Hamilton's Ricci Flow

Bennett Chow

Peng Lu

Lei Ni

Chapter 3

Closed 3-manifolds with Positive Ricci Curvature

In this chapter we present a proof of Hamilton's classification of closed 3-manifolds with positive Ricci curvature. The idea of the proof of Hamilton's theorem is to get estimates for various geometric quantities associated to the evolving metric, such as the curvature and its derivatives, which will show the metric limits to a constant positive sectional curvature metric. In Section 2 we consider the maximum principle for 2-tensors, which we shall apply to the Ricci tensor to get pointwise estimates for its pinching. The maximum principle for systems enables us to estimate the curvatures by a suitable analysis of this ODE system, which we carry out in Section 3. Then we discuss the gradient of the scalar curvature estimate, which unlike the pointwise pinching estimates for curvature, allows us to compare curvatures at different points. Finally we state the exponential convergence results for closed 3-manifolds with positive Ricci curvature and present their proof.

1. Hamilton's 3-manifolds with positive Ricci curvature theorem

Now we present the theorem which started Ricci flow. First note that by (2.11) the **evolution of the volume form** is given by

$$\frac{\partial}{\partial t} d\mu = -R d\mu \tag{3.1}$$

and $\operatorname{Vol}(g) \doteqdot \int_{M^n} d\mu$ evolves by

$$\frac{d}{dt}\operatorname{Vol}(g(t)) = -\int_M R d\mu. \tag{3.2}$$

Since this is not zero in general, we modify (normalize) the Ricci flow equation to make the volume constant. In particular, we define the **normalized Ricci flow** by

$$\frac{\partial}{\partial t}\tilde{g}_{ij} = -2\tilde{R}_{ij} + \frac{2}{n}\tilde{r}\tilde{g}_{ij}, \tag{3.3}$$

where $\tilde{r} = \operatorname{Vol}(\tilde{g})^{-1} \cdot \int_{M^n} \tilde{R}\, d\tilde{\mu}$ is the **average scalar curvature**. We then have (again use (2.11))

$$\frac{d}{dt}\operatorname{Vol}(\tilde{g}(t)) = 0 \tag{3.4}$$

under the normalized Ricci flow. Given a solution $g(t)$, $t \in [0, T)$, of the Ricci flow, the metrics

$$\tilde{g}\left(\tilde{t}\right) \doteqdot c(t)\, g(t), \tag{3.5}$$

where

$$c(t) \doteqdot \exp\left(\frac{2}{n}\int_0^t r(\tau)\, d\tau\right), \qquad \tilde{t}(t) \doteqdot \int_0^t c(\tau)\, d\tau, \tag{3.6}$$

are a solution of the normalized Ricci flow with $\tilde{g}(0) = g(0)$. Hence solutions of the normalized Ricci flow differ from solutions of the Ricci flow only by rescalings in space and time.

Exercise 3.1. *Prove that $\tilde{g}\left(\tilde{t}\right)$ is a solution of the normalized Ricci flow.*

The remainder of this chapter will be devoted to proving the following.

Theorem 3.2 (Hamilton, 3-manifolds with positive Ricci curvature)**.** *Let (M^3, g_0) be a closed Riemannian 3-manifold with positive Ricci curvature. Then there exists a unique solution $g(t)$ of the normalized Ricci flow with $g(0) = g_0$ for all $t \geq 0$. Furthermore, as $t \to \infty$, the metrics $g(t)$ converge exponentially fast in every C^m-norm to a C^∞ metric g_∞ with constant positive sectional curvature.*

2. The maximum principle for tensors

Since the Ricci and Riemann curvature tensors satisfy heat-type equations, just as the scalar curvature does, one can apply the maximum principle, which we develop in this section, to derive estimates. Given that a symmetric 2-tensor satisfies a heat-type equation, we would like to know when the nonnegativity of the 2-tensor is preserved as time evolves. A result in this direction is provided by Hamilton's **maximum principle for tensors**. A word about notation: if α is a symmetric 2-tensor, then $\alpha \geq 0$ means that α is nonnegative definite.

Theorem 3.3 (Maximum principle for symmetric 2-tensors)**.** *Let $g(t)$ be a smooth 1-parameter family of Riemannian metrics on a closed manifold M^n. Let $\alpha(t)$ be a symmetric 2-tensor satisfying*

$$\frac{\partial}{\partial t}\alpha \geq \Delta_{g(t)}\alpha + \nabla_{X(t)}\alpha + \beta,$$

where $X(t)$ is a time-dependent vector field and

$$\beta(x,t) = \beta(\alpha(x,t), g(x,t))$$

is a symmetric $(2,0)$-tensor which is locally Lipschitz in all its arguments. Suppose β satisfies the ***null-eigenvector assumption*** *that if A_{ij} is a nonnegative symmetric 2-tensor at a point (x,t) and if V is such that $A_{ij}V^j = 0$, then*

$$\beta_{ij}(A,g)V^iV^j \geq 0.$$

If $\alpha(0) \geq 0$, then $\alpha(t) \geq 0$ for all $t \geq 0$ as long as the solution exists.

Proof. Suppose that (x_1, t_1) is a point where there exists a vector V such that $\left(\alpha_{ij}V^j\right)(x_1,t_1) = 0$ for the first time (so $\left(\alpha_{ij}W^iW^j\right)(x,t) \geq 0$ for all W, $x \in M$, and $t \leq t_1$). Choose V to be constant in time. We then have at (x_1, t_1)

$$\begin{aligned}\frac{\partial}{\partial t}\left(\alpha_{ij}V^iV^j\right) &= \left(\frac{\partial}{\partial t}\alpha_{ij}\right)V^iV^j \\ &\geq (\Delta\alpha_{ij})V^iV^j + X^k(\nabla_k\alpha_{ij})V^iV^j + \beta_{ij}V^iV^j \\ &\geq (\Delta\alpha_{ij})V^iV^j + X^k(\nabla_k\alpha_{ij})V^iV^j.\end{aligned}$$

To handle the last line, we extend V in a neighborhood of x_1 by parallel translating it along geodesics (with respect to the metric $g(t_1)$) emanating from x_1. It is easy to see that $\nabla V(x_1,t_1) = 0$ and it can also be shown that $\Delta V(x_1,t_1) = 0$. Thus we have

$$\frac{\partial}{\partial t}\left(\alpha_{ij}V^iV^j\right) \geq \Delta\left(\alpha_{ij}V^iV^j\right) + X^k\nabla_k\left(\alpha_{ij}V^iV^j\right) \geq 0$$

by the first and second derivative tests. This shows that when α attains a zero eigenvalue for the first time, it wants to increase in the direction of any corresponding zero eigenvector. We can make the above argument rigorous by adding in an $\varepsilon > 0$ just as for the scalar maximum principle in Section 3 of Chapter 2. In particular, we can show that there exists $\delta > 0$ such that $\alpha \geq 0$ on $[0,\delta]$ by applying the above argument to the symmetric 2-tensor

$$A_\varepsilon(t) \doteqdot \alpha(t) + \varepsilon(\delta + t)g(t)$$

for $\varepsilon > 0$ sufficiently small and then letting $\varepsilon \to 0$. Tracking the dependence of δ on α, we find that on any compact time interval I we may apply this argument again to show that $\alpha \geq 0$ on $[\delta, 2\delta] \cap I$. Continuing this way, we conclude that $\alpha \geq 0$ on all of I. □

Exercise 3.4. *Complete the proof of Theorem 3.3.*

Recall that the evolution of the Ricci tensor is given by (2.43)

$$\frac{\partial}{\partial t} R_{ij} = \Delta_L R_{ij} = \Delta R_{ij} + 2R_{kij\ell} R_{k\ell} - 2R_{ik} R_{jk}.$$

So in order to prove that the **nonnegativity of the Ricci tensor is preserved** under the Ricci flow, all we need to do is to show that at any point and time where $R_{ij} W^i W^j \geq 0$ for all W and $R_{ij} V^j = 0$ for some V, we have

$$\left(R_{kij\ell} R_{k\ell} - R_{ik} R_{jk} \right) V^i V^j \geq 0.$$

Unfortunately, when $n \geq 4$, this is not possible in general. The main reason for this is that the Riemann curvature tensor cannot be recovered solely from the Ricci tensor (indeed, this is why the Weyl tensor does not vanish in general when $n \geq 4$). The exception to this is when $n = 3$, in which case

$$R_{ijk\ell} = R_{i\ell} g_{jk} + R_{jk} g_{i\ell} - R_{ik} g_{j\ell} - R_{j\ell} g_{ik} - \frac{R}{2} \left(g_{i\ell} g_{jk} - g_{ik} g_{j\ell} \right).$$

Substituting this into (2.43), we obtain the following.

Lemma 3.5 (3d evolution of Ricci)**.** *If $n = 3$, then under the Ricci flow we have*

$$\boxed{\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + 3R R_{ij} - 6R_{ip} R_{jp} + \left(2 |\mathrm{Rc}|^2 - R^2 \right) g_{ij}}. \tag{3.7}$$

Exercise 3.6. *Prove (3.7).*

By the maximum principle for tensors we have

Corollary 3.7 (Nonnegative Ricci is preserved)**.** *If $\left(M^3, g(t) \right)$, $t \in [0, T)$, is a solution to the Ricci flow on a closed 3-manifold with $\mathrm{Rc}(g(0)) \geq 0$, then $\mathrm{Rc}(g(t)) \geq 0$ for all $t \geq 0$ as long as the solution exists.*

Proof. We easily check that the tensor

$$\beta_{ij} = 3R R_{ij} - 6R_{ip} R_{jp} + \left(2 |\mathrm{Rc}|^2 - R^2 \right) g_{ij}$$

satisfies the null-eigenvector assumption with respect to $\alpha_{ij} = R_{ij}$. In particular, if at a point and time Rc has a null-eigenvector V (we do not need $\mathrm{Rc} \geq 0$ here), then $2 |\mathrm{Rc}|^2 - R^2 \geq 0$ and

$$\beta_{ij} V^i V^j = \left(2 |\mathrm{Rc}|^2 - R^2 \right) |V|^2 \geq 0.$$

□

Exercise 3.8 (Preservation of Ricci pinching)**.**

(1) *Show that nonnegative sectional curvature* $\frac{1}{2}Rg_{ij} - R_{ij} \geq 0$ *is preserved under the Ricci flow on a closed* 3*-manifold.*

(2) *Show that if* $R > 0$*, then the inequality* $R_{ij} \geq \varepsilon R g_{ij}$ *is preserved for any* $\varepsilon \geq 0$ *(of course,* $\varepsilon \leq 1/3$*).*

3. Curvature pinching estimates

Now that we know that the nonnegativity of the Ricci tensor is preserved under the Ricci flow on closed 3-manifolds, we are interested in a more precise understanding of the Ricci tensor as the metric evolves. A useful tool is the maximum principle for tensors as discussed in Section 2. This principle has been abstracted to the following setting.

Recall that Rm is a section of the bundle $\pi : E \to M$, where $E \doteqdot \Lambda^2 M^n \otimes_S \Lambda^2 M^n$. Throughout this book, Rm will sometimes be considered as a section of E and other times as an operator $\mathrm{Rm} : \Lambda^2 M^n \to \Lambda^2 M^n$. The bundle E has a natural bundle metric and connection induced by the Riemannian metric and connection on TM. Let $E_x \doteqdot \pi^{-1}(x)$ be the fiber over x. For each $x \in M$, consider the system of ODE on E_x corresponding to the PDE (2.70) obtained by dropping the Laplacian term:

$$\boxed{\frac{d}{dt}\mathbf{M} = \mathbf{M}^2 + \mathbf{M}^{\#}}, \tag{3.8}$$

where $\mathbf{M} \in E_x$ is a symmetric $N \times N$ matrix, where $N = \frac{n(n-1)}{2} = \dim \mathfrak{so}(n)$. The **maximum principle for systems** (for a proof of a more general version which applies to sections of vector bundles satisfying heat-type equations, see §4 of [277] or [153]) says the following. A set K in a vector space is said to be **convex** if for any $X, Y \in K$, we have $sX + (1-s)Y \in K$ for all $s \in [0,1]$. A subset K of the vector bundle E is said to be **invariant under parallel translation** if for every path $\gamma : [a,b] \to M$ and vector $X \in K \cap E_{\gamma(a)}$, the unique parallel section $X(s) \in E_{\gamma(s)}$, $s \in [a,b]$, along $\gamma(s)$ with $X(a) = X$ is contained in K.

Theorem 3.9 (Maximum principle applied to the curvature operator)**.** *Let* $g(t)$, $t \in [0,T)$, *be a solution to the Ricci flow on a closed manifold* M^n. *Let* $K \subset E$ *be a subset which is invariant under parallel translation and whose intersection* $K_x \doteqdot K \cap E_x$ *with each fiber is closed and convex. Suppose the* ODE *(3.8) has the property that for any* $\mathbf{M}(0) \in K$, *we have* $\mathbf{M}(t) \in K$ *for all* $t \in [0,T)$. *If* $\mathrm{Rm}(0) \in K$, *then* $\mathrm{Rm}(t) \in K$ *for all* $t \in [0,T)$.

Since if $\mathrm{Rm} \geq 0$, then $\mathrm{Rm}^2 \geq 0$ and $\mathrm{Rm}^{\#} \geq 0$ (see Lemma 2.57), by (2.70) and the above theorem, we have the following.

Corollary 3.10 (Rm ≥ 0 is preserved)**.** *If* $(M^n, g(t))$, $t \in [0,T)$, *is a solution to the Ricci flow on a closed manifold with* $\mathrm{Rm}(g(0)) \geq 0$, *then* $\mathrm{Rm}(g(t)) \geq 0$ *for all* $t \in [0,T)$.

In dimension 3, if $\mathbf{M}(0)$ is diagonal, then $\mathbf{M}(t)$ remains diagonal. Let $\lambda_1(\mathbf{M}) \leq \lambda_2(\mathbf{M}) \leq \lambda_3(\mathbf{M})$ be the eigenvalues of $\mathbf{M}$. Under the ODE the ordering of the eigenvalues is preserved and we have by (2.74)

$$\begin{aligned} \frac{d\lambda_1}{dt} &= \lambda_1^2 + \lambda_2\lambda_3, \\ \frac{d\lambda_2}{dt} &= \lambda_2^2 + \lambda_1\lambda_3, \\ \frac{d\lambda_3}{dt} &= \lambda_3^2 + \lambda_1\lambda_2. \end{aligned} \tag{3.9}$$

With this setup, we can come up with a number of closed, fiberwise convex sets K, invariant under parallel translation, which are preserved by the ODE. Each such set corresponds to an *a priori* estimate for the curvature Rm.

The following sets $K \subset E$ are invariant under parallel translation and for each $x \in M$, K_x is closed, convex and preserved by the ODE (3.9).

(1) Given $C_0 \in \mathbb{R}$, let $K = \{\mathbf{M} : \lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) + \lambda_3(\mathbf{M}) \geq C_0\}$. The trace $\lambda_1 + \lambda_2 + \lambda_3 : E_x \to \mathbb{R}$ is a linear function, which implies that K is closed and convex. That K is preserved by the ODE (3.9) follows from

$$\begin{aligned} \frac{d}{dt}(\lambda_1 + \lambda_2 + \lambda_3) &= \frac{1}{2}\left[(\lambda_1+\lambda_2)^2 + (\lambda_1+\lambda_3)^2 + (\lambda_2+\lambda_3)^2\right] \\ &\geq \frac{2}{3}(\lambda_1+\lambda_2+\lambda_3)^2 \geq 0. \end{aligned} \tag{3.10}$$

(We interjected the first inequality since we will find it useful later.) Hence,

(**lower bound of scalar is preserved**) if $R \geq C_0$ at $t = 0$ for some $C_0 \in \mathbb{R}$, then

$$\boxed{R \geq C_0} \tag{3.11}$$

for all $t \geq 0$. This is something we have already seen in Corollary 2.11.

(2) Let $K = \{\mathbf{M} : \lambda_1(\mathbf{M}) \geq 0\}$. Each K_x is closed and convex since $\lambda_1 : E_x \to \mathbb{R}$ is a concave function. Indeed, $\lambda_1(\mathbf{M}) = \min_{|V|=1} \mathbf{M}(V,V)$ so that

$$\lambda_1(s\mathbf{M}_1 + (1-s)\mathbf{M}_2) \geq s\lambda_1(\mathbf{M}_1) + (1-s)\lambda_1(\mathbf{M}_2)$$

for all $s \in [0,1]$. We see that K is preserved by the ODE since

$$\frac{d\lambda_1}{dt} = \lambda_1^2 + \lambda_2\lambda_3 \geq 0$$

whenever $\lambda_1 \geq 0$. That is, if $\mathbf{M}(t)$ is a solution of the ODE (3.9) with $\lambda_1(\mathbf{M}(0)) \geq 0$, then $\lambda_1(\mathbf{M}(t)) \geq 0$ for all $t \geq 0$. This implies (this is a special case of Corollary 3.10)

(**nonnegative sectional curvature is preserved**) the condition

$$\boxed{\mathrm{Rm} \geq 0} \tag{3.12}$$

is preserved under the Ricci flow. Since any 2-form on a 3-manifold is the wedge product of two 1-forms, this is equivalent to the sectional curvature being nonnegative.

(3) Let $K = \{\mathbf{M} : \lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) \geq 0\}$. Since $\lambda_1 + \lambda_2$ is concave,

$$(\lambda_1 + \lambda_2)(\mathbf{M}) = \min\{\mathbf{M}(V_1, V_1) + \mathbf{M}(V_2, V_2) : \{V_1, V_2\} \text{ orthonormal}\},$$

we have that K is closed and convex. We compute

$$\frac{d}{dt}(\lambda_1 + \lambda_2) = \lambda_1^2 + \lambda_2^2 + (\lambda_1 + \lambda_2)\lambda_3 \geq 0$$

whenever $\lambda_1 + \lambda_2 \geq 0$. From this we see that

(**nonnegative Ricci is preserved**)

$$\boxed{\mathrm{Rc} \geq 0} \tag{3.13}$$

is preserved under the Ricci flow since the smallest eigenvalue of Rc is $\frac{1}{2}[\lambda_1(\mathrm{Rm}) + \lambda_2(\mathrm{Rm})]$.

(4) Given $C \geq 1/2$, let

$$K = \{\mathbf{M} : \lambda_3(\mathbf{M}) \leq C(\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}))\}.$$

Since λ_3 is convex ($\lambda_3(\mathbf{M}) = \max_{|V|=1}\mathbf{M}(V,V)$) and $\lambda_1 + \lambda_2$ is concave, we have that K_x is convex for all $x \in M$. That each K_x is preserved by the ODE follows from the calculation

$$\frac{d}{dt}[\lambda_3 - C(\lambda_1 + \lambda_2)] = \lambda_3(\lambda_3 - C(\lambda_1 + \lambda_2)) - C\left(\lambda_1^2 - \frac{1}{C}\lambda_1\lambda_2 + \lambda_2^2\right).$$

In particular, if $\lambda_3 - C(\lambda_1 + \lambda_2) = 0$ and $C \geq 1/2$, then

$$\frac{d}{dt}[\lambda_3 - C(\lambda_1 + \lambda_2)] \leq 0.$$

Suppose $\mathrm{Rc}(g(0)) > 0$. Since M^3 is compact, there exists $C \geq 1/2$ such that at $t = 0$

$$\lambda_3(\mathrm{Rm}) \leq C(\lambda_1(\mathrm{Rm}) + \lambda_2(\mathrm{Rm})). \tag{3.14}$$

That is, $\operatorname{Rm}(g(0)) \subset K$. By the maximum principle for tensors, $\operatorname{Rm}(g(t)) \subset K$ and inequality (3.14) is true for all $t \geq 0$. Now (3.14) implies $\operatorname{Rc} \geq C^{-1}\lambda_3(\operatorname{Rm})\, g \geq \frac{1}{3}C^{-1}Rg$. Thus,

(**Ricci pinching is preserved**) the conditions that $R > 0$ and

$$\boxed{\operatorname{Rc} \geq \varepsilon Rg \quad (n=3)} \tag{3.15}$$

for some constant $\varepsilon > 0$ are preserved under the Ricci flow. In particular, since M^3 is compact, we have $\operatorname{Rc} > 0$ is preserved. (Compare with Exercise 3.8.)

Remark. Note that if (3.14) is satisfied and if

$$(\lambda_1(\operatorname{Rm}) + \lambda_2(\operatorname{Rm}))(x_0, t_0) < 0$$

for some $(x_0, t_0) \in M^3 \times [0, T)$, then since $C \geq 1/2$, we have $\lambda_3(\operatorname{Rm}) = \lambda_1(\operatorname{Rm}) = \lambda_2(\operatorname{Rm})$ at (x_0, t_0). Since $\lambda_1 + \lambda_2 < 0$ holds on a connected neighborhood U of x_0 at time t_0, we have $\lambda_1 = \lambda_2 = \lambda_3 = \frac{R}{3}$ in U (recall that the λ_i are twice the sectional curvatures). By the contracted Bianchi identity, we then have that R is constant on U. Since M^3 is connected, it is easy to conclude that

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{R}{3}$$

on all of M, where the scalar curvature R is a negative constant. Thus, if $\operatorname{Rm}(g(t_0)) \subset K$ for some t_0 and if $g(t_0)$ does not have constant negative sectional curvature, then $\operatorname{Rc} \geq 0$.

(5) Given $C_0 > 0$, $C_1 \geq 1/2$, $C_2 < \infty$ and $\delta > 0$, let

$$K = \left\{ \mathbf{M} : \begin{array}{c} \lambda_3(\mathbf{M}) - \lambda_1(\mathbf{M}) - C_2(\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) + \lambda_3(\mathbf{M}))^{1-\delta} \leq 0, \\ \lambda_3(\mathbf{M}) \leq C_1(\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M})), \\ \lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) + \lambda_3(\mathbf{M}) \geq C_0 \end{array} \right\}.$$

K is a convex set since $\lambda_3 - \lambda_1 - C_2(\lambda_1 + \lambda_2 + \lambda_3)^{1-\delta}$ is a convex function for $C_2 > 0$. Observe that if $\mathbf{M} \in K$, then $\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) > 0$ by the last two inequalities in the definition of K. We have already seen that the inequalities $\lambda_1 + \lambda_2 + \lambda_3 \geq C_0$ and $\lambda_3 \leq C_1(\lambda_1 + \lambda_2)$ are preserved under the ODE. Since $C_0 > 0$, we can compute

$$\frac{d}{dt}\log\left(\frac{\lambda_3-\lambda_1}{(\lambda_1+\lambda_2+\lambda_3)^{1-\delta}}\right)$$
$$=\delta(\lambda_1+\lambda_3-\lambda_2)-(1-\delta)\frac{(\lambda_1+\lambda_2)\lambda_2+(\lambda_2-\lambda_1)\lambda_3+\lambda_2^2}{\lambda_1+\lambda_2+\lambda_3}$$
$$\leq\delta(\lambda_1+\lambda_3-\lambda_2)-(1-\delta)\frac{\lambda_2^2}{\lambda_1+\lambda_2+\lambda_3}.$$

Note that

$$\frac{\lambda_2^2}{\lambda_1+\lambda_2+\lambda_3}\geq\frac{1}{6}\frac{(\lambda_1+\lambda_2)\lambda_2}{\lambda_3}\geq\frac{1}{6C_1}\lambda_2$$

since $\lambda_2+\lambda_3\leq 2\lambda_3\leq 2C_1(\lambda_1+\lambda_2)$, and we also have

$$\lambda_1+\lambda_3-\lambda_2\leq\lambda_3\leq C_1(\lambda_1+\lambda_2)\leq 2C_1\lambda_2.$$

Hence, choosing $\delta>0$ small enough so that $\frac{\delta}{1-\delta}\leq\frac{1}{12C_1^2}$, we have

$$\frac{d}{dt}\log\left(\frac{\lambda_3-\lambda_1}{(\lambda_1+\lambda_2+\lambda_3)^{1-\delta}}\right)\leq 0.$$

Since $\lambda_3-\lambda_1\geq\left|\operatorname{Rc}-\frac{1}{3}Rg\right|$, this implies the following:

(**Ricci pinching improves**) Suppose M^3 is closed and g_0 has positive Ricci curvature. There exist constants $C<\infty$ and $\delta>0$ such that

$$\boxed{\left|\operatorname{Rc}-\frac{1}{3}Rg\right|\leq CR^{1-\delta}\quad(n=3)}\,. \tag{3.16}$$

Remark 3.11. *The 2-tensor* $\operatorname{Rc}-\frac{1}{3}Rg$ *is the trace-free part of the Ricci tensor and*

$$\left|\operatorname{Rc}-\frac{1}{3}Rg\right|^2=|\operatorname{Rc}|^2-\frac{1}{3}R^2,$$

which vanishes everywhere exactly when g *is Einstein.*

Exercise 3.12 (3d trace-free part of Rc and Rm)**.** *Show that when* $n=3$,

$$\left|\operatorname{Rm}-\frac{1}{3}R\operatorname{Id}_{\Lambda^2}\right|^2=4\left|\operatorname{Rc}-\frac{1}{3}Rg\right|^2. \tag{3.17}$$

In summary, the main estimates we have proved for the curvatures are (3.11), (3.15) and (3.16).

Let $[0,T)$ denote the maximum time interval of existence of our solution. Recall that from applying the maximum principle to the evolution equation for scalar curvature (2.9), we have

$$R_{\min}(t) \geq \frac{1}{R_{\min}(0)^{-1} - \frac{2}{3}t}.$$

Since $R_{\min}(0) > 0$, we conclude $T \leq \frac{3}{2}R_{\min}(0)^{-1} < \infty$. In Section 5 we shall prove that

$$\sup_{M\times[0,T)} |\mathrm{Rm}| = \infty. \tag{3.18}$$

Intuitively speaking, we are in good shape now. Since the Ricci curvature is positive, the metric is shrinking: $\frac{\partial}{\partial t}g = -2\,\mathrm{Rc} < 0$. If we can show an appropriate gradient estimate for the scalar curvature, then we could conclude $\lim_{t\to T} R_{\min}(t) = \infty$. Assuming this, we then would have

$$\left|\frac{\mathrm{Rc}}{R} - \frac{1}{3}g\right|(x,t) \leq CR^{-\delta}(x,t),$$

which tends to 0 as $t \to T$ uniformly in x. To finish the proof of Theorem 3.2, we need to further show that the solution $\tilde{g}(\tilde{t})$ to the corresponding normalized Ricci flow exists for all time and the scale invariant quantity $\left|\frac{\widetilde{\mathrm{Rc}}}{\tilde{R}} - \frac{1}{3}\tilde{g}\right|$ decays exponentially to zero as $\tilde{t} \to \infty$. See the next section for the gradient of scalar curvature estimate. After that, we shall show that under the normalized Ricci flow the curvature tends to a constant. Finally we prove the long time existence and exponential convergence of the solution to a constant sectional curvature metric.

Remark 3.13 ($S^2 \times S^1$ example)**.** *It is instructive to keep in mind the example of the round product $S^2 \times S^1$ which has nonnegative Ricci curvature but not positive Ricci curvature. Under the Ricci flow the metric remains a round product. If the initial S^2 has radius r_0, then the radius at time t is $r(t) = \sqrt{r_0^2 - 2t}$. The radius of the circle S^1 remains constant since the Ricci curvature in the circle direction is zero. Note that at any point and time the curvature operator takes the form*

$$\mathrm{Rm} = \begin{pmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular,

$$\left|\mathrm{Rm} - \frac{1}{3}R\,\mathrm{Id}_{\Lambda^2}\right|^2 = \frac{2}{3}R^2.$$

4. Gradient bounds for the scalar curvature

A fundamental estimate used in combination with the 'Ricci pinching improves' estimate is the **gradient estimate for the scalar curvature**. In

Section 6 of Chapter 5 we shall give a proof of the sequential convergence of the Ricci flow to constant sectional curvature which bypasses this estimate; so the reader who is only interested in the topological consequences of Theorem 3.2 may skip this section.

Proposition 3.14 (Scalar curvature gradient estimate)**.** *Let $\left(M^3, g(0)\right)$ be a closed 3-manifold with positive Ricci curvature. For any $\varepsilon > 0$, there exists $C(\varepsilon)$ depending only on ε and $g(0)$ such that*

$$|\nabla R|^2 (x,t) \leq \varepsilon R(x,t)^3 + C(\varepsilon)$$

as long as the solution exists.

Remark 3.15. *We present the original proof in [**275**]. For a different proof using the Cheeger-Gromov-type compactness theorem for the Ricci flow, see Proposition 7.4; there the result is valid in all dimensions assuming that a pinching estimate for the Riemann curvature tensor holds for the solution.*

The rest of this section is devoted to proving the above gradient estimate. First we need a consequence of the contracted second Bianchi identity. Decomposing the 3-tensor $\nabla_i R_{jk}$ into its irreducible components, we let

$$\nabla_i R_{jk} \doteqdot E_{ijk} + F_{ijk},$$

where

$$E_{ijk} \doteqdot \frac{1}{20}\left(\nabla_j R g_{ik} + \nabla_k R g_{ij}\right) + \frac{3}{10}\nabla_i R g_{jk}. \tag{3.19}$$

Then $\langle E_{ijk}, F_{ijk}\rangle = 0$, $|E_{ijk}|^2 = \frac{7}{20}|\nabla_i R|^2$ and

$$|\nabla_i R_{jk}|^2 = |E_{ijk}|^2 + |F_{ijk}|^2 \geq \frac{7}{20}|\nabla_i R|^2 . \tag{3.20}$$

This estimate is better than the more elementary $|\nabla_i R_{jk}|^2 \geq \frac{1}{3}|\nabla_i R|^2$ which follows from the general estimate $|a_{ij}|^2 \geq \frac{1}{n}\left(g^{ij}a_{ij}\right)^2$ and $n = 3$ (see [**312**] for its generalization to higher dimensions).

Exercise 3.16 (Quicker proof with a worse constant)**.** *By using the inequality*

$$\left|\nabla_i R_{jk} - \frac{1}{3}\nabla_i R g_{jk}\right|^2 \geq \frac{1}{3}\left|\operatorname{div}\left(\operatorname{Rc} - \frac{1}{3}Rg\right)\right|^2$$

and the contracted second Bianchi identity, show that

$$|\nabla_i R_{jk}|^2 \geq \frac{37}{108}|\nabla_i R|^2 ,$$

which is weaker than (3.20).

In computing evolution equations for various quantities, the following general calculation is useful.

Exercise 3.17. *Let* $\square \doteqdot \frac{\partial}{\partial t} - \Delta$. *Prove that if* f *and* h *are functions of space and time and if* $p, q \in \mathbb{R}$, *then*

$$\begin{aligned} \square \left(\frac{f^p}{h^q} \right) &= p \frac{f^{p-1}}{h^q} \square f - q \frac{f^p}{h^{q+1}} \square h \\ &\quad - p(p-1) \frac{f^{p-2}}{h^q} |\nabla f|^2 - q(q+1) \frac{f^p}{h^{q+2}} |\nabla h|^2 \\ &\quad + 2pq \frac{f^{p-1}}{h^{q+1}} \langle \nabla f, \nabla h \rangle . \end{aligned} \tag{3.21}$$

In particular, taking $p = q = 1$, *we obtain*

$$\square \left(\frac{f}{h} \right) = \frac{1}{h} \square f - \frac{f}{h^2} \square h + \frac{2}{h} \left\langle \nabla h, \nabla \left(\frac{f}{h} \right) \right\rangle . \tag{3.22}$$

So that the reader knows that he or she needs to fill in some details, we give the following.

Exercise 3.18. *Prove (3.23) and (3.25) below.*

Now we present the proof of the gradient of scalar curvature estimate. We compute

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) \left(\frac{|\nabla R|^2}{R} \right) &= -\frac{2}{R^3} |R \nabla_i \nabla_j R - \nabla_i R \nabla_j R|^2 \\ &\quad + \frac{4}{R} \left\langle \nabla R, \nabla |\mathrm{Rc}|^2 \right\rangle - 2 \frac{|\mathrm{Rc}|^2}{R^2} |\nabla R|^2 \\ &\leq 16 |\nabla \mathrm{Rc}|^2 - 2 \frac{|\mathrm{Rc}|^2}{R^2} |\nabla R|^2 , \end{aligned} \tag{3.23}$$

where we used $|\mathrm{Rc}| \leq R$ and $|\nabla R| \leq \sqrt{3} |\nabla \mathrm{Rc}| \leq 2 |\nabla \mathrm{Rc}|$. Since

$$\left(\frac{\partial}{\partial t} - \Delta \right) (R^2) = -2 |\nabla R|^2 + 4R |\mathrm{Rc}|^2 ,$$

and $|\mathrm{Rc}|^2 \geq \frac{1}{3} R^2$, this implies that for any $\varepsilon \leq 1/3$, we have

$$\left(\frac{\partial}{\partial t} - \Delta \right) \left(\frac{|\nabla R|^2}{R} - \varepsilon R^2 \right) \leq 16 |\nabla_i R_{jk}|^2 - \frac{4}{3} \varepsilon R^3 .$$

To handle the bad (positive) $|\nabla_i R_{jk}|^2$ term on the RHS, we bring in the equation

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right) = -2\left(|\nabla_i R_{jk}|^2 - \frac{1}{3}|\nabla_i R|^2\right) \tag{3.24}$$

$$- 2R^3 + \frac{26}{3}R\,|\mathrm{Rc}|^2 - 8\,\mathrm{Trace}_g\left(\mathrm{Rc}^3\right)$$

$$\leq -\frac{2}{21}|\nabla_i R_{jk}|^2 + 4R\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right), \tag{3.25}$$

which has a good $|\nabla_i R_{jk}|^2$ term on the RHS. Note that (3.20) was used to obtain the last inequality. Combining the above formulas yields

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|\nabla R|^2}{R} - \varepsilon R^2 + 168\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right)\right) \leq C(\varepsilon),$$

where we use the fact that there exist $\delta > 0$ and $C < \infty$ such that $|\mathrm{Rc}|^2 - \frac{1}{3}R^2 \leq CR^{2-\delta}$ (3.16). Since the solution exists only for a finite time, we have

$$\frac{|\nabla R|^2}{R} - \varepsilon R^2 + 168\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right) \leq C.$$

Dropping the positive $168\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right)$ term (which was only included to produce good negative terms on the RHS of the evolution equation), we obtain

$$\frac{|\nabla R|^2}{R} \leq C + \varepsilon R^2.$$

Proposition 3.14 follows easily.

The paths toward obtaining various estimates are often/usually not unique. Here is a variation on the above theme.

Exercise 3.19 (∇R estimate again)**.** *Let $\left(M^3, g(0)\right)$ be a closed 3-manifold with positive Ricci curvature. Prove the following variant of the gradient of scalar curvature estimate. There exists a constant $\delta > 0$ depending only on $g(0)$ such that for any $\beta > 0$*

$$\frac{|\nabla R|^2}{R^3} \leq \beta R^{-\delta} + CR^{-3}, \tag{3.26}$$

where $C < \infty$ depends only on β and $g(0)$.

HINT: Let

$$V \doteqdot \frac{|\nabla R|^2}{R} + \frac{37}{2}\left(8\sqrt{3}+1\right)\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right)$$

and show that

$$\begin{aligned}\frac{\partial}{\partial t}V &\leq \Delta V - |\nabla \operatorname{Rc}|^2 + \frac{7400\sqrt{3}+925}{3} R\left(|\operatorname{Rc}|^2 - \frac{1}{3}R^2\right) \\ &\leq \Delta V - |\nabla \operatorname{Rc}|^2 + CR^{3-2\delta},\end{aligned}$$

where we used (3.16) to get the last inequality. Then use the equation

$$\frac{\partial}{\partial t}R^{2-\delta} = \Delta\left(R^{2-\delta}\right) - (2-\delta)(1-\delta)R^{-\delta}|\nabla R|^2 + 2(2-\delta)R^{1-\delta}|\operatorname{Rc}|^2$$

to derive

$$\frac{\partial}{\partial t}\left(V - \beta R^{2-\delta}\right) \leq \Delta\left(V - \beta R^{2-\delta}\right) + C,$$

where C depends only on β and $g(0)$. Estimate (3.26) follows from this.

Exercise 3.20. *Show that*

$$F \doteqdot \frac{|\operatorname{Rc}|^2 - \frac{1}{3}R^2}{R^{2-\delta}}$$

satisfies

$$\begin{aligned}\frac{\partial F}{\partial t} &= \Delta F + \frac{2(1-\delta)}{R}\langle \nabla R, \nabla F\rangle - \frac{2}{R^{4-\delta}}\left|R\nabla_i R_{jk} - \nabla_i R \cdot R_{jk}\right|^2 \\ &\quad - \frac{\delta(1-\delta)}{R^{4-\delta}}\left(|\operatorname{Rc}|^2 - \frac{1}{3}R^2\right)|\nabla R|^2 \\ &\quad + \frac{2}{R^{3-\delta}}\left(\delta|\operatorname{Rc}|^2\left(|\operatorname{Rc}|^2 - \frac{1}{3}R^2\right) - J\right),\end{aligned} \tag{3.27}$$

where $J \doteqdot 2|\operatorname{Rc}|^4 + R\left(R^3 - 5R|\operatorname{Rc}|^2 + 4\operatorname{Tr}_g\left(\operatorname{Rc}^3\right)\right)$.

Remark 3.21. *The above formula leads to another proof of (3.16). In particular, the key to proving that F is uniformly bounded is to show that if* $\operatorname{Rc} \geq \varepsilon R g$ *and* $R > 0$, *then*

$$J \geq 2\varepsilon^2|\operatorname{Rc}|^2\left(|\operatorname{Rc}|^2 - \frac{1}{3}R^2\right).$$

We leave it to the reader to verify this.

5. Curvature tends to constant

In this section we apply the gradient estimate and the Bonnet-Myers theorem to show that the global pinching of the scalar curvature tends to 1 as we approach the singularity time.

Lemma 3.22 (Global scalar curvature pinching)**.** *We have*

$$\lim_{t\to T}\frac{R_{\max}(t)}{R_{\min}(t)} = 1. \tag{3.28}$$

In fact, there exist constants $C < \infty$ and $\gamma > 0$ depending only on $g(0)$ such that

$$\frac{R_{\min}(t)}{R_{\max}(t)} \geq 1 - CR_{\max}(t)^{-\gamma} \tag{3.29}$$

for all $t \in [0, T)$.

Remark 3.23. *By Theorem 6.3 (whose proof is independent of our discussion here), we have $\lim_{t\to T} \max_{M^n} |\mathrm{Rm}(\cdot, t)| = \infty$. Since $n = 3$ and $\mathrm{Rc} > 0$, we have (1.62) and $R \geq |\mathrm{Rc}|$; hence $\lim_{t\to T} R_{\max}(t) = \infty$. This and (3.29) imply (3.28) and*

$$\lim_{t\to T} R_{\min}(t) = \infty.$$

Proof. By (3.26), $\inf_{M^3\times[0,T)} R > 0$, and $\lim_{t\to T} R_{\max}(t) = \infty$, there exist constants $C < \infty$ and $\delta > 0$ such that

$$|\nabla R(x, t)| \leq CR_{\max}(t)^{3/2-\delta}$$

for all $x \in M^3$ and $t \in [0, T)$. Given $t \in [0, T)$, there exists $x_t \in M^3$ such that $R_{\max}(t) = R(x_t, t)$. Given $\eta > 0$, to be chosen sufficiently small later, for any point $x \in B_{g(t)}\left(x_t, \frac{1}{\eta\sqrt{R_{\max}(t)}}\right)$ we have

$$R_{\max}(t) - R(x, t) \leq \frac{1}{\eta\sqrt{R_{\max}(t)}} \max_{M^3} |\nabla R(t)| \leq \frac{C}{\eta} R_{\max}(t)^{1-\delta}$$

so that

$$R(x, t) \geq R_{\max}(t)\left(1 - \frac{C}{\eta} R_{\max}(t)^{-\delta}\right) \tag{3.30}$$

for all $x \in B_{g(t)}\left(x_t, \frac{1}{\eta\sqrt{R_{\max}(t)}}\right)$. We claim that this ball is all of M^3, from which (3.29) follows. The argument goes like this. Since $\lim_{t\to T} R_{\max}(t) = \infty$, by (3.30), there exists $\tau < T$ such that for $t \in [\tau, T)$ we have

$$R(x, t) \geq R_{\max}(t)(1 - \eta)$$

for all $x \in B_{g(t)}\left(x_t, \frac{1}{\eta\sqrt{R_{\max}(t)}}\right)$. Now the Bonnet-Myers Theorem 1.127 and the pinching estimate $\mathrm{Rc} \geq \varepsilon R g$, where $\varepsilon > 0$, show that for $\eta > 0$ sufficiently small $M^3 = B_{g(t)}\left(x_t, \frac{1}{\eta\sqrt{R_{\max}(t)}}\right)$. □

Lemma 3.24 (Global sectional curvature pinching). *For every $\varepsilon \in (0, 1)$, there exists $\tau < T$ such that for all $t \in [\tau, T)$ the sectional curvatures of $g(t)$ are positive and*

$$\min_{x\in M^3} \lambda_1(\mathrm{Rm})(x, t) \geq (1 - \varepsilon) \max_{x\in M^3} \lambda_3(\mathrm{Rm})(x, t).$$

Proof. By (3.16), there exist $C < \infty$ and $\delta > 0$ such that

$$\frac{\lambda_1 (\mathrm{Rm})}{\lambda_3 (\mathrm{Rm})} (x,t) \geq 1 - C \frac{R^{1-\delta}}{\lambda_3 (\mathrm{Rm})} (x,t) \geq 1 - 3C R_{\min} (t)^{-\delta}$$

for all $x \in M^3$ and $t \in [0,T)$. We leave it as an exercise to the reader to show that this implies that for any $\varepsilon > 0$, there exists $\tau < T$ such that for any $x, y \in M^3$ and $t \in [\tau, T)$ we have

$$\lambda_1 (\mathrm{Rm}) (x,t) \geq (1-\varepsilon) \lambda_3 (\mathrm{Rm}) (y,t) . \tag{3.31}$$

The idea for deriving the comparison (3.31) is to make an intermediate comparison with the scalar curvature of the point in question, whether it be x or y. □

Recall that the Rauch-Klingenberg-Berger topological sphere Theorem 1.153 implies that if N^3 is simply connected and g on N^3 satisfies

$$\min_{x \in N^3} \lambda_1 (\mathrm{Rm}) (x) > \frac{1}{4} \max_{x \in N^3} \lambda_3 (\mathrm{Rm}) (x) ,$$

then N^3 is diffeomorphic to the 3-sphere. Hence the above lemma implies that the universal cover $\tilde{M}^3$ is diffeomorphic to the 3-sphere.

6. Exponential convergence of the normalized flow

Now we revert back to the normalized flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{3} r g_{ij}.$$

This flow is equivalent to the original Ricci flow by rescaling time and space (i.e., the metrics); see Section 1. It is useful to know how evolution equations change upon normalizing the Ricci flow. We say that a tensor quantity X depending on the metric g has **degree** k in g if $X(cg) = c^k X(g)$ for any $c > 0$.

Exercise 3.25 (Degrees of tensors)**.** *Show that the* $(4,0)$*-tensor* Rm *has degree* 1, Rc *has degree* 0, R *has degree* -1, *and* $d\mu$ *has degree* $n/2$ *(if* $\dim M = n$*).*

The following result concerning the normalized Ricci flow introduced in Section 1 is not hard to prove.

Lemma 3.26 (Going from unnormalized to normalized RF)**.** *If an expression* $X = X(g)$ *formed algebraically from the metric and the Riemann curvature tensor by contractions has degree* k *and if under the Ricci flow*

$$\frac{\partial X}{\partial t} = \Delta X + Y, \tag{3.32}$$

then the degree of Y is $k-1$ and the evolution under the normalized Ricci flow $\frac{\partial}{\partial \tilde{t}}\tilde{g}_{ij} = -2\tilde{R}_{ij} + \frac{2}{n}\tilde{r}\tilde{g}_{ij}$ *of* $\tilde{X} \doteqdot X(\tilde{g})$ *is given by*

$$\frac{\partial \tilde{X}}{\partial \tilde{t}} = \tilde{\Delta}\tilde{X} + \tilde{Y} + k\frac{2}{n}\tilde{r}\tilde{X}. \tag{3.33}$$

Remark 3.27. *The above lemma also holds when the equalities in (3.32) and (3.33) are replaced by inequalities going the same way.*

Exercise 3.28. *Prove Lemma 3.26.*

Next we study the maximum and average scalar curvatures under the Ricci flow.

Lemma 3.29.

$$R_{\max}(t) \geq \frac{1}{2(T-t)}$$

and in particular,

$$\int_0^T R_{\max}(t)\, dt = \infty.$$

Proof. First recall that $\lim_{t\to T} R_{\max}(t) = \infty$. We compute (using Section 6 in Appendix B)

$$\frac{d}{dt}R_{\max}(t) \leq 2 \max_{M\times\{t\}} |\mathrm{Rc}|^2 \leq 2R_{\max}(t)^2.$$

Hence $\frac{d}{dt}R_{\max}(t)^{-1} \geq -2$ so that $-R_{\max}(t)^{-1} \geq -2(T-t)$ and the lemma follows. □

Exercise 3.30. *Show that if* $[0,\tilde{T})$ *is the maximal time interval of existence of the normalized Ricci flow, then* $\int_0^{\tilde{T}} \tilde{r}(\tilde{t})\, d\tilde{t} = \infty$.

HINT: Show that

$$\int_0^{\tilde{t}_0} \tilde{r}(\tilde{t})\, d\tilde{t} = \int_0^{t_0} r(t)\, dt.$$

Each of the following estimates represents in some way the fact that under the normalized Ricci flow the metrics converge to constant curvature exponentially fast. The order in which they are stated reflects a natural order in which they are proved.

Lemma 3.31 (Estimates for the normalized RF)**.** *If* $(M^3, g(0))$ *is a closed 3-manifold with positive Ricci curvature, then under the normalized Ricci flow we have the following estimates. Let* $[0,\tilde{T})$ *denote the maximal time interval of existence of the normalized Ricci flow. There exist constants* $C<\infty$ *and* $\delta>0$ *such that*

(1)
$$\lim_{\tilde{t}\to\tilde{T}} \frac{\tilde{R}_{\max}\left(\tilde{t}\right)}{\tilde{R}_{\min}\left(\tilde{t}\right)} = 1,$$
(2)
$$\widetilde{\mathrm{Rc}} \geq \delta \tilde{R}\tilde{g},$$
(3)
$$\tilde{R}_{\max}\left(\tilde{t}\right) \leq C,$$
(4)
$$\tilde{T} = \infty,$$
(5)
$$\tilde{R}_{\min}\left(\tilde{t}\right) \geq \frac{1}{C},$$
and hence $\operatorname{diam}\left(\tilde{g}\left(t\right)\right) \leq C$,

(6)
$$\left|\widetilde{\mathrm{Rc}} - \frac{1}{3}\tilde{R}\tilde{g}\right| \leq Ce^{-\delta\tilde{t}},$$
(7)
$$\tilde{R}_{\max}\left(\tilde{t}\right) - \tilde{R}_{\min}\left(\tilde{t}\right) \leq Ce^{-\delta\tilde{t}},$$
(8)
$$\left|\widetilde{\mathrm{Rc}} - \frac{1}{3}\tilde{r}\tilde{g}\right| \leq Ce^{-\delta\tilde{t}}. \tag{3.34}$$

Proof. Parts (1) and (2) follow from the corresponding estimates for the unnormalized Ricci flow since the inequalities are scale-invariant.

Part (3): Since $\tilde{R}_{ij} \geq 0$, by the Bishop-Gromov volume comparison theorem, we have $\text{const} = \operatorname{Vol}\left(\tilde{g}\left(\tilde{t}\right)\right) \leq C \operatorname{diam}\left(\tilde{g}\left(\tilde{t}\right)\right)^3$ for a universal constant C. Now since $\widetilde{\mathrm{Rc}} \geq \varepsilon \tilde{R}_{\max} \cdot \tilde{g}$ for some $\varepsilon > 0$ (combine (1) and (2)), by the Bonnet-Myers theorem, we have

$$\operatorname{diam}\left(\tilde{g}\left(\tilde{t}\right)\right) \leq C\tilde{R}_{\max}\left(\tilde{t}\right)^{-1/2} \tag{3.35}$$

and we conclude $\tilde{R}_{\max}\left(\tilde{t}\right) \leq C$.

Part (4) is left as an exercise. Use $\int_0^{\tilde{T}} \tilde{r}\left(\tilde{t}\right) d\tilde{t} = \infty$.

Part (5): By Klingenberg's injectivity radius estimate (see Theorem 1.115) and replacing $\left(M^3, \tilde{g}\left(\tilde{t}\right)\right)$ by their universal covering Riemannian manifolds $\left(\tilde{M}^3, \tilde{\tilde{g}}\left(\tilde{t}\right)\right)$, we have

$$\operatorname{inj}\left(\tilde{\tilde{g}}\left(\tilde{t}\right)\right) \geq \varepsilon \tilde{R}_{\max}\left(\tilde{t}\right)^{-1/2}$$

for some universal constant $\varepsilon > 0$. Since $\operatorname{sect}\left(\widetilde{\tilde{g}}\left(\tilde{t}\right)\right) \leq C\tilde{R}_{\max}\left(\tilde{t}\right)$, this implies $\operatorname{Vol}\left(\widetilde{\tilde{g}}\left(\tilde{t}\right)\right) \geq \varepsilon\tilde{R}_{\max}\left(\tilde{t}\right)^{-3/2}$ for some other constant $\varepsilon > 0$. Hence we have

$$\text{const} = \operatorname{Vol}\left(\tilde{g}\left(\tilde{t}\right)\right) \geq \delta'\tilde{R}_{\max}\left(\tilde{t}\right)^{-3/2},$$

where $\delta' > 0$ depends also on $\left|\pi_1\left(M^3\right)\right| < \infty$. Hence $\tilde{R}_{\max}\left(\tilde{t}\right) \geq \frac{1}{C}$ and the same estimate holds for $\tilde{R}_{\min}\left(\tilde{t}\right)$ by (1). Now by (3.35) we also have a uniform upper bound for the diameter of $\tilde{g}\left(\tilde{t}\right)$.

Part (6): Let

$$\tilde{f} \doteqdot \frac{\left|\widetilde{\mathrm{Rc}} - \frac{1}{3}\tilde{R}\tilde{g}\right|^2}{\tilde{R}^2}.$$

$\tilde{f}$ satisfies the same equation as for its counterpart $f \doteqdot \left|\mathrm{Rc} - \frac{1}{3}Rg\right|^2 \big/ R^2$ for the unnormalized flow. This equation is the following (see Exercise 3.33 below):

$$\frac{\partial f}{\partial t} = \Delta f + 2\left\langle \nabla \log R, \nabla f\right\rangle - \frac{2}{R^4}\left|R\nabla_i R_{jk} - \nabla_i R R_{jk}\right|^2 + 4P, \tag{3.36}$$

where

$$P \doteqdot \frac{1}{R^3}\left(\frac{5}{2}R^2\left|\mathrm{Rc}\right|^2 - 2R\operatorname{Trace}_g\left(\mathrm{Rc}^3\right) - \frac{1}{2}R^4 - \left|\mathrm{Rc}\right|^4\right).$$

This is actually the same P as in (10.82), where now $v_{ij} = R_{ij}$ and $\rho = 0$. Note that when $\mathrm{Rc} = \frac{1}{3}Rg$, we have $P = 0$. One can show that if $R_{ij} \geq \varepsilon R g_{ij}$, where $R > 0$ and $\varepsilon \geq 0$, then

$$P \leq -\varepsilon^2 \frac{\left|\mathrm{Rc} - \frac{1}{3}Rg\right|^2}{R}; \tag{3.37}$$

see Exercise 3.34 below. Hence we have

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial \tilde{t}} &\leq \tilde{\Delta}\tilde{f} + 2\left\langle \tilde{\nabla}\log\tilde{R}, \tilde{\nabla} f\right\rangle - 4\varepsilon^2\frac{\left|\widetilde{\mathrm{Rc}} - \frac{1}{3}\tilde{R}\tilde{g}\right|^2}{\tilde{R}} \\ &\leq \tilde{\Delta}\tilde{f} + 2\left\langle \tilde{\nabla}\log\tilde{R}, \tilde{\nabla} f\right\rangle - \delta\left(\tilde{R}_{\min}\right)\tilde{f}.\end{aligned} \tag{3.38}$$

The desired exponential decay estimate for $\tilde{f}$ now follows from the maximum principle to (3.38) using $\tilde{R}_{\min} \geq \frac{1}{C} > 0$.

Part (7): We go back to (3.23) and (3.25). Adding these two equations implies that

$$\psi \doteqdot \frac{\left|\nabla R\right|^2}{R} + 168\left(\left|\mathrm{Rc}\right|^2 - \frac{1}{3}R^2\right)$$

satisfies, under the unnormalized Ricci flow,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\psi \leq 672R\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right).$$

Hence, for the normalized Ricci flow, the corresponding quantity $\tilde{\psi}$ satisfies

$$\left(\frac{\partial}{\partial \tilde{t}} - \tilde{\Delta}\right)\tilde{\psi} \leq Ce^{-\delta\tilde{t}} - \frac{4}{3}\tilde{r}\tilde{\psi},$$

where we used $672\tilde{R}\left(\left|\widetilde{\mathrm{Rc}}\right|^2 - \frac{1}{3}\tilde{R}^2\right) \leq Ce^{-\delta\tilde{t}}$. Since $\frac{4}{3}\tilde{r} \geq \delta_1$ for some $\delta_1 \in (0, \delta]$, we can conclude that

$$\left(\frac{\partial}{\partial \tilde{t}} - \tilde{\Delta}\right)\left(e^{\delta_1\tilde{t}}\tilde{\psi} - C\tilde{t}\right) \leq 0$$

and hence $\tilde{\psi} \leq Ce^{-\delta_1\tilde{t}}\left(1+\tilde{t}\right)$. This gives us the gradient estimate $\left|\tilde{\nabla}\tilde{R}\right| \leq Ce^{-\frac{49}{100}\delta_1\tilde{t}}$. Since the diameters of $\tilde{g}\left(\tilde{t}\right)$ are uniformly bounded, we obtain (7) by integrating the gradient estimate along minimal geodesics.

Part (8) follows from (6) and (7). □

Exercise 3.32. *Prove part (4) of Lemma 3.31.*

Exercise 3.33. *Prove (3.36).*

Hint: Use (3.25), (3.21) and the evolution equation for R.

Exercise 3.34. *Prove (3.37).*

Hint: First show that

$$\begin{aligned} R^3P &= -\left(a^4+b^4+c^4+abc^2+ab^2c+a^2bc\right) \\ &\quad + ab^3+a^3b+ac^3+a^3c+bc^3+b^3c \\ &= -(a-b)^2\left(a^2+(a+b)(b-c)\right) - c^2(a-c)(b-c), \end{aligned}$$

where a, b, c denote the eigenvalues of Rc.

By (3.34) and Lemma 6.10, we obtain the following.

Corollary 3.35. *There exists a constant $C < \infty$ such that*

$$\frac{1}{C}\tilde{g}(0) \leq \tilde{g}\left(\tilde{t}\right) \leq C\tilde{g}(0)$$

for all $\tilde{t} \in [0, \infty)$, and the metrics $\tilde{g}\left(\tilde{t}\right)$ converge uniformly on compact sets to a continuous metric $\tilde{g}(\infty)$ as $\tilde{t} \to \infty$.

Next we study the higher derivatives of the curvature. We shall use the following interpolation inequality for tensors (see Corollaries 12.6 and 12.7 of [275]).

Lemma 3.36. *For any $k \in \mathbb{N}$ there exists a constant C depending only on k and n such that for any p-tensor $\alpha_{i_1 \cdots i_p}$*

(1)
$$\int_M \left|\nabla^j \alpha\right|^{2k/j} d\mu \le C \max_M |\alpha|^{2\left(\frac{k}{j}-1\right)} \int_M \left|\nabla^k \alpha\right|^2 d\mu$$
for any $j = 1, \ldots, k-1$, and

(2)
$$\int_M \left|\nabla^j \alpha\right|^2 d\mu \le C \left(\int_M \left|\nabla^k \alpha\right|^2 d\mu\right)^{j/k} \left(\int_M |\alpha|^2 \, d\mu\right)^{1-(j/k)}$$
for any $j = 0, \ldots, k$.

Under the normalized Ricci flow, the higher derivatives of the Ricci curvature decay exponentially.

Lemma 3.37.

$$\left|\tilde{\nabla}^k \widetilde{\mathrm{Rc}}\right| \le C e^{-\delta \tilde{t}} \tag{3.39}$$

for all $k \in \mathbb{N}$.

Remark 3.38. *Since we are in dimension 3, the statement with $\widetilde{\mathrm{Rc}}$ replaced by $\widetilde{\mathrm{Rm}}$ is exactly the same.*

Proof. We first show that under the unnormalized Ricci flow

(3.40)
$$\frac{d}{dt}\int_M \left|\nabla^k \mathrm{Rm}\right|^2 d\mu \le -2\int_M \left|\nabla^{k+1} \mathrm{Rm}\right|^2 d\mu + C \max_M |\mathrm{Rm}| \int_M \left|\nabla^k \mathrm{Rm}\right|^2 d\mu$$

for some constant $C < \infty$. To prove this, by Exercise 6.29 we have

$$\frac{\partial}{\partial t}\left|\nabla^k \mathrm{Rm}\right|^2 \le \Delta \left|\nabla^k \mathrm{Rm}\right|^2 - 2\left|\nabla^{k+1} \mathrm{Rm}\right|^2 + \sum_{\ell=0}^{k} \left|\nabla^\ell \mathrm{Rm}\right| \left|\nabla^{k-\ell} \mathrm{Rm}\right| \left|\nabla^k \mathrm{Rm}\right|.$$

Thus we compute

$$\begin{aligned}
&\frac{d}{dt}\int_M \left|\nabla^k \mathrm{Rm}\right|^2 d\mu + 2\int_M \left|\nabla^{k+1} \mathrm{Rm}\right|^2 d\mu \\
&\le \sum_{\ell=0}^{k} \int_M \left|\nabla^\ell \mathrm{Rm}\right| \left|\nabla^{k-\ell} \mathrm{Rm}\right| \left|\nabla^k \mathrm{Rm}\right| d\mu \\
&\le \sum_{\ell=0}^{k} \left(\int_M \left|\nabla^\ell \mathrm{Rm}\right|^{\frac{2k}{\ell}} d\mu\right)^{\frac{\ell}{2k}} \left(\int_M \left|\nabla^{k-\ell} \mathrm{Rm}\right|^{\frac{2k}{k-\ell}} d\mu\right)^{\frac{k-\ell}{2k}} \qquad (3.41) \\
&\qquad \times \left(\int_M \left|\nabla^k \mathrm{Rm}\right|^2 d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 3.36(1), we have

$$\left(\int_M \left|\nabla^\ell \operatorname{Rm}\right|^{\frac{2k}{\ell}} d\mu\right)^{\frac{\ell}{2k}} \leq C \max_M |\operatorname{Rm}|^{\left(1-\frac{\ell}{k}\right)} \left(\int_M \left|\nabla^k \operatorname{Rm}\right|^2 d\mu\right)^{\frac{\ell}{2k}}$$

$$\left(\int_M \left|\nabla^{k-\ell} \operatorname{Rm}\right|^{\frac{2k}{k-\ell}} d\mu\right)^{\frac{k-\ell}{2k}} \leq C \max_M |\operatorname{Rm}|^{\frac{\ell}{k}} \left(\int_M \left|\nabla^k \operatorname{Rm}\right|^2 d\mu\right)^{\frac{k-\ell}{2k}}.$$

Applying these inequalities to (3.41) implies (3.40).

Now for the normalized Ricci flow, since all the terms in (3.40) have the same degree of homogeneity, we have the same estimate:

$$\frac{d}{d\tilde{t}} \int_M \left|\tilde{\nabla}^k \widetilde{\operatorname{Rc}}\right|^2 d\tilde{\mu} \leq -2 \int_M \left|\tilde{\nabla}^{k+1} \widetilde{\operatorname{Rc}}\right|^2 d\tilde{\mu} + C \int_M \left|\tilde{\nabla}^k \widetilde{\operatorname{Rc}}\right|^2 d\tilde{\mu}, \tag{3.42}$$

where we used $\max_M \left|\widetilde{\operatorname{Rc}}\right| \leq C$ and dimension 3 to replace $\widetilde{\operatorname{Rm}}$ by $\widetilde{\operatorname{Rc}}$. We apply Lemma 3.36(2) to get

$$\int_M \left|\tilde{\nabla}^k \widetilde{\operatorname{Rc}}\right|^2 d\mu \leq C \left(\int_M \left|\tilde{\nabla}^{k+1} \widetilde{\operatorname{Rc}}\right|^2 d\mu\right)^{\frac{k}{k+1}} \left(\int_M \left|\widetilde{\operatorname{Rc}} - \frac{\tilde{r}}{3}\tilde{g}\right|^2 d\mu\right)^{\frac{1}{k+1}}$$

using the fact that $\left|\tilde{\nabla}^m \widetilde{\operatorname{Rc}}\right|^2 = \left|\tilde{\nabla}^m \left(\widetilde{\operatorname{Rc}} - \frac{\tilde{r}}{3}\tilde{g}\right)\right|^2$ for $m \in \mathbb{N}$. Hence

$$\begin{aligned}\frac{d}{d\tilde{t}} \int_M \left|\tilde{\nabla}^k \widetilde{\operatorname{Rc}}\right|^2 d\tilde{\mu} &\leq C \left(\int_M \left|\widetilde{\operatorname{Rc}} - \frac{\tilde{r}}{3}\tilde{g}\right|^2 d\mu\right)^{\frac{1}{k+1}} \\ &\leq C e^{-\delta \tilde{t}}\end{aligned}$$

and we have

$$\int_M \left|\tilde{\nabla}^k \widetilde{\operatorname{Rc}}\right|^2 d\tilde{\mu} \leq C$$

independent of $\tilde{t}$. We can now apply Lemma 3.36(1) again to obtain

$$\int_M \left|\tilde{\nabla}^j \widetilde{\operatorname{Rc}}\right|^{2k/j} d\tilde{\mu} \leq C \max_M \left|\widetilde{\operatorname{Rc}} - \frac{\tilde{r}}{3}\tilde{g}\right|^{2\left(\frac{k}{j}-1\right)} \int_M \left|\tilde{\nabla}^k \widetilde{\operatorname{Rc}}\right|^2 d\tilde{\mu}$$

for any $j = 1, \ldots, k-1$. Hence, given any $j, p \in \mathbb{N}$, we may choose $k = pj$ to conclude

$$\begin{aligned}\int_M \left|\tilde{\nabla}^j \widetilde{\operatorname{Rc}}\right|^{2p} d\tilde{\mu} &\leq C \max_M \left|\widetilde{\operatorname{Rc}} - \frac{\tilde{r}}{3}\tilde{g}\right|^{2(p-1)} \int_M \left|\tilde{\nabla}^{pj} \widetilde{\operatorname{Rc}}\right|^2 d\tilde{\mu} \\ &\leq C e^{-\delta \tilde{t}}\end{aligned}$$

for some $C < \infty$ and $\delta > 0$. Since all of the metrics $\tilde{g}\left(\tilde{t}\right)$ are uniformly equivalent for $t \in [0, \infty)$, the Sobolev constant is uniformly bounded and it follows that for any $k \in \mathbb{N}$, there exists $C < \infty$ and $\delta > 0$ such that

$$\left|\tilde{\nabla}^k \widetilde{\mathrm{Rc}}\right| \leq C e^{-\delta \tilde{t}}.$$

□

From the above lemma and the fact that we can estimate the derivatives of the metrics in terms of the estimates for the derivatives of the Ricci tensor, one can complete the proof of Theorem 3.2.

Completion of the proof of Theorem 3.2. By Corollary 3.35 the metrics $\tilde{g}\left(\tilde{t}\right)$ are uniformly equivalent and converge uniformly on compact sets to a continuous metric $\tilde{g}\left(\infty\right)$ as $\tilde{t} \to \infty$. On the other hand, the estimates (3.39) imply the exponential convergence in each C^k-norm of $\tilde{g}\left(\tilde{t}\right)$ to $\tilde{g}\left(\infty\right)$. This implies $\tilde{g}\left(\infty\right)$ is C^∞. Furthermore, by (3.34) we conclude

$$\left|\widetilde{\mathrm{Rc}} - \frac{1}{3}\tilde{r}\tilde{g}\right|\left(\infty\right) \equiv 0.$$

That is, $\tilde{g}\left(\infty\right)$ has constant positive sectional curvature. □

7. Notes and commentary

The original proof of Theorem 3.2 is given in Hamilton [275]. Some of the solutions of the exercises in this chapter may be found in the text of [163].

Section 3. There is an interesting question related to (3.15).

Conjecture 3.39 (Hamilton)**.** *If $\left(M^3, g\right)$ is a complete Riemannian 3-manifold with* $\mathrm{Rc} \geq \varepsilon R g$, *where* $R > 0$ *and* $\varepsilon > 0$, *then* M^3 *is compact.*

Chen and Zhu [122] (see also [153] for an exposition) proved that if $\left(M^3, g\right)$ is a complete Riemannian 3-manifold with bounded nonnegative sectional curvature and $\mathrm{Rc} \geq \varepsilon R g$ with $\varepsilon > 0$, then M^3 is either compact or flat. A related question is the following (see Chapter 10 for more on differential Harnack inequalities).

Problem 3.40. *If $\left(M^3, g\left(t\right)\right)$ is a complete solution to the Ricci flow on a 3-manifold with nonnegative Ricci curvature which is bounded on compact time intervals, can one prove a trace differential Harnack inequality? One could hope for an inequality similar to (10.45).*

Note that a result related to the first problem above, due to Hamilton [285], is the following.

Theorem 3.41. *If $M^n \subset \mathbb{R}^{n+1}$ is a C^∞ complete, strictly convex hypersurface with $h_{ij} \geq \varepsilon H g_{ij}$ for some $\varepsilon > 0$, then M^n is compact.*

Analogous to the second problem posed above is the following.

Problem 3.42. *Does there exist a Harnack inequality for solutions to the mean curvature flow with nonnegative mean curvature and second fundamental form which is bounded on compact time intervals? In particular, can one prove a* differential *Harnack inequality? Here one hopes for an inequality similar to (10.86).*

Comparison with mean curvature flow (MCF). It is interesting to compare the Ricci flow with the MCF; scattered throughout the rest of the book we shall see analogies and relations between the two flows. Let $X_t : M^{n-1} \to \mathbb{R}^n$, $t \in [0,T)$, be a solution to the **mean curvature flow**:

$$\frac{\partial X}{\partial t}(p,t) = -H(p,t)\,\nu(p,t), \quad p \in M^{n-1},\ t \in [0,T), \tag{3.43}$$

where H is the mean curvature and ν is the unit outward normal. This is the gradient flow for the volume functional.

Lemma 3.43 (Basic evolutions under MCF, [311])**.** *We have the following evolution equations for the induced metric g_{ij}, second fundamental form h_{ij}, and H:*

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2Hh_{ij}, \\
\frac{\partial}{\partial t} h_{ij} &= \Delta h_{ij} - 2Hh_{ik}h_{kj} + |h|^2 h_{ij}, \\
\frac{\partial}{\partial t} H &= \Delta H + |h|^2 H.
\end{aligned}$$

Proof. We leave this as an exercise. Use the formulas $H = g^{ij}h_{ij}$,

$$\begin{aligned}
g_{ij} &= \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle, \\
h_{ij} &= \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial \nu}{\partial x^j} \right\rangle = -\left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right\rangle.
\end{aligned}$$

While carrying out the computation, keep in mind that the inner product of a tangential vector with a normal vector is zero. □

Exercise 3.44. *Suppose that we have the flow $\frac{\partial X}{\partial t} = -\beta\nu$, where β is some function. Compute the evolution equations for g_{ij}, ν, h_{ij} and H.*

From Lemma 3.43 one can show (see [311]) using the maximum principle for tensors that if $H \geq 0$, then the inequalities

$$\alpha H g_{ij} \leq h_{ij} \leq \beta H g_{ij}$$

are preserved under the MCF. Compare this with the 'Ricci pinching is preserved' estimate (3.15).

The Codazzi equations $\nabla_i h_{jk} = \nabla_j h_{ik}$ imply that one can improve the estimate $|\nabla_i h_{jk}|^2 \geq \frac{1}{n-1} |\nabla_i H|^2$ to[1]

$$|\nabla_i h_{jk}|^2 \geq \frac{3}{n+1} |\nabla_i H|^2 \tag{3.44}$$

(this is an improvement only when $n \geq 3$); see [311] for details. Compare this with (3.20).

Remark 3.45. *If the hypersurface M^{n-1} is totally umbillic, so that $h = \frac{H}{n-1} g$, then from (3.44) we have $\frac{1}{n-1} |\nabla H|^2 \geq \frac{3}{n+1} |\nabla H|^2$. When $n \geq 3$, this implies $|\nabla H| = 0$. (Compare with Exercise 1.184.)*

Huisken's pinching theorem says that if M^{n-1} is a closed convex hypersurface evolving by the mean curvature flow, then

$$\left| h_{ij} - \frac{1}{n-1} H g_{ij} \right| \leq C H^{1-\delta}. \tag{3.45}$$

Compare with the 'Ricci pinching improves' estimate (3.16). Pointwise estimates are not sufficient to obtain this since, under the ODE corresponding to the PDE for h_{ij}, the pinching is preserved but not improved. In [311] an iteration argument is used to obtain (3.45).

[1] As usual, $|\nabla_i h_{jk}|^2 \doteq g^{ip} g^{jq} g^{kr} \nabla_i h_{jk} \nabla_p h_{qr}$.