A Retrospective Look at Ricci Flow: Lecture 4

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Short Course at Xiamen University
March 20-30, 2023

UC San Diego

Abstract

This is the fourth talk in the short course
"A Retrospective Look at Ricci Flow" given via Tencent
http://tianyuan.xmu.edu.cn/cn/MiniCourses/2077.html
at Xiamen University from March 20 to 30, 2023.

Lecture 4: 3-Dimensional Singularity Models

In this talk we discuss the classification of 3-dimensional singularity models.

References:

Hamilton, R. S. *The formation of singularities in the Ricci flow.* JDG 1995.
Perelman, G. *The entropy formula for the Ricci flow and its geometric applications.*

Review

- Ricci flow on surfaces: Hamilton's surface entropy, trace Harnack estimate, potential function, shrinking Ricci soliton estimate.
- Higher-dimensional singularity analysis: Intuitive examples of neckpinch and degenerate neckpinch, Perelman's entropy, reduced distance (we did not discuss), no local collapsing, (local) derivatives of curvature estimates, Cheeger–Gromov compactness theorem, rescaling to get singularity models, Type I and II singular solutions, choosing sequences of space-time points, shrinking and steady Ricci solitons.
- Hamilton's 3-manifolds with $\text{Ric} > 0$ theorem: Statement of pinching improves estimate; singularity model must be S^3/Γ.

Outline of today's talk

- Evolution equation for Rm in dimension 3. The associated system of ODEs.
  - Ricci pinching is preserved.
  - Ricci pinching improves.
  - Hamilton–Ivey estimate.
- Classification of singularity models in dimension 3.
  - Strong maximum principle for systems and a splitting theorem.
  - Compact singularity models.
  - Noncompact singularity models.
    - Hamilton's trace Harnack estimate.
    - Hamilton's matrix Harnack estimate.
    - 3-dimensional eternal solutions are steady solitons.
    - Classification of 3-dimensional shrinking solitons.
    - Classification of 3-dimensional steady solitons.
    - Generalized Smale Conjecture
    - The Lai-Hamilton flying wings

Evolution equation for Rm, I

Under the Ricci flow $\partial_t g = -2\,\mathrm{Ric}$, the Riemann curvature tensor Rm evolves by a heat-type equation of the form:

$$\partial_t \mathrm{Rm} = \Delta \mathrm{Rm} + Q(\mathrm{Rm}),$$

where Q is a quadratic map. In dimension 3, we can express $Q(\mathrm{Rm})$ as follows. Firstly, consider Rm as a curvature operator

$$\mathrm{Rm} : \Lambda^2 M \to \Lambda^2 M,$$

where

$$\mathrm{Rm}(\alpha)(X, Y) := \sum_{i,j=1}^{n} \mathrm{Rm}(X, Y, e_i, e_j)\, \alpha(e_j, e_i)$$

for any 2-form α and where $\{e_i\}_{i=1}^n$ is an orthonormal frame. Note that Rm is a self-adjoint linear map on each fiber $\Lambda_x^2 M$, $x \in M^3$.

Evolution equation for Rm, II

Again, we assume that we are in dimension 3. The Hodge star operator provides a vector bundle isomorphism

$$* : \Lambda^2 M \to \Lambda^1 M \cong T^* M.$$

Thus, we can consider Rm as a self-adjoint bundle map

$$\mathrm{Rm} : T^* M \to T^* M.$$

For $x \in M^3$, choose an orthonormal basis of eigenvectors for Rm, so that Rm is a diagonal matrix: $(\partial_t - \Delta)\,\mathrm{Rm} = Q(\mathrm{Rm})$, where:

$$\mathrm{Rm} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad Q(\mathrm{Rm}) := \begin{pmatrix} a^2 + bc & 0 & 0 \\ 0 & b^2 + ac & 0 \\ 0 & 0 & c^2 + ab \end{pmatrix}.$$

For experts: In the calculation of $\partial_t \,\mathrm{Rm}$, we have swept under the rug what is known as "**Uhlenbeck's trick**".

(Weak) Maximum Principle for Systems, I: Ricci pinching

The behavior of the ODE system $\frac{d\mathbf{R}}{dt}(t) = Q(\mathbf{R}(t))$ associated to the PDE for Rm yields estimates for Rm. This ODE is:

$$da/dt = a^2 + bc,$$

$$db/dt = b^2 + ac,$$

$$dc/dt = c^2 + ab.$$

By analyzing this ODE system and using what is called the **parabolic maximum principle for systems**, one can show the following estimates for solutions to the Ricci flow on a closed 3-manifold with an initial metric g_0 with positive Ricci curvature:

1. (**Ricci pinching.**) The inequality $\mathrm{Ric} \geq \varepsilon R g$, where $\varepsilon > 0$, is preserved. That is, the symmetric 2-tensor $\mathrm{Ric} - \varepsilon R g$ being positive semi-definite is preserved. In the second lecture we mentioned the related estimate that the maximum of $\frac{|\mathrm{Ric}|^2}{R^2}$ is non-increasing.

Maximum Principle for Systems, II: Hamilton–Ivey estimate

2. (**Hamilton–Ivey estimate.**) Let $a < b < c$ denote the eigenvalues of Rm. If at $t = 0$ we have $\mathrm{Rm} \geq -C\,\mathrm{id}$, where $C > 0$ and $\mathrm{id} : \Lambda^2 M \to \Lambda^2 M$ is the identity, then: If $a < 0$ at a space-time point (x, t), then at that point (x, t) we have:

$$|a| \ln |a| \leq R + |a|(3 + \ln C).$$

So, if $|\,\mathrm{Rm}\,|$ is large at a point (x, t), then R is large and positive at the point (x, t), and also if $a < 0$ there, then $|a|/R$ is small at (x, t). Therefore, an important consequence is:

Theorem

*Any 3-**dimensional singularity model** must have non-negative sectional curvature.*

Remark: More generally, by localizing the estimate B.-L. Chen proved that any **3-dimensional complete ancient solution** must have $\mathrm{Rm} \geq 0$.

Proof of Ricci pinching is preserved

Note that the principal Ricci curvatures are sums of principal sectional curvatures, so that $\mathrm{Ric} = \begin{pmatrix} b+c & 0 & 0 \\ 0 & a+c & 0 \\ 0 & 0 & a+b \end{pmatrix}$.

Under the **ODE** $\frac{d\mathbf{R}}{dt}(t) = Q(\mathbf{R}(t))$, one computes for any $K \in \mathbb{R}$,

$$\frac{d}{dt}(c - K(a+b)) = c(c - K(a+b)) - K(a^2 - K^{-1}ab + b^2).$$

In particular, whenever $c - K(a+b) = 0$, where $C \geq 1/2$, we have

$$\frac{d}{dt}(c - K(a+b)) \leq 0.$$

From this calculation and the **parabolic maximum principle for systems**, we have an estimate qualitatively the same as Ricci pinching. The largest sectional curvature c is bounded by a constant times the smallest Ricci curvature $a+b$. So:

If $\mathrm{Ric}_{\min}(x,0) \geq K^{-1}\,\mathrm{sect}_{\max}(x,0)$ for all $x \in M^3$,
then $\mathrm{Ric}_{\min}(x,t) \geq K^{-1}\,\mathrm{sect}_{\max}(x,t)$ for all $x \in M^3$ and $t \geq 0$.

Proof that positive Ricci in 3D tends toward Einstein

Recall that Hamilton proved that there exists $\delta > 0$ and a constant C such that

$$R^{-2}\left|\operatorname{Ric} - \tfrac{1}{3}Rg\right|^2 \leq C\,R^{-\delta}. \qquad (\#)$$

This estimate follows from a monotonicity formula for the ODE $\frac{d\mathbf{R}}{dt}(t) = Q(\mathbf{R}(t))$ and the maximum principle for systems. Namely, to prove $(\#)$ one computes that under the ODE:

$$\frac{d}{dt}\ln\frac{c-a}{(a+b+c)^{1-\delta}} \leq \delta(a+c-b) - (1-\delta)\frac{b^2}{a+b+c}.$$

Without going through the details, one easily shows that

$$\frac{b^2}{a+b+c} \geq \frac{1}{6K}b, \qquad a+c-b \leq 2K\,b.$$

and hence, by choosing $\delta > 0$ so that $\frac{\delta}{1-\delta} \leq \frac{1}{12K^2}$, we obtain

$$\frac{d}{dt}\ln\frac{c-a}{(a+b+c)^{1-\delta}} \leq 0.$$

The idea of the proof of the Hamilton–Ivey estimate

The **Hamilton–Ivey estimate** says that: If $a < 0$ at a space-time point (x, t), then at that point (x, t) we have:

$$|a| \ln |a| \leq R + |a|(3 - \ln(C^{-1} + t)). \qquad (*)$$

This estimate is better than the one we previously stated because of the t term on the right-hand side. For $a < 0$, define $\phi(\mathbf{R}) := \frac{a+b+c}{|a|} - \ln |a|$. One can show that

$$\frac{d}{dt}\phi(\mathbf{R}) \geq |a|.$$

Using this, one proves that whenever $a \leq -\frac{1}{1+t}$, we have:

$$\frac{d}{dt}\left(\frac{a+b+c}{|a|} - \ln |a| - \ln(1+t)\right) \geq 0.$$

The Hamilton–Ivey estimate $(*)$ follows from this and the maximum principle for systems.

The splitting theorem for 3-dimensional ancient solutions

The **strong maximum principle** applied to the system of parabolic PDE $\partial_t \operatorname{Rm} = \Delta \operatorname{Rm} + Q(\operatorname{Rm})$ implies the following.

Theorem (Hamilton)

Let $(M^3_\infty, g_\infty(t))$ be a 3-dimensional singularity model with non-negative sectional curvature. If M^3_∞ is simply-connected, then either:

(1) *$(M^3_\infty, g_\infty(t))$ is the product of $\mathbb{R}$ an ancient solution to the Ricci flow on surfaces with positive curvature.*

(2) *$g_\infty(t)$ has positive sectional curvature.*

We can strengthen this result by invoking **Perelman's No Local Collapsing Theorem**, which says that for (1), the ancient solution to the Ricci flow on surfaces must be the **shrinking round 2-sphere**. I.e., for (1), the singularity model is the **round cylinder**.

Noncollapsed ancient solutions on surfaces

As stated in the previous slide:

Theorem

Any noncollapsed ancient solution to the Ricci flow on oriented surfaces must be the shrinking round 2-sphere.

Proof (by Hamilton). If the ancient solution $(M^2, g(t))$ is Type II, i.e., if

$$\sup_{M^2\times(-\infty,0]} |t|R = \infty,$$

then Hamilton proved that there exists a sequence (x_i, t_i) with $t_i \to -\infty$ for which the rescaled limit exists and is the cigar steady Ricci soliton. Since the cigar soliton is collapsed, we have a contradiction.

On the other hand, if the ancient solution $(M^2, g(t))$ is Type I, i.e., if

$$\sup_{M^2\times(-\infty,0]} |t|R < \infty,$$

Hamilton proved that there is a sequence $(x_i, t_i \to -\infty)$ for which the rescaled limit exists and is the round sphere; then so is $g(t)$.

Compact singularity models must be shrinking solitons, I

Zhenlei Zhang proved the following result in all dimensions.

Theorem

*Let $(M^n_\infty, g_\infty(t))$ be a singularity model of a Ricci flow $(M^n, g(t))$. If M^n_∞ is **compact** (this implies that M^n_∞ is diffeomorphic to M^n), then $(M^n_\infty, g_\infty(t))$ is a **shrinking gradient Ricci soliton**.*

In particular, in dimension 3, we have the following consequence.

Corollary

*Let $(M^3_\infty, g_\infty(t))$ be a singularity model of a Ricci flow $(M^3, g(t))$. If M^3_∞ is **compact** (this implies that M^3_∞ is diffeomorphic to M^3), then $(M^3_\infty, g_\infty(t))$ is a **shrinking spherical space form**; i.e., $g_\infty(t)$ has constant positive sectional curvature.*

So we want to now study **non-compact 3D singularity models**.

Compact singularity models must be shrinking solitons, II

The idea of the proof of Z. Zhang's theorem is to show that any **compact** singularity model must have constant Perelman's entropy. Since

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = 2\tau \int_{M^n} \left| \mathrm{Ric} + \nabla^2 f - \frac{1}{2\tau} g \right|^2 u \, d\mu \geq 0,$$

and since $\mathcal{W}(g(t), f(t))$ is constant on the singularity model, we have that

$$\mathrm{Ric} + \nabla^2 f = \frac{1}{2\tau} g$$

for f being the minimizer of the entropy of the singularity model. Thus, there exist functions $f_\infty(t)$ such that $(M^n, g_\infty(t), f_\infty(t))$ is a shrinking gradient Ricci soliton.

Compact singularity models must be shrinking solitons, III

Can one classify compact shrinking solitons?

This question is undoubtedly too difficult. In particular, Einstein metrics with constant positive scalar curvature are compact shrinking solitons. And currently there seems no hope of classifying Einstein metrics with constant positive scalar curvature, even in dimension 4.

Is there a notion of stability which is both useful for Ricci flow and for which one classify stable compact shrinking solitons?

Can one show that generic Ricci flows avoid unstable compact (and noncompact) shrinking solitons?

There is work of H.-D. Cao, etal. on stable shrinking gradient Ricci solitons.

Noncompact 3D singularity models

Let $(M^3_\infty, g_\infty(t))$ be a **noncompact 3D singularity model**. Then $(M^3_\infty, g_\infty(t))$ is either **(1)** isometric to

$$S^2 \times \mathbb{R} \quad \text{or} \quad (S^2 \times \mathbb{R})/\mathbb{Z}_2,$$

where the $\mathbb{Z}_2$ action on $S^2 \times \mathbb{R}$ is generated by the involution $(x, y) \mapsto (-x, -y)$, or **(2)**

$(M^3_\infty, g_\infty(t))$ has **positive sectional curvature**.

By Perelman's No Local Collapsing Theorem, there exists $\kappa > 0$ such that if $(x, t) \in M^3_\infty \times (-\infty, 0)$ and $0 < r < \infty$ satisfy $R \leq r^{-2}$ in $B(x, t, r)$, then

$$\mathrm{Vol}_t\, B(x, t, r) \geq \kappa r^3.$$

Hamilton's trace Harnack estimate

Hamilton's **matrix Harnack estimate** is a differential Harnack estimate that holds for complete solutions $(M^n, g(t))$, $t \in [0, T)$, to the Ricci flow with bounded non-negative curvature operator.

A consequence of this is Hamilton's **trace Harnack estimate**: For any vector V,

$$\partial_t R + \frac{R}{t} + 2\nabla R \cdot V + 2\,\mathrm{Ric}(V, V) \geq 0.$$

In particular, by taking $V = 0$, we obtain:

$$\partial_t (tR) = t\left(\partial_t R + \frac{R}{t}\right) \geq 0.$$

As a special case, if $(M^n, g(t))$, $t \in (-\infty, 0]$, is an ancient solution, then we obtain

$$\partial_t R(x,t) \geq 0 \quad \text{for all } x \in M^n,\ t \in (-\infty, 0).$$

In particular, if $R \leq C$ at $t = 0$, then we have

$$R(x,t) \leq C \quad \text{for all } x \in M^n,\ t \in (-\infty, 0).$$

That is, we have **control of the curvature backwards in time**.

Hamilton's matrix Harnack estimate

Under the same assumptions as in the previous slide (complete with bounded $\mathrm{Rm} \geq 0$), Hamilton's **matrix Harnack estimate** says that: For any 1-form W and any 2-form U,

$$M_{ij} W^i W^j + 2 P_{pij} U^{pi} W^j + R_{pijq} U^{pi} U^{qj} \geq 0,$$

where

$$M_{ij} := \triangle R_{ij} - \frac{1}{2} \nabla_i \nabla_j R + 2 R^p_{\ell ij} R^\ell_p - R^p_i R_{pj} + \frac{1}{2t} R_{ij}$$

and

$$P_{kij} := \nabla_k R_{ij} - \nabla_i R_{kj}.$$

Let $\{\omega^a\}_{a=1}^n$ be an orthonormal coframe. Taking $W = \omega^a$, $U = \omega^a \wedge V$, and summing over $1 \leq a \leq n$, we obtain the trace Harnack estimate:

$$\partial_t R + \tfrac{R}{t} + 2 \nabla R \cdot V + 2 \,\mathrm{Ric}(V, V) \geq 0.$$

Remark: Hamilton's matrix Harnack quadratic itself is a space-time curvature tensor; see Sun-Chin Chu and C.; Perelman.

3-dimensional eternal solutions are steady solitons

Suppose that $(M^3, g(t))$, $t \in [0, T)$, is a 3D Type II singular solution. Recall from the previous lecture that by the singularity model existence theorem, we obtain a **singularity model** $(M^n_\infty, g_\infty(t))$ defined on the **eternal** time interval $(-\infty, \infty)$ as a complete limit with bounded curvature and such that

$$R_\infty(x_\infty, 0) = \sup_{M_\infty \times (-\infty,\infty)} R_\infty(x, t).$$

As we have observed in today's lecture, since we are in dimension 3, $g_\infty(t)$ has non-negative sectional curvature; i.e., $\mathrm{Rm} \geq 0$. Thus Hamilton's **matrix and trace Harnack estimates** hold for the singularity model $(M^n_\infty, g_\infty(t))$. Using the **strong maximum principle**, Hamilton proved that the singularity model $g_\infty(t)$ is a **steady gradient Ricci soliton**; i.e., there exist functions $f_\infty(t)$ such that:

$$\mathrm{Ric}_{g_\infty(t)} + \nabla^2_{g_\infty(t)} f_\infty(t) = 0.$$

Classification of shrinking solitons in dimension 3

Since any compact shrinking soliton in dimension 3 must be a spherical space form, we consider the **non-compact case**.

Munteanu and Wang proved the following result.

Theorem

*If (M^n, g, f) is a shrinking gradient Ricci soliton with positive sectional curvature, then M^n is **compact**.*

In dimension 3, the following result was originally proved by Perelman, and Cao, Chen and Zhu, and Ni and Wallach.

Corollary

*If (M^3, g, f) is a **noncompact shrinking gradient Ricci soliton**, then (M^3, g) is either isometric to:*

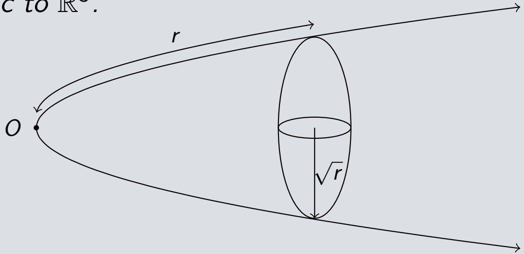
$$\mathbb{R}^3, \quad S^2 \times \mathbb{R} \quad \text{or} \quad (S^2 \times \mathbb{R})/\mathbb{Z}_2.$$

Classification of noncollapsed steady solitons in dimension 3

We have the following theorem of Brendle, which was originally asserted without proof by Perelman.

Theorem

*Any 3-dimensional noncollapsed steady gradient Ricci soliton must be $\mathbb{R}^3$ or the rotationally symmetric **Bryant steady gradient Ricci soliton**. In particular, M^3 is necessarily noncompact and diffeomorphic to $\mathbb{R}^3$.*



Classification of noncollapsed ancient solutions in dimension 3

Later, Brendle proved the following stronger result which was conjectured by Perelman.

Theorem (Brendle)

Any 3-dimensional complete noncollapsed ancient solution with positive sectional curvature must be the rotationally symmetric Bryant soliton.

Combined with the earlier work of Hamilton and Perelman, this has the following **classification of 3D singularity models**.

Corollary

Any 3-dimensional singularity model must be either a spherical space form S^3/Γ, $S^2 \times \mathbb{R}$, $(S^2 \times \mathbb{R})/\mathbb{Z}_2$, *or the Bryant soliton.*

Bamler and Kleiner's Theorem, I

Let $\mathrm{Diff}(M^n)$ denote the diffeomorphism group of a smooth manifold.

Let $\mathrm{Isom}(M^n, g)$ denote the isometry group of a Riemannian manifold.

Using Ricci flow, Bamler and Kleiner proved the following.

Theorem (Generalized Smale Conjecture)

For any spherical space form S^3/Γ, the inclusion map

$$\mathrm{O}(4) \cong \mathrm{Isom}(S^3/\Gamma) \hookrightarrow \mathrm{Diff}(S^3/\Gamma)$$

*is a **homotopy equivalence**.*

Bamler and Kleiner's Theorem, II

One of the ideas of Bamler and Kleiner's proof is to consider the **linearized Ricci flow**, which is the Ricci flow $\frac{\partial}{\partial t} g = -2 \operatorname{Ric}$ coupled to the equation

$$\frac{\partial}{\partial t} h = \Delta_L h,$$

where $h = h(t)$ is a family of symmetric 2-tensors and where Δ_L is the **Lichnerowicz Laplacian**.

Recall that the Lichnerowicz Laplacian is defined by

$$(\Delta_L h)_{ij} := (\Delta h)_{ij} + 2R_{kijl} h_{kl} - R_{ik} h_{kj} - R_{jk} h_{ik}.$$

If h were an (antisymmetric) 2-form, instead of a symmetric 2-tensor, then we would have that

$$\Delta_L h = \Delta_d := -(d\delta + \delta d)$$

equals the Hodge Laplacian (Bochner–Weitzenböck formula).

Linearized Ricci flow

Recall that the **linearized Ricci flow** is the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ coupled to the Lichnerowicz Laplacian heat equation

$$\frac{\partial}{\partial t} h_{ij} = (\Delta_L h)_{ij} = (\Delta h)_{ij} + 2R_{kijl} h_{kl} - R_{ik} h_{kj} - R_{jk} h_{ik}.$$

Can one use the linarized Ricci flow to study higher (≥ 4) dimensions?

Can one formulate a useful notion of generic Ricci flow, where one has a more restrictive class of forming singularity models?

Remark: Hamilton and C. proved a **linear trace Harnack estimate** for complete solutions with bounded $\mathrm{Rm} \geq 0$.

The Lai–Hamilton flying wing steady solitons, I

Hamilton conjectured that there exist flying wing steady gradient Ricci solitons in dimension three. Hamilton's conjecture has been proved by Yi Lai.

Theorem (Lai–Hamilton flying wings, I)

For each $\alpha \in (0, \pi)$ there exists a 3-dimensional steady gradient Ricci soliton (M^3, g, f) with the following properties.

1. *M^3 is diffeomorphic to $\mathbb{R}^3$.*
2. *g has bounded positive sectional curvature.*
3. *g is not noncollapsed.*
4. *The asymptotic cone of (M^3, g) is the cone over an interval of length α, a.k.a. a sector of angle α.*
5. *Un-rescaled limits at infinity are $\mathbb{R}^2 \times S^1$ or the product of $\mathbb{R}$ and the cigar soliton.*

The Lai–Hamilton flying wing steady solitons, II

The following is a diagram from Yi Lai's paper "A FAMILY OF 3D STEADY GRADIENT SOLITONS THAT ARE FLYING WINGS", arXiv:2010.07272.

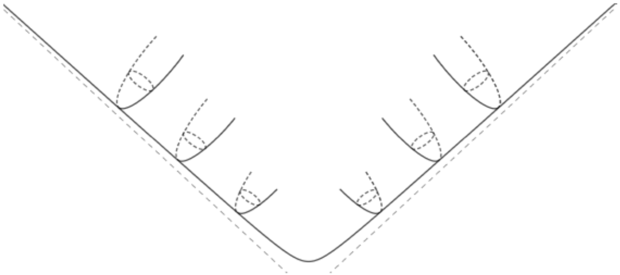


FIGURE 1. A 3d flying wing

Flying wing shrinking solitons?

Bamler: Do there exist flying wing shrinking gradient Ricci solitons in dimension 4?

- For the flying steady Ricci soliton in dimension 4, the **unrescaled** pointed limit for a sequence $x_i \to \infty$ along an edge of the wing converges the product of $\mathbb{R}$ and the 3D Bryant soliton.
- Bamler has asked the question of whether there exists a 4D noncompact shrinking gradient Ricci soliton with $|\,\mathrm{Rm}\,| \leq C(r+1)^2$ such that $|\,\mathrm{Rm}\,| \approx C(r+1)^2$ along the two edges of the wing and such that the the **rescaled** pointed limit for a sequence $x_i \to \infty$ along an edge of the wing converges the product of $\mathbb{R}$ and the 3D Bryant soliton. If so, he conjectures that this shrinking soliton is a singularity model.

THANK YOU!