A Retrospective Look at Ricci Flow: Lecture 3

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Abstract

This is the third talk in the short course
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Lecture 3: Ricci Solitons

In this talk we complete the discussion of Hamilton's seminal 1982 result classifying compact 3-dimensional manifolds with positive Ricci curvature from the point of view of singularity analysis.

References:

Reference: Bennett Chow, *Ricci Solitons in Low Dimensions*, AMS, to appear.

Ricci flow singularity analysis: The idea of looking under a microscope

Let $(M^n, g(t))$, $[0, T)$, where $T \in (0, \infty)$, be a singular solution to the Ricci flow on a closed manifold. By Hamilton's long-time existence theorem, we have that

$$\sup_{(x,t)\in M^n \times [0,T)]} |\operatorname{Rm}|(x,t) = \infty.$$

We want to look under a microscope at neighborhoods of the space-time points of a sequence $\{(x_i, t_i)\}$. Let

$$K_i := |\operatorname{Rm}|(x_i, t_i) = |\operatorname{Rm}_{g(t_i)}|(x_i)$$

and assume that $K_i \to \infty$. This last condition says that we are seeing a glimpse of the forming singularity.

What understanding of the singularity formation is sufficient for (1) proving convergence as $t \to T$ or (2) defining a Ricci flow past the singularity time T?

What does it mean to look under a microscope?

Consider the metric $g(t_i)$, which is the solution to the Ricci flow at time t_i. We have that the norm of the curvature of this metric at the point x_i is equal to K_i, which tends to infinty. To rectify this defect, we consider the **rescaled metric**

$$K_i g(t_i).$$

This metric has the property that the norm of its curvature at the point x_i is equal to 1: Since

$$|\operatorname{Rm}_{K_i g(t_i)}| = K_i^{-1} |\operatorname{Rm}_{g(t_i)}|,$$

we have

$$|\operatorname{Rm}_{K_i g(t_i)}|(x_i) = K_i^{-1} |\operatorname{Rm}_{g(t_i)}|(x_i) = 1.$$

So we have geometric control of the metric $K_i g(t_i)$ **at** the point x_i.

But do we have geometric control (i.e., curvature bound) of the metrics $g(t)$ for times t near t_i and at points x near x_i?

The goal: To see a limiting shape

Recall that $K_i := |\mathrm{Rm}|(x_i, t_i)$ and the rescaled metrics are $K_i g(t_i)$. We would like to take a limit of a subsequence of $K_i g(t_i)$ to obtain a smooth Riemannian metric g_∞ **in the limit** on a manifold M^n_∞ possibly different than the original manifold M^n. The limit Riemannian manifold (M^n_∞, g_∞) is called a **singularity model** (as we will see later, it is actually only a single time-slice of a singularity model solution to Ricci flow).

How do we find suitable sequences of space-time points $\{(x_i, t_i)\}$ so that there is an associated singularity model?

Ideally, one would like to prove the existence of singularity models associated to *all space-time sequences* of points $\{(x_i, t_i)\}$, but in practice one should aim for whatever *class of sequences* is sufficient for topological applications.

Sequences of rescaled Ricci flows

For the rescaled metrics, not only do we want to see "still frames" $(M^n, K_i g(t_i))$, we also want to see the whole "movie"; i.e., instead of a single metric, a whole Ricci flow. For this reason we **rescale** the whole Ricci flow based at the time t_i as follows: Define

$$g_i(t) := K_i g(t_i + K_i^{-1} t).$$

It is easy to check that each family of metrics $g_i(t)$ is a solution to the Ricci flow on M^n and the time interval of existence of each flow is

$$[-K_i t_i, K_i(T - t_i)).$$

We have

$$g_i(0) = K_i g(t_i).$$

Observe that since $K_i \to \infty$ and $t_i \to T > 0$, we have that $-K_i t_i \to -\infty$. Thus, if we have a limit solution $g_\infty(t)$, then it is defined at least on the **ancient** time interval $(-\infty, 0]$.

Selecting sequences of points, I: Doubling time estimate

We are interested in proving the existence of singularity models. Firstly, we recall some **basic estimates**. Recall that in dimension 2, under the Ricci flow the scalar curvature satisfies the equation

$$\partial_t R = \Delta R + R^2.$$

In all dimensions, we have the following inequality:

$$\partial_t |\,\mathrm{Rm}\,| \leq \Delta |\,\mathrm{Rm}\,| + 8 |\,\mathrm{Rm}\,|^2.$$

Thus, by the maximum principle, if $|\,\mathrm{Rm}\,|(x, t_0) \leq K_0$ for all $x \in M^n$, then for all $x \in M^n$ and $t \in \left[t_0, t_0 + \frac{1}{8K_0}\right)$ we have

$$K(x,t) := |\,\mathrm{Rm}\,|(x,t) \leq \frac{K_0}{1 - 8K_0(t - t_0)}.$$

In particular, for all $x \in M^n$ and $t \in \left[t_0, t_0 + \frac{1}{16K_0}\right]$ we have

$$K(x,t) = |\,\mathrm{Rm}\,|(x,t) \leq 2K_0.$$

That is, on a sufficiently short time interval, the maximum curvature norm at most doubles.

Selecting sequences of points, II: Derivatives of curvatures estimate

Now suppose that for all $x \in M^n$ and $t \in \left[t_0, t_0 + \frac{1}{16K_0}\right]$ we have
$$K(x,t) = |\operatorname{Rm}|(x,t) \leq 2K_0.$$

By an estimate of Bando, we have:

Theorem (Bernstein-type derivatives of curvature estimates)

Under the hypotheses above, for all $k \geq 1$,
$$|\nabla^k \operatorname{Rm}|(x,t) \leq C K_0^{1+\frac{k}{2}}$$
for all $x \in M^n$ and $t \in \left[t_0 + \frac{1}{32K_0}, t_0 + \frac{1}{16K_0}\right]$.

That is, given a curvature bound for a time interval, we have derivatives of curvatures bounds for the second half of the time interval. This useful in view of the **Arzelà–Ascoli Theorem**.

Remark. W.-X. Shi proved an important **local version** of Bando's derivatives of curvatures estimates.

Selecting sequences of points, III: Types of singular solutions

One way to select sequences of points about which to rescale our solution to Ricci flow is according to the **singularity type**.

Consider a shrinking round sphere singular solution $(S^n, g(t))$, $t \in [0, T)$, where $T \in (0, \infty)$ is the time at which the sphere shrinks to a point. We have for this "model solution" that

$$(T - t)|\operatorname{Rm}|(x, t) \equiv c_n,$$

where c_n is some constant depending only on n. Motivated by this, make the following definition. A singular solution to the Ricci flow is a **Type I singular solution** if it satisfies

$$\sup_{M^n \times [0,T)]} (T - t)|\operatorname{Rm}|(x, t) < \infty.$$

We say that it is a **Type II singular solution** if

$$\sup_{M^n \times [0,T)]} (T - t)|\operatorname{Rm}|(x, t) = \infty.$$

Singularity models, I: The definition

Recall that given a singular solution $(M^n, g(t))$, $t \in [0, T)$, and a sequence of space-time points $\{(x_i, t_i)\}$, we have the rescaled solutions:

$$g_i(t) := K_i g(t_i + K_i^{-1} t), \quad t \in \big[-K_i t_i, K_i(T - t_i)\big)$$

on M^n. Since we are centering our microscope at the point x_i, we consider the **pointed** sequence of solutions of Ricci flow:

$$(M^n, g_i(t), x_i),, \quad t \in \big[-K_i t_i, K_i(T - t_i)\big).$$

Goal: **To show that there exists a subsequence such that**

$(M^n, g_i(t), x_i)$ **converges to a smooth complete limit solution** $(M^n_\infty, g_\infty(t), x_\infty)$.

$(M^n_\infty, g_\infty(t), x_\infty)$ is called a **singularity model**.

Singularity models, II: Singularities are complicated

A given singular solution to the Ricci flow $(M^n, g(t))$, $t \in [0, T)$, may have many different singularity models, either because:

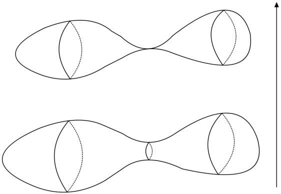
1. Singularities are forming at the time T at different locations in space.

2. The singularity forming at the time T and at a given location is complicated. This includes "bubbling" phenomena.

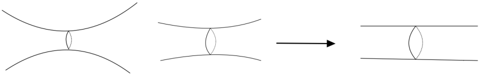
Next, we look at singularity formation at a given location in space from an **intuitive perspective**, first explained by Hamilton in his 1995 "*Formation of Singularities*" paper.

Singularity models, III: Neckpinch singularities

This figure shows a **Type I neckpinch singularity** forming at the singularity time T. The metric on the top is at the time T and the metric on the bottom is at an earlier time.



The figure below shows rescalings of the forming neckpinch, which limit to the round cylinder. The time t increases from left to right.



Singularity models, V: Round cylinder models

- Recall that **singularity models** are actually defined as the limit of rescaled solutions to the Ricci flow, not just rescaled Riemannian metrics, and the limit is itself a **solution to the Ricci flow**.
- **For the neckpinch, the associated singularity model is a shrinking round cylinder.**

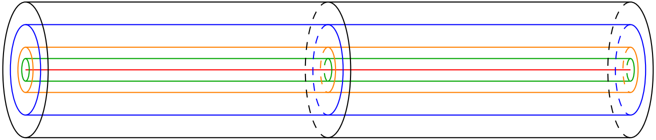
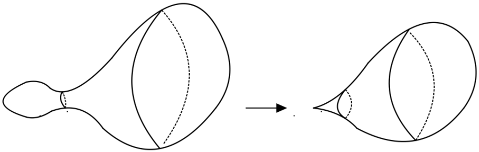


Figure: Snapshots of a shrinking $S^{n-1} \times \mathbb{R}$ cylinder, where $n \geq 3$.

Singularity models, VI: Degenerate neckpinch singularities

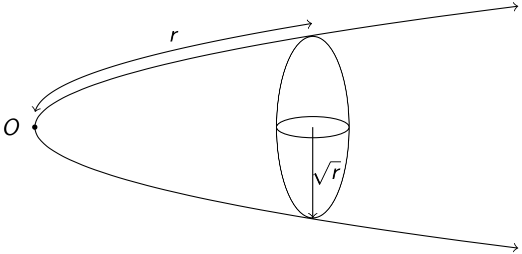
The **Type II degenerate neckpinch singularity** is similar to a neckpinch singularity except that the region to one side of the pinching neck also shrinks to a point, and a cusp-like singularity forms.



The figure shows the formation of a degenerate neckpinch singularity. The time t increases from left to right.

Singularity models, VII: Models for the degenerate neckpinch

- For a degenerate neckpinch in dimension at least 3, associated singularity models are (1) the **shrinking cylinder** and (2) the **Bryant soliton**.
- The Bryant steady gradient Ricci soliton on $\mathbb{R}^n$ has positive curvature operator and opens up like a paraboloid.



Singularity models, VIII

The Type I neckpinch and Type II degenerate neckpinch singularities are intuitive examples that can be made rigorous. In particular, Angenent and Knopf have analyzed rotationally symmetric **neckpinch** singularity formation and Gu and Zhu have proved the existence of rotationally symmetric **degenerate neckpinch** singularity formation.

However, we are still left with the question:

How do we begin to analyze singularity formation for the Ricci flow?

A key tool is the **Cheeger–Gromov compactness theorem**, which we now recall.

Cheeger–Gromov compactness theorem, I

Let $\{(M_i^n, g_i, x_i)\}$ be a sequence of pointed, complete Riemannian manifolds. Under what geometric conditions does there exist a subsequence such that (M_i^n, g_i, x_i) converges to a complete limit $(M_\infty^n, g_\infty, x_\infty)$ as $i \to \infty$?

The **Cheeger–Gromov compactness theorem** says that uniformly bounded curvatures on the M_i^n, together with uniform no local collapsing at x_i, is a sufficient condition.

Hamilton proved the Cheeger–Gromov compactness theorem for **solutions to the Ricci flow** (instead of Riemannian manifolds). In the next slide we give the statement of Hamilton's compactness theorem for Ricci flow.

Cheeger–Gromov compactness theorem, II

In 1995 Hamilton proved the following.

Theorem (Cheeger–Gromov-type compactness theorem)

Let $\{(M_i^n, g_i(t), x_i)\}$, $t \in I$, be a sequence of complete ***Ricci flows*** *defined on a common time open interval I containing 0. Suppose that (**curvature bound**)*

$$|\operatorname{Rm}_{g_i(t)}| \leq C \quad \text{on} \;\; M_i^n \times I$$

*for some constant C independent of i. Suppose also that (**no local collapsing**)*

$$\operatorname{Vol} B(x_i, 0, 1) \geq \kappa$$

for some positive constant κ independent of i.
Then there is a subsequence such that $(M_i^n, g_i(t), x_i)$ ***converges to a complete solution to the Ricci flow*** *$(M_\infty^n, g_\infty(t), x_\infty)$, $t \in I$, with curvature norm bounded by C and $\operatorname{Vol} B(x_\infty, 0, 1) \geq \kappa$.*

Perelman's no local collapsing, I: The statement

Recall that Perelman's no local collapsing says the following.

Theorem (Perelman's no local collapsing)

For any finite time compact Ricci flow $(M, (g_t)_{t \in [0,T)})$, *there exists* $\kappa > 0$ *depending only on* $g(0), T, \rho$ *such that if* (x, t) *and* $0 < r \leq \rho$ *satisfy* $R \leq r^{-2}$ *in* $B(x, t, r)$, *then*

$$\mathrm{Vol}_t\, B(x, t, r) \geq \kappa\, r^n,$$

where Vol_t *denotes the volume with respect to* $g(t)$.

If $|\,\mathrm{Rm}\,| \leq c_n r^{-2}$ for a suitable constant c_n depending only on n, then $R \leq r^{-2}$. So, if $|\,\mathrm{Rm}\,| \leq C$ on M^n, then by taking $r = \sqrt{c_n/C}$ we have that $|\,\mathrm{Rm}\,| \leq c_n r^{-2}$ on all of M^n. Thus, if a solution to the Ricci flow satisfies $|\,\mathrm{Rm}\,| \leq C$, then

$$\mathrm{Vol}_t\, B(x, t, r) \geq \kappa\, r^n.$$

Perelman's no local collapsing, II: Singularity models

One important significance of Perelman's no local collapsing theorem is that, **for singularity analysis**, it reduces the hypotheses of Hamilton's Cheeger–Gromov compactness theorem to only that of bounded curvature. We have (**singularity models**):

Theorem (Existence of singularity models)

*Let $\{(M^n, g_i(t), x_i)\}$, $(\alpha_i, 0)$, be a sequence of **rescalings** of a compact Ricci flow $(M^m, g(t))$, $t \in [0, T)$, where $\alpha_i \to -\infty$. Suppose that*

$$|\operatorname{Rm}_{g_i(t)}| \leq C \quad \text{on } M_i^n \times (\alpha_i, 0)$$

*for some constant C independent of i. Then there exists a subsequence such that $(M_i^n, g_i(t), x_i)$ **converges to a complete ancient solution to the Ricci flow** $(M_\infty^n, g_\infty(t), x_\infty)$, $t \in (-\infty, 0)$, with $|\operatorname{Rm}|$ bounded by C and $\operatorname{Vol} B(x_\infty, 0, 1) \geq \kappa$.*

Choosing sequences of points, I: The issue

To effect the Ricci flow compactness theorem, we need to answer the following question:

Question: **How big of a space-time neighborhood of the point $(x_i, 0)$ is the curvature of $g_i(t)$ uniformly bounded?**

Note that the point $(x_i, 0)$ in the solution $g_i(t)$ corresponds to the point (x_i, t_i) in the solution $g(t)$.

If we choose the sequence of space-time points $\{(x_i, t_i)\}$ in a suitable way, we can answer this question in an affirmative way.

We effect the choice depending on whether the singular solution is **Type I** or **Type II**.

Choosing sequences of points, II: Type I singular solutions

We first recall the following **curvature gap estimate** for singular solutions

Lemma

If $(M^n, g(t))$ *is a solution to the Ricci flow on a closed manifold on a maximal time interval* $[0, T)$, *where* $T < \infty$, *then*

$$(T - t) \max_{x \in M^n} |\mathrm{Rm}(x, t)| \geq \frac{1}{8}.$$

Proof. Recall that

$$\partial_t |\mathrm{Rm}| \leq \Delta |\mathrm{Rm}| + 8 |\mathrm{Rm}|^2 .$$

By the maximum principle, the quantity $K(t) := \max_{x \in M^n} |\mathrm{Rm}|(x, t)$ satisfies $\frac{dK}{dt} \leq 8K^2$. From this and the fact that $\lim_{t \to T} K(t)^{-1} = 0$, we conclude that

$$K(t)^{-1} \leq 8(T - t).$$

Choosing sequences, III: Type I singular solutions, cont.

In the Type I case, by the lemma we can choose a sequence (x_i, t_i) with $t_i \to T$ and

$$(T - t_i)\, K_i = (T - t_i)\, |\mathrm{Rm}|\, (x_i, t_i) \geq \frac{1}{8}.$$

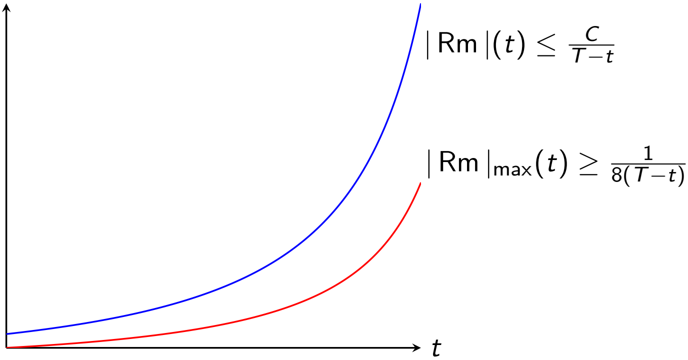
Recall that $g_i(t) = K_i g(t_i + K_i^{-1} t)$. We have the curvature estimates:

$$\begin{aligned} |\, \mathrm{Rm}_{g_i(t)}\, | &= K_i^{-1} |\, \mathrm{Rm}_{g(t_i + K_i^{-1} t)}\, | \leq K_i^{-1} \frac{C}{T - (t_i + K_i^{-1} t)} \\ &= \frac{C}{(T - t_i)\, K_i - t} \leq \frac{C}{\frac{1}{8} - t} \end{aligned}$$

for $t \in [-t_i K_i, \frac{1}{8})$. By the **existence of singularity models** theorem, a subsequence of $\{(M^n, g_i(t), x_i)\}$ converges to a **complete ancient solution** $(M_\infty^n, g_\infty(t), x_\infty)$ defined on $(-\infty, \frac{1}{8})$ with $\left|\mathrm{Rm}_{g_\infty(t)}\right| \leq \frac{C}{\frac{1}{8} - t}$. This is our singularity model.

Choosing sequences, IV: Type I singular solutions, cont.

The following diagram shows why the sequence selection method works for Type I singular solutions.



Choosing sequences of points, V: Type II singular solutions

In the Type II case, we first choose any sequence of times $T_i \nearrow T$. Then we "pretend" that T_i is the final time and choose points and times $(x_i, t_i) \in M^n \times [0, T_i]$ such that

$$(T_i - t_i) \,|\, \mathrm{Rm} \,|\, (x_i, t_i) = \max_{M^n \times [0, T_i]} (T_i - t) \,|\, \mathrm{Rm} \,|\, (x, t) \,.$$

Again let $K_i := |\, \mathrm{Rm} \,|\, (x_i, t_i)$ and again define:

$$g_i\,(t) := K_i g(t_i + K_i^{-1} t).$$

Then

$$|\, \mathrm{Rm}_{\,g_i}|\,(x, t) = K_i^{-1} |\, \mathrm{Rm}_{\,g}|(x, t_i + K_i^{-1} t) \leq \frac{(T_i - t_i)\, K_i}{(T_i - t_i)\, K_i - t}$$

for all $x \in M^n$ and $t \in [-t_i K_i, (T_i - t_i)\, K_i)$ and $(T_i - t_i)\, K_i \to \infty$.

Choosing sequences, VI: Type II singular solutions, cont.

Recall that the rescaled solutions $g_i(t) := K_i g(t_i + K_i^{-1} t)$ satisfy

$$|\operatorname{Rm}|_{g_i}(x,t) \leq \frac{(T_i - t_i) K_i}{(T_i - t_i) K_i - t}$$

for all $x \in M^n$ and $t \in [-t_i K_i, (T_i - t_i) K_i)$ and $(T_i - t_i) K_i \to \infty$. By the singularity model existence theorem, we obtain a **singularity model** $(M_\infty^n, g_\infty(t), x_\infty)$ defined on the **eternal** time interval $(-\infty, \infty)$ as a complete limit with bounded curvature and such that

$$1 = |\operatorname{Rm}_{g_\infty}|(x_\infty, 0) = \sup_{M_\infty \times (-\infty,\infty)} |\operatorname{Rm}_{g_\infty}|(x,t).$$

Convergence of 3-manifolds with positive Ricci curvature, I

Let $(M^3, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed 3-manifold with $g(0)$ having positive Ricci curvature and defined on a maximal time interval. Since $R_{g(0)} > 0$ and $\partial_t R \geq \Delta R + \frac{2}{3} R^2$, we have that $0 < T < \infty$ (maximum time is **finite**). So, by definition, $g(t)$ is either a Type I or a Type II singular solution. In either case, we obtain subconvergence to a complete ancient solution $(M^3_\infty, g_\infty(t), x_\infty)$ defined at least on the time interval $(-\infty, \frac{1}{8})$ (possibly $(-\infty, \infty)$) and satisfying $|\operatorname{Rm}_{g_\infty}| \leq \frac{C}{\frac{1}{8} - t}$ in the Type I case and $|\operatorname{Rm}_{g_\infty}| \leq C$ in the Type II case. By the way we have rescaled the sequence, we have that

$$|\operatorname{Rm}_{g_\infty}|(x_\infty, 0) = 1.$$

By the strong maximum principle, we have that the scalar curvature of the limit is positive: $R_{g_\infty} > 0$ on M^3_∞.

Convergence of 3-manifolds with positive Ricci curvature, II

Now, by using the "Ricci pinching improves" estimate for the solution $(M^3, g(t))$, $t \in [0, T)$, to the Ricci flow at the times t_i:

$$\frac{\left|\operatorname{Ric} - \frac{1}{3}Rg\right|}{R}(t_i) \leq CR^{-\delta}(t_i),$$

we can show that the singularity model (M^3_∞, g_∞) satisfies

$$\left|\operatorname{Ric}_{g_\infty} - \tfrac{1}{3}R_{g_\infty} g_\infty\right| \equiv 0.$$

This is because, by the Cheeger–Gromov convergence of $g_i := g_i(0)$ to $g_\infty := g_\infty(0)$, we have for i large that on $U_i \subset M^3_\infty$,

$$\frac{\left|\operatorname{Ric}_{g_\infty} - \frac{1}{3}R_{g_\infty} g_\infty\right|}{R_{g_\infty}} \overset{\text{CG}}{\approx} \frac{\left|\operatorname{Ric}_{g_i} - \frac{1}{3}R_{g_i} g_i\right|}{R_{g_i}} \leq CK_i^{-\delta} R_{g_i}^{-\delta} \to 0$$

(since $g_i = K_i g(t_i)$ implies that $R_{g_i} = K_i^{-1} R(t_i)$). Here, $\{U_i\}_{i=1}^{\infty}$ is an exhaustion of M^3_∞ by open subsets.

Convergence of 3-manifolds with positive Ricci curvature, II

Recall from the previous slide that the singularity model (M^3_∞, g_∞) satisfies

$$\mathrm{Ric}_{g_\infty} = \tfrac{1}{3} R_{g_\infty} g_\infty.$$

By the contracted second Bianchi identity, this implies that R_{g_∞} is a positive constant on M^3_∞. Indeed,

$$\frac{1}{2}\nabla_{g_\infty} R_{g_\infty} = \mathrm{div}(\mathrm{Ric}_{g_\infty}) = \tfrac{1}{3}\,\mathrm{div}(R_{g_\infty} g_\infty) = \frac{1}{3}\nabla_{g_\infty} R_{g_\infty},$$

so that $\nabla_{g_\infty} R_{g_\infty} \equiv 0$. Hence the Ricci curvature of g_∞ is a positive constant, which in turn implies that the sectional curvature of g_∞ is a positive constant since we are in dimension 3. Since the metric g_∞ is complete, we conclude that M^3_∞ **is diffeomorphic to a spherical space form, and hence so is** M^3.

THANK YOU!