A Retrospective Look at Ricci Flow: Lecture 5

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Abstract

This is the fifth talk in the short course
"A Retrospective Look at Ricci Flow" given via Tencent
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Lecture 5: Singularity Formation in Higher Dimensions

This talk is an introduction to the study of singularity formation for the Ricci flow in higher dimensions.

References:

Bamler, R. H. *Recent developments in Ricci flows.*
Perelman, G. *The entropy formula for the Ricci flow and its geometric applications.*

Introduction

One of the main goals in Ricci flow is to understand singularity formation and singularity models. The reason for this to be able to define Ricci flows past singularity times via some version of Ricci flow with surgery. Singularity formation is well understood in dimensions two and three. This is due to the works of Hamilton, Perelman, Brendle, and Bamler. A fascinating question is to understand singularity formation in dimension four. In the past few years, great progress and breakthroughs have been made by Bamler. A general open question is: As in dimension three, can Ricci flow be used to obtain topological results in dimension four?

Review of Ricci solitons

Ricci solitons model singularities. Recall a **shrinking gradient Ricci soliton** (**shrinking soliton** for short) is a triple (M^n, g, f) satisfying the equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_{\nabla f} g = \mathrm{Ric} + \nabla^2 f = \frac{1}{2} g.$$

A special case of this is when f is constant, in which case we get the **Einstein metric** equation $\mathrm{Ric} = \frac{1}{2} g$. By **Myers' Theorem**, such Einstein manifolds must be **compact**. One is interested in both compact and noncompact shrinking gradient Ricci solitons, especially **noncompact** ones. Examples of noncompact shrinking gradient Ricci solitons are **cylinders** $(S^k/\Gamma) \times \mathbb{R}^{n-k}$, where $k \geq 2$.

The **steady soliton** equation, as e.g. the 2D cigar soliton, is

$$\mathrm{Ric} + \nabla^2 f = 0.$$

Bamler's Higher-Dimensional Theory, I

An aim of Ricci flow in higher dimensions is to prove the **existence of Ricci flow past singularity times** (e.g. Ricci flow with surgery). To this end there is recent striking progress due to Richard Bamler.

Theorem (Bamler)

*Let $(M^4, g(t))$, $t \in [0, T)$, be a Ricci flow on a closed 4-manifold that develops a singularity at time $T < \infty$. If M^4 is not diffeomorphic to a shrinking Ricci soliton, then there **exists an associated singularity model** which is one of the following:*

1. 2-***cylinder***: $S^2 \times \mathbb{R}^2$,
2. 3-***cylinder***: $(S^3/\Gamma) \times \mathbb{R}$,
3. ***Cone***: *A Riemannian cone with non-negative scalar curvature, that is either a flat orbifold $\mathbb{R}^4/\Gamma$ or the asymptotic cone of an asymptotically conical shrinking Ricci soliton.*

Bamler's Higher-Dimensional Theory, II

The result on the previous slide is a corollary of an extensive **theory of higher-dimensional singularity formation** for Ricci flow developed by Bamler. The goal of Bamler's theory is to enable one to continue the Ricci flow past singularities, where one has some understanding of the topological change as the time passes singularity times.

Some highlights of Bamler's theory are:

- Role of the conjugate heat kernel, the pointed Nash entropy, and Wasserstein 1-distance in the analysis of Ricci flow.
- A general theory of metric flows and a general compactness theorem associated to this theory.
- A structure theory for singularity formation inspired by Cheeger–Colding theory.

Ricci solitons

For singularity analysis in all dimensions, Ricci solitons play an important role. We will now discuss some of what is known about the geometry and topology of Ricci solitons. We first discuss shrinking solitons and then steady solitons.

Low-dimensional classification of shrinking Ricci solitons

Just as the study of 3D Ricci flow is related to the study of 2D Ricci flow, the study of 4D Ricci flow is related to the study of 3D Ricci flow.

In view of the importance of **shrinking gradient Ricci solitons** in the Ricci flow theories of Hamilton, Perelman, and Bamler, we discuss some aspects of the qualitative study of shrinking solitons.

In **dimension** 2, the only complete shrinking solitons on orientable surfaces are the Gaussian soliton of the flat $\mathbb{R}^2$ with the potential function $f(x) = \frac{|x|^2}{4}$ and the round sphere S^2 with constant scalar curvature $R = 1$.

In **dimension** 3, the only complete shrinking solitons on orientable 3-manifolds are the Gaussian soliton of the flat $\mathbb{R}^3$ with the potential function $f(x) = \frac{|x|^2}{4}$, spherical space forms S^3/Γ, the cylinder $S^2 \times \mathbb{R}$, and its quotient $(S^2 \times \mathbb{R})/\mathbb{Z}_2$.

Estimates for shrinking Ricci solitons, I

What can we say about the geometry of Ricci solitons?

Let (M^n, g, f) be a **complete non-compact** shrinking soliton, so that $\mathrm{Ric} + \nabla^2 f = \frac{1}{2} g$. Recall that

$$R + |\nabla f|^2 = f.$$

Then there exists an associated **complete Ricci flow** defined on the **ancient** time interval $(-\infty, 1)$.

B.-L. Chen proved that for any ancient complete Ricci flow, we have $R \geq 0$. Thus,

$$|\nabla f|^2 \leq f, \quad \text{i.e.,} \quad \left|\nabla \sqrt{f}\right|^2 \leq \frac{1}{4}, \quad \text{i.e.,} \quad \left|\nabla (2\sqrt{f})\right| \leq 1.$$

Fix a point $o \in M^n$. Then, for any $x \in M^n$, we have

$$f(x) \leq \frac{1}{4}\left(d(x, o) + C\right)^2,$$

where $C = 2\sqrt{f}(o)$. So we have a good **upper bound for the potential function** f. This implies $R(x) \leq \frac{1}{4}\left(d(x, o) + C\right)^2$.

Estimates for shrinking Ricci solitons, II

Huai-Dong Cao and Detang Zhou (with a contribution by Munteanu), proved the following important **lower bound for the potential function** of a complete non-compact shrinking soliton:

$$f(x) \geq \frac{1}{4}(d(x, o) - C)_+^2,$$

where $A_+ := \max\{A, 0\}$ for $A \in \mathbb{R}$. Therefore **f attains its minimum on M^n**. Choosing o to be a minimum point of f, one can prove that

$$f(x) \geq \frac{1}{4}(d(x, o) - 5n)_+^2;$$

this improvement was by Robert Haslhofer and Reto Müller. Cao and Zhou's estimate is important in the analysis of shrinking solitons.

Estimates for shrinking Ricci solitons, III

In fact $f(o) = \min f \leq \frac{n}{2}$. So we have the qualitatively sharp estimates for the potential function:

$$\frac{1}{4}(d(x,o) - 5n)_+^2 \leq f(x) \leq \frac{1}{4}\left(d(x,o) + \sqrt{2n}\right)^2.$$

What about the curvature of complete shrinking solitons?

Firstly, recall that $R \geq 0$. One can show that either (M^n, g, f) is the Gaussian shrinking soliton on $\mathbb{R}^n$ or $R > 0$. So, from now on we assume that $R > 0$.

Peng Lu, Bo Yang, and C. proved the following: There exists a constant $c > 0$ depending on the shrinking soliton such that

$$R(x) \geq c(1 + d(x,o))^{-2}.$$

What upper bounds are known for shrinking Ricci solitons?

Estimates for four-dimensional shrinking solitons, I

Again recall that

$$R + |\nabla f|^2 = f.$$

This implies that

$$R(x) \leq f(x) \leq \frac{1}{4}\left(d(x, o) + \sqrt{2n}\right)^2.$$

Michael Freedman, Henry Shin, Yongjia Zhang, and C. showed that in dimension 4, one can improve this as follows.

Theorem

If (M^4, g, f) is a shrinking gradient Ricci soliton that is also a singularity model, then there exists a constant C depending on (M^4, g, f) such that

$$|\operatorname{Rm}|(x) \leq C\left(d(x, o) + C\right)^2.$$

Estimates for four-dimensional shrinking solitons, II

The idea of the proof is as follows. Suppose the quadratic growth estimate for $|\,\mathrm{Rm}\,|$ is false. Then there exists $x_i \to \infty$ such that

$$K_i := |\,\mathrm{Rm}\,|(x_i) \gg C d^2(x_i, o) \geq R(x_i).$$

Use point selection to obtain a complete limit $(M^4_\infty, g_\infty, x_\infty)$. Since we are rescaling by the factors K_i, for the limit, we have $R_{g_\infty} \equiv 0$. We obtain a limit solution $g_\infty(t)$ to Ricci flow, so the equation $\partial_t R = \Delta R + 2|\,\mathrm{Ric}\,|^2$ for $g_\infty(t)$ implies that $\mathrm{Ric}_{g_\infty} \equiv 0$. By Perelman's **no local collapsing theorem**, g_∞ has Euclidean volume growth; i.e., $\mathrm{Vol}(B_r) \geq cr^4$ for some $c > 0$ and all $r > 0$. By a theorem of **Cheeger and Naber**, (M^4_∞, g_∞) is an ALE. By the Cheeger–Gromov convergence and since we have a 4D shrinker model, this implies that there is an unbounded number of disjoint embeddings of the Ricci flat ALE in the original compact 4-manifold. By topological and geometric ((co)homology and characteristic classes) methods, one can obtain a contradiction.

Estimates for four-dimensional shrinking solitons, III

There are examples of Kähler shrinking Ricci solitons (including in dimension 4) that are **asymptotically conical**. These are called **FIK solitons** (for Feldman, Ilmanen, and Knopf). For such shrinking solitons, we have

$$R(x) \approx c\, d(x, o)^{-2}$$

as $x \to \infty$. So the lower scalar curvature bound is qualitatively sharp.

Is the upper scalar curvature bound qualitatively sharp?

Bamler has asked the following question:

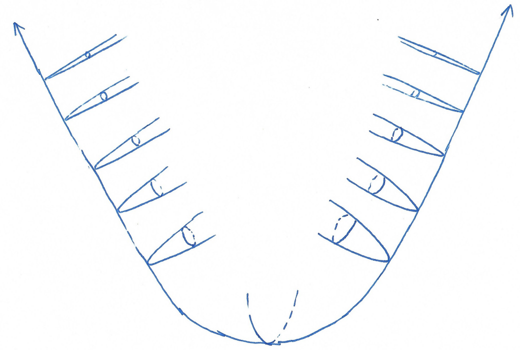
Does there exist flying wing shrinking solitons with curvature growing quadratically in certain directions?

Such shrinking solitons would be somewhat analogous to the Lai–Hamilton flying steady solitons.

A shrinking flying wing?

A hypothetical shrinking flying wing is pictured below.

In the central region, as points tend to infinity, the shrinker looks like $S^2 \times \mathbb{R}^2$, where S^2 has a fixed radius. Along the edges, the rescalings tend to the product of a 3D Bryant steady soliton and $\mathbb{R}$.



Estimates for four-dimensional shrinking solitons, III

What is the structure of 4-dimensional shrinking gradient Ricci soliton singularity models?

Ovidiu Munteanu and Jiaping Wang proved the following result without assuming the shrinking soliton is a singularity model.

Theorem (Munteanu and Wang)

Any complete non-compact 4-dimensional shrinking gradient Ricci soliton with $R(x) \to 0$ as $x \to \infty$ must satisfy the estimate

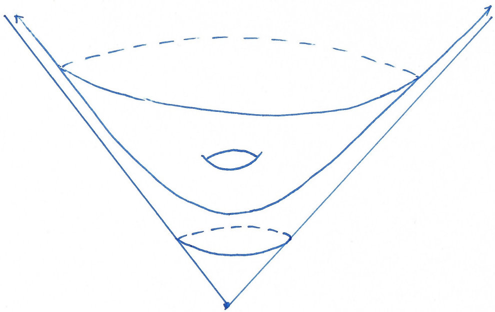
$$|\operatorname{Rm}|(x) \leq C\left(d(x, o)+1\right)^{-2}.$$

This implies that the shrinking soliton is ***asymptotically conical****.*

We say (M^n, g) is **asymptotic to a Riemannian cone** $(C_0\Sigma^{n-1}, g_{\text{cone}})$ if there are r_0 and a diffeomorphism $\Phi : C_{r_0}\Sigma \to M^n \setminus K$, where K is a compact set, such that $(C_{\lambda^{-1}r_0}\Sigma, \lambda^{-2}\lambda^*\Phi^*g)$ converges as $\lambda \to \infty$ to $(C_0\Sigma, g_{\text{cone}})$ in C^2_{loc}.

Asymptotically conical shrinking solitons, I

Picturing an asymptotically conical shrinking gradient Ricci soliton. The shrinker (M^n, g, f) is asymptotic to a cone $C(\Sigma^{n-1})$ over an $(n-1)$-dimensional closed Riemannian manifold (Σ^{n-1}, h).



Asymptotically conical shrinking solitons, II

Brett Kotschwar and Lu Wang proved that any asymptotically conical shrinking soliton is determined by its asymptotic cone.

Theorem (Kotschwar and Wang)

Let (M_1^n, g_1, f_1) and (M_2^n, g_2, f_2) be complete shrinking gradient Ricci solitons. If there exists a topological end E_1 in M_1^n and a topological end E_2 in M_2^n that are asymptotic to the same Riemannian cone, then the universal covers of (M_1^n, g_1) and (M_2^n, g_2) are ***isometric****.*

We remark that any asymptotic cone $C(\Sigma^{n-1})$ has nonnegative scalar curvature and the $(n-1)$-manifold Σ^{n-1} has positive scalar curvature. In particular, if $n = 4$, then by Perelman's theorem, Σ^3 is diffeomorphic to the connected sum of spherical space forms S^3/Γ's and $S^2 \times S^1$'s.

Asymptotically conical shrinking solitons, III

Open problem: **What can prove toward a classification of 4-dimensional asymptotically conical shrinking solitons?**

In particular, if $C(\Sigma^3)$ is the asymptotic cone of a 4-dimensional asymptotically conical shrinking soliton: **What can one say about the geometry of Σ^3 besides that it has positive scalar curvature?**

For the FIK Kähler shrinker example, in dimension 4, Σ^3 is a Berger sphere with positive sectional curvature.

Are there examples of 4-dimensional asymptotically conical shrinking soliton such that any of the following is <u>not</u> true?

- Σ^3 has positive Ricci curvature (this implies that Σ^3 is diffeomorphic to S^3/Γ).
- Σ^3 has constant scalar curvature.
- Σ^3 is a locally homogeneous space (this implies constant scalar curvature).

Asymptotic geometry of shrinking solitons, I

What is the structure of a 4-dimensional shrinking soliton which does not satisfy $R(x) \to 0$ as $x \to \infty$?

One direction regarding this question is to understand the asymptotic geometry at infinity of 4-dimensional shrinking solitons. Munteanu and Wang have a number of results in this direction. Some of the results that they have proved are as follows. Let (M^4, g, f) be a 4D shrinking soliton with bounded scalar curvature.

- Then $|\mathrm{Rm}| \leq CR$ and hence is bounded.
- If $R \geq c > 0$ along a topological end E of M^4, then either:
  - g is smoothly asymptotic along E to $(S^3/\Gamma) \times \mathbb{R}$, or
  - For any integral curve γ to ∇f and any $t_i \to \infty$, the pointed sequence $(M^4, g, \gamma(t_i))$ subconverges to $S^2 \times \mathbb{R}^2$ or $(S^2 \times \mathbb{R}^2)/\mathbb{Z}_2$.

Asymptotic geometry of shrinking solitons, II

Munteanu and Wang Dichotomy Conjecture:

For any 4D shrinking soliton, either:

1. $R(x) \to 0$ as $x \to \infty$ or
2. $R \geq c$ on M^4 for some $c > 0$.

(1) Recall that in the case where $R \to 0$, Munteanu and J. Wang proved that the shrinker is asymptotically conical ($|\operatorname{Rm}|$ has quadratic decay) and in this case, the shrinker is determined up to isometry by its asymptotic cone by Kotschwar and L. Wang.

(2) Cylinders are examples with R equal to a positive constant. There is a new (Kähler) 4D shrinker by Bamler, Cifarelli, Conlon, and Deruelle on the connected sum of $\mathbb{C}P^2$ and $S^2 \times \mathbb{R}^2$, which is asymptotic at infinity to $S^2 \times \mathbb{R}^2$. Together with the work of Conlon, Deruelle, and Sun, they completed the classification of complete noncompact Kähler 4D shrinkers with bounded scalar curvature. **Are there other 4D shrinkers with** $0 < c \leq R \leq C$**?**

Estimates for steady Ricci solitons

Recall that B.-L. Chen proved that any complete steady gradient Ricci soliton must have non-negative scalar curvature.

Recall also that the steady soliton equation is $\mathrm{Ric} + \nabla^2 f = 0$. Assume that g is not Ricci flat. Then we have that $R + |\nabla f|^2 = 1$ and $R > 0$. In particular, $|\nabla f| \leq 1$. This implies that

$$f(o) - d(x, o) \leq f(x) \leq f(o) + d(x, o).$$

P. Lu, B. Yang, and C. showed that if $\lim_{x\to\infty} f = -\infty$, then there exists a constant $c > 0$ such that

$$R(x) \geq c\mathrm{e}^{f(x)} \geq c^2 \mathrm{e}^{-d(x,o)}.$$

O. Munteanu, C.-J. Sung, and J. Wang improved this estimate by replacing the hypothesis by f being bounded from above. The estimate is **sharp** in the sense that the cigar steady Ricci soliton satisfies $R(x) \approx c\mathrm{e}^{-d(x,o)}$ at infinity.

Estimates for four-dimensional steady Ricci solitons, I

For a non-Ricci-flat steady gradient Ricci soliton, Hamilton proved that:

$$R + |\nabla f|^2 = 1.$$

In particular, we have $R \leq 1$.

Michael Freedman, Henry Shin, Yongjia Zhang, and C. showed that in dimension 4 and for singularity models, one can improve this as follows.

Theorem

If (M^4, g, f) is a complete non-compact steady gradient Ricci soliton that is also a singularity model, then there exists a constant C depending on (M^4, g, f) such that

$$|\operatorname{Rm}|(x) \leq C.$$

Estimates for four-dimensional steady Ricci solitons, II

The best lower bound for the scalar curvature of a steady soliton is exponentially decaying because of the cigar soliton example. However, the cigar soliton is not non-collapsed.

Using **Bamler's (2020) theory**, P.-Y. Chan, Z. Ma, Y. Zhang, and C. proved the following.

Theorem

If (M^4, g, f) is a 4-dimensional ***steady*** *gradient Ricci soliton singularity model and if g is not Ricci-flat, then there exists a constant $c > 0$ such that*

$$R(x) \geq \frac{c}{1 + d(x,o)^2}.$$

Open problem: **Can one improve the exponent 2 to to the exponent 1 in the denominator?** Note: 1 is the exponent for the Bryant soliton and the Appleton solitons on plane bundles over S^2.

The geometry and topology at infinity of steady solitons, I: Topological ends

Regarding the topology of steady solitons, Munteanu and Wang proved the following result.

Theorem

Any complete non-compact ***steady*** *gradient Ricci soliton must either be connected at infinity (i.e., has exactly one topological end) or must split as the product of* $\mathbb{R}$ *with a compact Ricci flat manifold.*

In contrast, it is unknown in dimensions at least 4 how many ends shrinking gradient Ricci solitons can have, even in the special case of asymptotically conical shrinking solitons.

The geometry and topology at infinity of steady solitons, II: Volume growth upper bound

Firstly, Cao and Zhou proved that **shrinking** gradient Ricci solitons have at most Euclidean volume growth: There exists a constant C_n depending only on n such that

$$\operatorname{Vol} B_r(o) \leq C_n r^n,$$

where o is a minimum point of f.

Bamler proved the following:

Theorem

For any singularity model $(M^n_\infty, g_\infty(t))$ *there exists a constant* C *such that*

$$\operatorname{Vol} B_r(o) \leq C r^n.$$

Question: **What lower bounds are there for the volume growth of Ricci solitons?**

The geometry and topology at infinity of steady solitons, III: Volume growth lower bound

R. Bamler, P.-Y. Chan, Z. Ma, and Y. Zhang proved the following sharp volume growth lower bound.

Theorem

*For any **steady** gradient Ricci soliton singularity model, there exists a constant $c > 0$ such that*

$$\mathrm{Vol}\, B_r(o) \geq c r^{\frac{n+1}{2}}.$$

This estimate is qualitatively sharp because of the Bryant soliton example.

The geometry and topology at infinity of steady solitons, IV: Tangent flow at infinity

Bamler introduced the notion of **tangent flow at infinity** for ancient solutions. This represents an asymptotic limit of the ancient solution based on the conjugate heat kernel. We may think of such limits as corresponding to base points about which the heat kernel is concentrated, and roughly where the volume is largest in some sense. The tangent flow at infinity is a **shrinking soliton**.

R. Bamler, Y. Deng, Z. Ma, Y. Zhang, and C. proved the following.

Theorem

*Let $(M^4, g(t))$ be a 4D steady Ricci soliton singularity model. Then its **tangent flow at infinity** is isometric to either:*

1. *$\mathbb{R}^4/\Gamma$, where $\Gamma \neq 1$, and in this case $(M^4, g(t))$ must be a static **Ricci flat ALE**, or*
2. *$(S^3/\Gamma) \times \mathbb{R}$, $S^2 \times \mathbb{R}^2$, or $(S^2 \times \mathbb{R}^2)/\mathbb{Z}_2$.*

The geometry and topology at infinity of steady solitons, IV: Tangent flow at infinity, continued

Furthermore, for 4D steady soliton models they proved:

1. The tangent flow at infinity is **unique**.
2. If the tangent flow at infinity of the 4D steady soliton singularity model is $(S^3/\Gamma) \times \mathbb{R}$, then (M^4, g) is qualitatively asymptotic to the quotient of the 4D Bryant soliton by some Γ in the sense that outside of a compact set, the steady soliton has positive curvature operator and linear curvature decay.

What more can we say about the geometry and topology of a 4D steady soliton singularity model with an $(S^3/\Gamma) \times \mathbb{R}$ tangent flow at infinity?

So far, the only known such examples are the Appleton cohomogeneity-one steady solitons on plane bundles over S^2. **Are there other examples?**

Concluding remarks

In general, one would like to understand the high curvature regions of 4D Ricci flows. In particular (Bamler), one would like to prove a **"canonical neighborhood" theorem** for the high curvature regions of 4D Ricci flow singular solutions that is sufficient to enable the formulation of Ricci flow past singularity times with suitable control on the topology change of the 4-manifold at the singularity times.

In dimensions 2 and 3, all singularity models are noncollapsed shrinking or steady Ricci solitons. In higher dimensions, all known singularity models are Ricci solitons. A few general questions are:

- Do there exist non-soliton singularity models?
- What can one prove about the geometry and topology of 4D singularity models? For 4D shrinking & steady soliton models?

Undoubtedly, **there will be many important open questions and unsolved problems for years to come in Ricci flow**.

THANK YOU!