\partial_t - \Delta)\,\|\nabla f\|^2 = -2\|\nabla^2 f\|^2 + r\|\nabla f\|^2,$$

we obtain

$$(\partial_t - \Delta)\,(\overline{R} + \|\nabla f\|^2) = -2\left\|\nabla^2 f - \tfrac{1}{2}(\Delta f)g\right\|^2 + r(\overline{R} + \|\nabla f\|^2).$$

In particular,

$$(\partial_t - \Delta)\,(\overline{R} + \|\nabla f\|^2) \leq r(\overline{R} + \|\nabla f\|^2).$$

By applying the parabolic maximum principle to this equation, we obtain the estimate

$$\overline{R}(x,t) \leq (\overline{R} + \|\nabla f\|^2)(x,t) \leq C\mathrm{e}^{rt}.$$

We conclude that $$|\overline{R}(x,t)| \leq C\mathrm{e}^{rt}.$$

The convergence theorem for $\chi(M^2) < 0$

When $r < 0$, which by the Gauss–Bonnet formula is equivalent to $\chi(M^2) < 0$, one can use the estimate $|\overline{R}(x,t)| \leq C e^{rt}$ as a pillar to prove the following **long-time existence** and **convergence** theorem.

Theorem (Uniformization of $\chi < 0$ surfaces by Ricci flow)

Let (M^2, g_0) be a closed oriented Riemannian surface with negative Euler characteristic $\chi(M^2) < 0$. Then there ***exists*** *a solution $g(t)$ to the normalized Ricci flow for all time $t \in [0, \infty)$ with $g(0) = g_0$. As $t \to \infty$, $g(t)$* ***converges*** *in each C^k-norm to a C^∞ metric g_∞* ***with constant scalar curvature*** *equal to $\frac{4\pi\chi(M^2)}{\mathrm{Area}(g_0)}$.*

Higher derivatives of curvature estimate

Besides the curvature decay estimate, we need decay estimates for the higher derivatives of the curvature. This is provided by the following.

Lemma (Higher derivatives of curvature estimate)

Under the normalized Ricci flow on a closed surface M^2 with $\chi(M^2) < 0$ and for each positive integer k, there exists a positive constants C_k depending only on the initial metric g_0 and k such that

$$\|\nabla^k R\|^2(x,t) \leq C_k \, \mathrm{e}^{\frac{r}{2}t} \tag{0.4}$$

for all $x \in M^2$ and $t \in [0, T)$.

This result is used to prove that $g(t)$ converges as $t \to \infty$ to a smooth metric g_∞ in each C^k-norm, where $k \geq 0$.

The convergence theorem for $\chi(M^2) = 0$

When $r = 0$, which by the Gauss–Bonnet formula is equivalent to $\chi(M^2) = 0$, i.e., M^2 is diffeomorphic to a torus, one has the estimate $|R(x,t)| \leq C$. By maximum principle estimates or by integral estimates, one actually prove decay estimates for the curvature and its derivatives. This yields the following result.

Theorem (Uniformization of $\chi = 0$ surfaces by Ricci flow)

*Let (M^2, g_0) be a closed oriented Riemannian surface with zero Euler characteristic $\chi(M^2) = 0$. Then there **exists** a solution $g(t)$ to the normalized Ricci flow for all time $t \in [0, \infty)$ with $g(0) = g_0$. As $t \to \infty$, $g(t)$ **converges** in each C^k-norm to a C^∞ metric g_∞ **with zero scalar curvature**.*

The Ricci flow on the 2-sphere, I: Introduction

The remaining case of the Ricci flow is when $r > 0$, which is equivalent to $\chi(M^2) > 0$, in which case M^2 is diffeomorphic to S^2. The result we are aiming to prove is the following.

Theorem (Uniformization of $\chi > 0$ surfaces by Ricci flow)

Let (M^2, g_0) be a closed oriented Riemannian surface with positive Euler characteristic $\chi(M^2) > 0$. Then there ***exists*** *a solution $g(t)$ to the normalized Ricci flow for all time $t \in [0, \infty)$ with $g(0) = g_0$. As $t \to \infty$, $g(t)$* ***converges*** *in each C^k-norm to a C^∞ metric g_∞* ***with positive scalar curvature****.*

The Ricci flow on the 2-sphere, II: Need for a new estimate

From now on, we assume that M^2 is diffeomorphic to S^2, so that $\chi(M^2) > 0$ and $r > 0$. In this case, we may use the equation

$$\partial_t R(t) = \Delta_{g(t)} R(t) + R(t)^2 - r\, R(t) \geq \Delta_{g(t)} R(t) - r\, R(t)$$

yields the following lower bound for curvature:

$$R(x,t) \geq -C\mathrm{e}^{-rt}.$$

In other words, as $t \to \infty$, the curvature tends to non-negative if not already positive. On the other hand, the improved upper bound yielded:

$$R(x,t) \leq C\mathrm{e}^{rt},$$

which has exponential growth since $r > 0$.

The Ricci flow on the 2-sphere, III: Harnack estimate (a)

A fundamental Bochner formula in geometric analysis says: For any function $u : M^n \to \mathbb{R}$,

$$\frac{1}{2}\Delta|\nabla u|^2 = \langle \nabla u, \nabla \Delta u \rangle + |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u).$$

In particular, if (M^m, g) satisfies Ric ≥ 0, then

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \langle \nabla u, \nabla \Delta u \rangle + |\nabla^2 u|^2.$$

A fundamental application of this inequality is the following Liouville Theorem for harmonic functions.

Theorem (Yau)

Let (M^n, g) be a complete Riemannian manifold with non-negative Ricci curvature. If $u : M^n \to \mathbb{R}$ is a positive harmonic function, i.e., $\Delta u = 0$, then u must be a constant.

The Ricci flow on the 2-sphere, IV: Harnack estimate (b)

The method of proof of the Yau's theorem for harmonic functions (satisfying an elliptic equation) applies to positive solutions of the heat equation (a parabolic equation). Namely, Li and Yau proved the following result.

Theorem (Li–Yau inequality)

Let (M^n, g) be a complete Riemannian manifold with non-negative Ricci curvature. If $u : M^n \times [0, \infty) \to \mathbb{R}$ be a positive solution to the heat equation:

$$\partial_t u = \Delta u.$$

Then we have the following ***differential Harnack estimate****:*

$$\partial_t \ln u - |\nabla \ln u|^2 = \Delta \ln u \geq -\frac{n}{2t}.$$

The Ricci flow on the 2-sphere, V: Harnack estimate (c)

Let (x_1, t_1) and (x_2, t_2) be points in $M^n \times [0, \infty)$ with $t_1 < t_2$. If we integrate the differential Harnack estimate $\partial_t \ln u - |\nabla \ln u|^2 \geq -\frac{n}{2t}$ along space-time paths of the form

$$t \mapsto (\gamma(t), t),$$

where $\gamma : [t_1, t_2] \to M^n$ is a constant speed minimal geodesic from x_1 to x_2, then we obtain the following inequality for positive solutions to the heat equation on complete Riemannian manifolds with non-negative Ricci curvature:

$$\frac{u(x_2, t_2)}{u(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-n/2} \mathrm{e}^{-\frac{d(x_1, x_2)^2}{4(t_2 - t_1)}},$$

where $d(x_1, x_2)$ denotes the distance between x_1 and x_2 with respect to g.

The Ricci flow on the 2-sphere, VI: Harnack estimate (d)

There is a similar Harnack estimate for the Ricci flow on surfaces due to Hamilton. Let $g(t)$ be a solution to the normalized Ricci flow on S^2 with positive scalar curvature. The Harnack quantity is:

$$\begin{aligned} Q &:= \frac{\partial}{\partial t} \ln R - \|\nabla \ln R\|^2 \\ &= \Delta \ln R + R - r. \end{aligned}$$

We now explain the main aspects of the derivation of the Harnack estimate, which follows the method of Li and Yau. Firstly, we compute the evolution equation for Q as:

$$\partial_t Q \geq \Delta Q + 2 \langle \nabla \ln R, \nabla Q \rangle + Q^2 + rQ.$$

Note the similarity between this equation and the equation satisfied by the scalar curvature R.

The Ricci flow on the 2-sphere, VII: Harnack estimate (e)

Regarding $\partial_t Q \geq \Delta Q + 2 \langle \nabla \ln R, \nabla Q \rangle + Q^2 + rQ$, the parabolic maximum principle says that the solution to the ODE $\frac{dq}{dt} = q^2 + rq$ with $q(0) \leq \min_{x \in S^2} Q(x, 0)$ is a lower bound for Q; that is,

$$\partial_t \ln R - \|\nabla \ln R\|^2 = Q(x, t) \geq q(t) := -\frac{Cr\mathrm{e}^{rt}}{C\mathrm{e}^{rt} - 1}.$$

This is the differential Harnack estimate for the Ricci flow on surfaces. By integrating this estimate, we obtain:

Theorem (Hamilton's Harnack estimate)

Let $(S^2, g(t))$ be a solution to the normalized Ricci flow on surfaces with $R > 0$. Let $x_1, x_2 \in S^2$ and $t_1 < t_2$. Then for any path $\gamma : [t_1, t_2] \to M^2$ with $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$, we have

$$\frac{R(x_2, t_2)}{R(x_1, t_1)} \geq \mathrm{e}^{-C'(t_2 - t_1)} \exp\left(- \int_{t_1}^{t_2} \frac{1}{4} \|\gamma'(t)\|^2_{g(t)} dt \right).$$

Hamilton's surface entropy, I: Entropy for the heat equation

Let (M^n, g) be a closed Riemannian manifold and let $f : M^n \to \mathbb{R}$ be a positive function with $\int_{M^n} f \, d\mu = 1$. The **relative entropy** of the probability distribution $f \, d\mu$ is defined as

$$N(f) := -\int_{M^n} f \ln(f) \, d\mu. \tag{0.5}$$

If $f(t) : M^n \to \mathbb{R}$ is a solution to the **heat equation** $\partial_t f = \Delta f$, then

$$\begin{aligned} \frac{dN}{dt} &= -\int_{M^n} \left(\ln(f) \frac{\partial f}{\partial t} + f \frac{\partial}{\partial t} \ln(f) \right) d\mu \\ &= -\int_{M^n} (\ln(f) \Delta f + \Delta f) \, d\mu \\ &= \int_{M^n} \frac{\|\nabla f\|^2}{f} d\mu \\ &\geq 0. \end{aligned}$$

Hamilton's surface entropy, II: Its definition and evolution

Given a closed Riemannian surface (S^2, g) with positive curvature, **Hamilton's surface entropy** is defined by

$$N(g) := \int_{M^2} R \ln R \, d\mu.$$

One computes that

$$\frac{d}{dt} N = -\int_{M^2} \frac{\|\nabla R\|^2}{R} d\mu + \int_{M^2} \overline{R}^2 d\mu,$$

where $\overline{R} = R - r$. Hamilton's surface entropy monotonicity formula says the following.

Theorem

If $(S^2, g(t))$ is a solution to the normalized Ricci flow on surfaces with positive curvature, then the entropy is monotonically non-increasing:

$$\frac{dN}{dt}(t) \le 0.$$

Hamilton's surface entropy, III: Monotonicity and its proof

We define the symmetric 2-tensor β by

$$\beta := -\frac{1}{2}\overline{R}g + \nabla^2 f,$$

where we recall that f is defined by $\Delta f = \overline{R}$. Hamilton's surface entropy monotonicity follows from the identity:

$$\begin{aligned}\frac{d}{dt}N(g(t)) &= -\int_{M^2} \frac{\|\nabla R\|^2}{R} d\mu + \int_{M^2} \overline{R}^2 d\mu \\ &= -4\int_{M^2} \frac{\|\operatorname{div}(\beta)\|^2}{R} d\mu - 2\int_{M^2} \|\beta\|^2 d\mu \\ &\leq 0.\end{aligned}$$

What are the origins of this proof?

Hamilton's surface entropy, IV: Origins of the proof

Recall that $\beta = -\frac{1}{2}\overline{R}g + \nabla^2 f$ appears on the right-hand side of the entropy monotonicity formula. We actually have the following:

$$(\partial_t - \Delta)\left(\overline{R} + \|\nabla f\|^2\right) = -2\,\|\beta\|^2 + r\left(\overline{R} + \|\nabla f\|^2\right).$$

This is actually the first occurrence of the tensor β in the study of Ricci flow on surfaces. Moreover, we have the following important monotonicity formula:

$$(\partial_t - \Delta)\,\|\beta\|^2 = -2\|\nabla\beta\|^2 - 2R\|\beta\|^2 \leq -2R\|\beta\|^2.$$

In particular, if we can prove that $R(x,t) \geq c > 0$ on $M \times [0,\infty)$, then we can conclude that

$$\|\beta\|^2 \leq C\mathrm{e}^{-2ct}.$$

How can we prove a uniform positive lower bound for $R(x,t)$?

A uniform positive lower bound for the scalar curvature

Using Hamilton's Harnack estimate and the surface entropy formula, one can prove that under the normalized Ricci flow on surfaces with positive curvature, the curvature remains uniformly bounded from above. That is, there exists a constant C such that

$$R(x,t) \leq C \quad \text{for all } x \in S^2,\ t \in [0,\infty).$$

Then, by using the Harnack estimate again together with a uniform diameter bound, one can prove that there exists a positive constant c such that

$$R(x,t) \geq c \quad \text{for all } x \in S^2,\ t \in [0,\infty).$$

As a result, we obtain

$$\|\beta\|^2 \leq C\mathrm{e}^{-2ct}.$$

Convergence of the normalized Ricci flow, I

There are a few steps to prove the convergence of the Ricci flow on S^2. Firstly, Hamilton considered the **modified Ricci flow** equation defined by:

$$\partial_t g = 2\beta = -\overline{R} g + 2\nabla^2 f = -\overline{R} g + \mathcal{L}_{\nabla f} g,$$

where $\mathcal{L}$ denotes the Lie derivative. Solutions to the normalized Ricci flow and the modified Ricci flow with the same initial metric are **isometric**.

Because of this, under the modified Ricci flow we obtain the same estimate for β:

$$\|\partial_t g\| = 4\|\beta\|^2 \leq C\mathrm{e}^{-2ct}.$$

Using this, one can show that solutions to the modified Ricci flow converge to a smooth metric g_∞ on S^2 as $t \to \infty$.

What are the geometric properties of g_∞?

Convergence of the normalized Ricci flow, II

The limit g_∞ under the modified Ricci flow, which is a smooth Riemannian metric on S^2, satisfies the equation

$$0 = \beta_\infty := -\frac{1}{2}\overline{R}_\infty g_\infty + \nabla^2_\infty f_\infty$$

for some function $f_\infty : S^2 \to \mathbb{R}$. This equation is called the **shrinking gradient Ricci soliton equation**. We call g_∞ a **shrinking soliton** for short. We can view this equation for g_∞ and f_∞ as:

$$-R_\infty g_\infty = -r\, g_\infty + \mathcal{L}_{\nabla_\infty f_\infty}(g_\infty).$$

Hamilton proved that on the 2-sphere the only solutions to this equation satisfy $f_\infty =$ constant. Thus, g_∞ satisfies

$$R_\infty = r.$$

One then shows that for the original normalized Ricci flow, the metrics $g(t)$ converge to a smooth constant curvature metric on S^2.

THANK YOU!