A Retrospective Look at Ricci Flow: Lecture 2

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Short Course at Xiamen University
March 20-30, 2023

UC San Diego

Abstract

This is the second talk in the short course
"A Retrospective Look at Ricci Flow" given via Tencent
http://tianyuan.xmu.edu.cn/cn/MiniCourses/2077.html
at Xiamen University from March 20 to 30, 2023.

Lecture 2: Three-Manifolds with Positive Ricci Curvature

In this talk we discuss Hamilton's seminal 1982 result classifying compact 3-dimensional manifolds with positive Ricci curvature by using the Ricci flow.

References:

Reference: Bennett Chow, Peng Lu, and Lei Ni, *Hamilton's Ricci flow*, Chapter 3, AMS 2006.

Introduction

In the first lecture we discussed the Ricci flow in dimension 2. In this lecture we begin to discuss the main result of Hamilton's seminal paper "Three-manifolds with positive Ricci curvature" from a more general perspective.

Short-time existence of the Ricci flow

A very nice property of Ricci flow is that for any metric on a closed manifold, there **exists a unique solution to the Ricci flow** starting at that initial metric. This result was proved by Hamilton in 1982. A simplified proof was given by DeTurck in 1983.

Theorem (Short-time existence)

*For any closed Riemannian manifold (M^n, g_0), there exists $T \in (0, \infty]$ and a unique family of metrics $g(t)$, $t \in [0, T)$, that satisfy the **Ricci flow***

$$\partial_t g(t) = -2 \operatorname{Ric}_{g(t)}$$

with the initial condition $g(0) = g_0$.

We need this result in order to use Ricci flow as a tool to improve metrics by deforming them to "better" metrics.

Evolution of R under the Ricci flow, I

Given that we have a solution $g(t)$ on M^n to the Ricci flow, how do we study the behavior of this solution?

The scalar curvature R is the easiest curvature to work with since it is a function. If $g(t)$ is a solution to the Ricci flow on a manifold M^n, then its scalar curvature $R(t)$ evolves by the following **heat-type equation**:

$$\partial_t R = \Delta R + 2|\operatorname{Ric}|^2.$$

This formula is useful for obtaining a lower bound for the scalar curvature. Indeed, we have in general for any 2-tensor α that

$$|\alpha|^2 \geq \frac{1}{n}\operatorname{trace}(\alpha)^2.$$

By taking $\alpha = \operatorname{Ric}$, since $\operatorname{trace}(\operatorname{Ric}) = R$ we obtain:

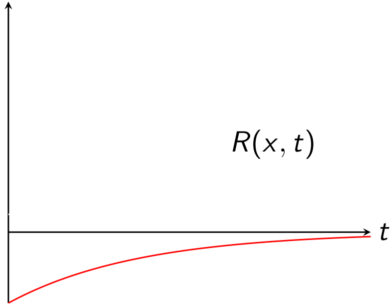
$$\partial_t R \geq \Delta R + \frac{2}{n} R^2.$$

The evolution equation for R is central to the study of Ricci flow.

Evolution of R under the Ricci flow, II

By applying the parabolic maximum principle to the inequality $\partial_t R \geq \Delta R + \frac{2}{n} R^2$, we obtain the estimate

$$R(x,t) \geq -\frac{n}{2t} \quad \text{for all } x \in M^n,\ t > 0.$$



In particular, the scalar curvature of a Ricci flow tends toward being non-negative.

The shrinking Ricci soliton equation, I

However, again we have the difficulty of obtaining **upper estimates for the scalar curvature**. Unlike in dimension $n = 2$, it does not seem useful to consider the potential function f defined by $\Delta f = R - r$ for higher-dimensional Riemannian manifolds, except in the Kähler case. Even so, there are lessons to be learned from the potential function.

So, to understand how to proceed in higher-dimensions, we re-examine the 2-dimensional case. Recall that under the Ricci flow on S^2, Hamilton first proved that (under the modified Ricci flow) the limit metric g_∞ (we will now call it g for simplicity) satisfies the equation (we change the sign for f from the last lecture)

$$2\,\mathrm{Ric} = Rg = -2\nabla^2 f + rg, \qquad (\text{so } -\Delta f = R - r).$$

Again, this is the fundamental equation satisfied by the limit metric g_∞.

The shrinking Ricci soliton equation, II

Recall from the last slide that the limit metric g_∞ of the Ricci flow on S^2 satisfies the equation

$$\mathrm{Ric} = -\nabla^2 f + \frac{r}{2} g,$$

where $r > 0$. Hamilton then showed that, since we are on the 2-sphere, f must be constant, so that $R \equiv r$ for g_∞.

Definition (Shrinking soliton)

If (M^n, g, f) in any dimension satisfies this equation (with $r = 1$):

$$\mathrm{Ric} + \nabla^2 f = \frac{1}{2} g,$$

then we say that it is a **shrinking (gradient Ricci) soliton**.

Recall that this equation generalizes the **Einstein metric** equation since that is the case where f is constant, so that $\mathrm{Ric} = \frac{1}{2} g$.

Properties of shrinking solitons, I

Let (M^n, g, f) be a shrinking soliton; i.e.,

$$\operatorname{Ric} + \nabla^2 f = \frac{1}{2} g.$$

By tracing this equation, we obtain

$$R + \Delta f = \frac{n}{2}.$$

The contracted **second Bianchi identity** says that

$$\operatorname{div}(\operatorname{Ric}) = \frac{1}{2} \nabla R.$$

By applying this to the shrinking soliton equation, one can show that (exercise!):

$$\nabla R = 2 \operatorname{Ric}(\nabla f).$$

This is a basic and useful equation for the study of Ricci solitons.

Properties of shrinking solitons, II

Let (M^n, g, f) be a shrinking soliton. We can substitute the equation $\mathrm{Ric} + \nabla^2 f = \frac{1}{2} g$ into the equation $\nabla R = 2\,\mathrm{Ric}(\nabla f)$ to obtain

$$\nabla R = 2\,\mathrm{Ric}\,(\nabla f) = -2\nabla^2 f \cdot \nabla f + g \cdot \nabla f = -\nabla |\nabla f|^2 + \nabla f.$$

Thus (by adding a suitable constant to f), we have that

$$R + |\nabla f|^2 = f.$$

This is another important equation in the study of shrinking solitons.

Ricci flow associated to a shrinking soliton

A fundamental fact is that, given a shrinking soliton (M^n, g, f), there is an associated **Ricci flow** $(M^n, g(t))$, $t \in (-\infty, 1)$, and time-dependent potential function $f(t)$ such that:

$$\partial_t g(t) = -2\,\mathrm{Ric}_{g(t)}\,,$$

$$\partial_t f(t) = |\nabla_{g(t)} f(t)|^2_{g(t)}\,,$$

and

$$\mathrm{Ric}_{g(t)} + \nabla^2_{g(t)} f(t) = \frac{1}{2\tau(t)} g(t)\,,$$

where $\tau(t) := 1 - t$. Consequently,

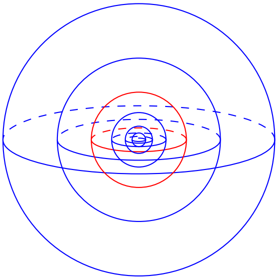
$$R_{g(t)} + \Delta_{g(t)} f(t) = \frac{n}{2\tau(t)},$$

$$R_{g(t)} + |\nabla_{g(t)} f(t)|^2_{g(t)} = \frac{f(t)}{\tau(t)}.$$

The equations for (M^n, g, f) are the case of $t = 0$, so that $\tau = 1$.

A sphere and its associated Ricci flow

As an example of a shrinking soliton and its associated Ricci flow, we consider a sphere.



$t \to -\infty$

The red sphere represents the shrinking soliton (S^n, g), where $\mathrm{Ric} = \frac{1}{2}g$. For the associated Ricci flow $(S^n, g(t))$, this is time $t = 0$. The spheres shrink to a point at time $t = 1$. As $t \to -\infty$, the spheres expand to infinity. We have $\mathrm{Ric}_{g(t)} = \frac{1}{2(1-t)}g(t)$.

Perelman's coupled Ricci flow equations

Hamilton's Ricci flow

$$\partial_t g = -2\,\mathrm{Ric}$$

is **coupled by Perelman** to the backward heat-type equation

$$\partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \qquad (*)$$

and the backward-time equation

$$\partial_t \tau = -1.$$

For the special case of a Ricci flow associated to a shrinking soliton (M^n, g, f), recall that we have $\partial_t f = |\nabla f|^2$. By subtracting the equation $R + \Delta f - \frac{n}{2\tau} = 0$, we obtain equation $(*)$. That is, a shrinking soliton solution to the Ricci flow satisfies Perelman's coupling for Ricci flow.

We have motivated Perelman's coupling of Ricci flow by Ricci solitons. How else might we motivate it?

Motivations from the Euclidean heat kernel

Recall that the **Euclidean heat kernel** is given by

$$H(x,t) = (4\pi t)^{-n/2} \mathrm{e}^{-\frac{|x|^2}{4t}} \qquad x \in \mathbb{R}^n,\ t > 0.$$

Taking the natural logarithm of this, we obtain

$$\ln H = -\frac{n}{2} \ln(4\pi t) - \frac{|x|^2}{4t}.$$

Hence, we have for the Euclidean heat kernel:

$$\partial_t \ln H - |\nabla \ln H|^2 = \Delta \ln H = -\frac{n}{2t}.$$

Recall that the **Li–Yau inequality** for a positive solution u to the heat equation on a complete Riemannian manifold with **non-negative Ricci curvature** is:

$$\partial_t \ln u - |\nabla \ln u|^2 = \Delta \ln u \geq -\frac{n}{2t}.$$

We see that this inequality is modeled on the Euclidean heat kernel.

Perelman's coupling and the Gaussian shrinking soliton

Let's return to Perelman's coupling of $\partial_t g = -2\,\mathrm{Ric}$, $\partial_t \tau = -1$, to

$$\partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}. \qquad (*)$$

Consider, as a special case, **Euclidean space** $(\mathbb{R}^n, g_{\mathbb{E}})$ as a static solution to the Ricci flow. Define

$$f(x,t) := \frac{|x|^2}{4\tau(t)},$$

where $\tau(t) = 1 - t$ for $t < 1$ and $x \in \mathbb{R}^n$. We easily compute that

$$\Delta f = \frac{2n}{4\tau} = \frac{n}{2\tau} \qquad \text{and} \qquad \partial_t f = \frac{|x|^2}{4\tau^2} = |\nabla f|^2.$$

Thus f satisfies $(*)$. If fact, $(\mathbb{R}^n, g_{\mathbb{E}}, f)$ is the **Gaussian shrinking soliton**:

$$\mathrm{Ric}_{\mathbb{E}} + \nabla^2_{\mathbb{E}} f = \nabla^2_{\mathbb{E}} f = \frac{1}{2\tau} g_{\mathbb{E}}.$$

All of this is a special case of the fact that a shrinking Ricci soliton satisfies Perelman's coupling of the Ricci flow.

Perelman's entropy monotonicity formula,I

Perelman's entropy is defined for a closed Riemannian manifold (M^n, g), a function f, and a positive real number τ as:

$$\mathcal{W}(g,f,\tau) := \int_{M^n} \left(\tau\left(R + |\nabla f|^2\right) + (f-n)\right)(4\pi\tau)^{-n/2}\, e^{-f}\, d\mu.$$

Integrating by parts yields

$$\mathcal{W}(g,f,\tau) = \int_{M^n} \left(\tau\left(R + 2\Delta f - |\nabla f|^2\right) + f - n\right)(4\pi\tau)^{-n/2}\, e^{-f}\, d\mu.$$

As a special case, consider the **entropy of the Gaussian soliton** defined by $(\mathbb{R}^n, f(t), g(t))$, where

$$f(x,t) = \tfrac{|x|^2}{4\tau(t)} \qquad \text{and} \qquad g(t) = g_{\mathbb{E}}.$$

Observe that

$$\Delta f = \tfrac{n}{2\tau} \qquad \text{and} \qquad \tau|\nabla f|^2 = f.$$

Thus, $\mathcal{W}(g(t), f(t), \tau(t)) \equiv 0$ for the Gaussian shrinking soliton.

Perelman's entropy monotonicity formula,II

Let $u := (4\pi\tau)^{-n/2} e^{-f}$. Under Perelman's coupling for the Ricci flow, we have

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = 2\tau \int_{M^n} \left|\mathrm{Ric} + \nabla^2 f - \frac{1}{2\tau} g\right|^2 u \, d\mu \geq 0.$$

That is, $\mathcal{W}(g(t), f(t), \tau(t))$ **is monotonically non-decreasing**. Moreover, if $\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = 0$, then

$$\mathrm{Ric}_{g(t)} + \nabla^2_{g(t)} f(t) = \frac{1}{2\tau(t)} g(t).$$

That is, $(M^n, g(t), f(t))$ is a shrinking soliton. In particular, Perelman's entropy is monotone increasing in general and is constant on shrinking Ricci solitons.

Proof of Perelman's entropy monotonicity formula

The proof of Perelman's entropy monotonicity formula is a calculation. Given (M^n, g, f, τ), **Perelman's Harnack quantity** is:

$$v := \left(\tau \left(R + 2\Delta f - |\nabla f|^2\right) + f - n\right) u,$$

where $u = (4\pi\tau)^{-n/2} e^{-f}$. One computes (using integration by parts) that

$$\mathcal{W}(g, f, \tau) = \int_{\mathcal{M}} v \, d\mu.$$

If $(M^n, g(t), f(t), \tau(t))$ satisfies **Perelman's coupling of Ricci flow**, then

$$\left(-\frac{\partial}{\partial t} - \Delta + R\right) v = -2\tau \left|\mathrm{Ric} + \nabla^2 f - \frac{1}{2\tau} g\right|^2 u.$$

This implies the **entropy monotonicity formula**:

$$-\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = -2\tau \int_{M^n} \left|\mathrm{Ric} + \nabla^2 f - \frac{1}{2\tau} g\right|^2 u \, d\mu \leq 0.$$

The conjugate heat operator

The **heat operator** is

$$\square := \frac{\partial}{\partial t} - \Delta.$$

Its **conjugate** (a.k.a. **adjoint**) **operator** is:

$$\square^* := -\frac{\partial}{\partial t} - \Delta + R.$$

The reason for the term R in $\square^*$ is that the metrics $g(t)$ depend on time and that under the Ricci flow the evolution of the volume form is given by:

$$\frac{\partial}{\partial t} d\mu = -R\, d\mu.$$

Indeed, using this, one calculates that for any functions ϕ and ψ on $M^n \times I$, where I is an interval,

$$\iint_{M^n \times I} \square\phi\, \psi\, dg\, d\mu = \iint_{M^n \times I} \phi\, \square^* \psi\, d\mu dt.$$

Perelman's differential Harnack estimate

Perelman's coupling for $(M^n, g(t), f(t), \tau(t))$ implies that $u = (4\pi\tau)^{-n/2} e^{-f}$ satisfies the conjugate heat equation:

$$\Box^* u = 0.$$

Recall also that $v := \left[\tau\left(R + 2\Delta f - |\nabla f|^2\right) + f - n\right] u$ satisfies

$$\Box^* v = -2\tau \left|\mathrm{Ric} + \nabla^2 f - \frac{1}{2\tau} g\right|^2 u \leq 0.$$

Given a Ricci flow $(M^n, g(t))$, assume that the coupling $(f(t), \tau(t))$ is defined for $\tau \in (0, T)$, for some $T > 0$. Further assume that as $\tau(t) \to 0$, we have that $u(\cdot, t)$ approaches a **Dirac delta function** δ_{x_0}, where $x_0 \in M^n$. That is, assume that u is a **conjugate heat kernel**. In this case, Perelman proved that

$$v = \left[\tau\left(R + 2\Delta f - |\nabla f|^2\right) + f - n\right] u \leq 0.$$

This is **Perelman's differential Harnack estimate**.

No Local Collapsing

The $\mathcal{W}$ entropy monotonicity formula yields estimates from below for the **volume of balls** on which the scalar curvature is bounded from above. This is called Perelman's **no local collapsing**. This result is fundamentally important in the study of singularity formation in Ricci flow.

Let $B(x, t, r)$ denote the ball of radius $r > 0$ centered at a point $x \in M^n$ with respect to the metric $g(t)$ at time t.

Theorem (Perelman's no local collapsing (κ-noncollapsing))

For any finite time compact Ricci flow $(M^n, (g_t)_{t\in[0,T)})$, there exists $\kappa > 0$ depending only on $g(0), T, \rho$ such that if (x, t) and $0 < r \leq \rho$ satisfy $R \leq r^{-2}$ in $B(x, t, r)$, then

$$\mathrm{Vol}_t\, B(x, t, r) \geq \kappa\, r^n,$$

where Vol_t denotes the volume with respect to $g(t)$.

Idea of the proof of Perelman's no local collapsing

Suppose that $R \leq r^{-2}$ in $B_r := B(x, t, r)$. Define $f(t)$ by

$$(4\pi r^2)^{-n/2} e^{-f(t)} := c\,\chi_{B_r},$$

where χ_{B_r} is the **characteristic function** of the ball B_r and where the constant c satisfies $c \operatorname{Vol}_t B_r = 1$. Then, by $\mathcal{W}$-monotonicity,

$$\mathcal{W}(g(t), f(t), r^2) \geq \mathcal{W}(g(0), f(0), r^2 + t) \geq C(g(0), T, \rho).$$

On the other hand,

$$\begin{aligned}\mathcal{W}(g(t), f(t), r^2) &= \int_M \left(r^2 R - \ln c - \frac{n}{2} \ln(4\pi r^2) - n \right) c\,\chi_{B_r}\, d\mu \\ &\leq 1 - \ln c - \frac{n}{2} \ln(4\pi r^2) - n \\ &\leq \ln \frac{\operatorname{Vol}_t B_r}{r^n} - \frac{n}{2} \ln(4\pi),\end{aligned}$$

and the result follows. A rigorous proof uses a cutoff function.

Commentary

What is remarkable about Perelman's entropy formula is that it only uses the evolution equations for the metric g, the scalar curvature R, and the volume form $d\mu$. So far, we have not had to understand how the Ricci tensor Ric or the Riemann curvature tensor Rm evolve under the Ricci flow.

On the other hand, Hamilton's original 1982 paper is based in large part on estimates for the Ricci tensor. We now discuss these estimates.

Hamilton's 3-manifolds with positive Ricci curvature, I

Recall that under the Ricci flow, the scalar curvature evolves by

$$\partial_t R = \Delta R + |\operatorname{Ric}|^2.$$

Next, we wish to state the **evolution equation for the Ricci tensor**. Given symmetric 2-tensors α and β, we define their **product** by

$$(\alpha \cdot \beta)(X, Y) := \frac{1}{2} \sum_{i=1}^{n} \big(\alpha(X, e_i)\beta(e_i, Y) + \alpha(Y, e_i)\beta(e_i, X)\big),$$

where $\{e_i\}_{i=1}^n$ is an orthonormal frame. The **square** of a symmetric 2-tensor is defined by

$$\alpha^2(X, Y) := (\alpha \cdot \alpha)(X, Y) = \sum_{i=1}^{n} \alpha(X, e_i)\alpha(e_i, Y).$$

Hamilton's 3-manifolds with positive Ricci curvature, II

The **Lichnerowicz Laplacian** Δ_L acting on symmetric 2-tensors is defined by

$$\Delta_L \alpha := \Delta\alpha + 2\,\mathrm{Rm}(\alpha) + 2\,\mathrm{Ric}\cdot\alpha,$$

where

$$\mathrm{Rm}(\alpha)(X,Y) := \sum_{i,j=1}^{n} \mathrm{Rm}(X, e_i, e_j, Y)\alpha(e_i, e_j).$$

We remark that if α were an (antisymmetric) 2-form, instead of a symmetric 2-tensor, then we would have that the Licherowicz Laplacian of α,

$$\Delta_L \alpha = \Delta_d \alpha := -(d\delta + \delta d)\alpha,$$

equals the Hodge Laplacian of α (Bochner–Weitzenböck formula).

If $(M^n, g(t))$ is a solution to the **Ricci flow**, then

$$\partial_t \,\mathrm{Ric} = \Delta_L \,\mathrm{Ric}\,.$$

Hamilton's 3-manifolds with positive Ricci curvature, III

For any Riemannian 3-manifold (M^3, g), its Riemann curvature tensor can be expressed in terms of Ric and g as follows:

$$R_{ijk\ell} = R_{i\ell} g_{jk} + R_{jk} g_{i\ell} - R_{ik} g_{j\ell} - R_{j\ell} g_{ik} - \frac{1}{2} R \left(g_{i\ell} g_{jk} - g_{ik} g_{j\ell} \right).$$

Here, we use index notation to represent the Riemann curvature tensor as $R_{ijk\ell}$ and the Ricci tensor as R_{ij}.

By using this formula, we see that for any solution $(M^3, g(t))$ of the Ricci flow on a 3-manifold, the Ricci tensor evolves by:

$$\partial_t \operatorname{Ric} = \Delta \operatorname{Ric} + 3R \operatorname{Ric} - 6 \operatorname{Ric}^2 + (2 \, |\mathrm{Rc}|^2 - R^2) g.$$

Using, this, Hamilton computed that

$$\left(\partial_t - \Delta - 2 \tfrac{\nabla R}{R} \cdot \nabla\right) \frac{|\operatorname{Ric}|^2}{R^2} \leq -\frac{4}{R^3} P,$$

where $P := \sum_{[ijk]=[123]} \lambda_i^2 (\lambda_i - \lambda_j)(\lambda_i - \lambda_k) \geq 0$ and $\lambda_1, \lambda_3, \lambda_3$ are the eigenvalues of the Ricci tensor.

Hamilton's 3-manifolds with positive Ricci curvature, IV

By applying the maximum principle to the equation on the previous slide, we obtain that if $\frac{|\operatorname{Ric}|^2}{R^2} \leq C$ at $t = 0$, then $\frac{|\operatorname{Ric}|^2}{R^2} \leq C$ for $t \geq 0$.

Observe that

$$0 \leq \frac{\left|\operatorname{Ric} - \frac{1}{3}Rg\right|^2}{R^2} = \frac{|\operatorname{Ric}|^2 - \frac{1}{3}R^2}{R^2} = \frac{|\operatorname{Ric}|^2}{R^2} - \frac{1}{3}.$$

Thus, $\frac{|\operatorname{Ric}|^2}{R^2} \geq \frac{1}{3}$, with equality if and only if $\operatorname{Ric} = \frac{1}{3}Rg$.

So our goal is to show that under the Ricci flow, as t approaches the singularity time T, we have $\frac{|\operatorname{Ric}|^2}{R^2} \to \frac{1}{3}$ for the metric $g(t)$.

Hamilton's 3-manifolds with positive Ricci curvature, V

Let (M^3, g_0) be a closed Riemannian 3-manifold with $\mathrm{Ric} > 0$. Since M^3 is compact, there exists $\varepsilon > 0$ such that $\mathrm{Ric} \geq \varepsilon R g$. Let $g(t)$, $t \in [0, T)$, be the solution to the Ricci flow with $g(0) = g_0$.

Hamilton proved that $g(t)$ satisfies $\mathrm{Ric} \geq \varepsilon R g$ for all $t \in [0, T)$. A metric that satisfies such an inequality is called **Ricci pinched**. For such metrics, for each point p in M, the minimum Ricci curvature at p is at least ε times the maximum Ricci curvature at p.

Hamilton then proved the following strong estimate: There exists $\delta > 0$ and a constant C such that

$$R^{-2} \left| \mathrm{Ric} - \tfrac{1}{3} R g \right|^2 \leq C R^{-\delta}.$$

In particular, if $\{(x_i, t_i)\}$ is a sequence of points in $M^3 \times [0, T)$ such that $R(x_i, t_i) \to \infty$, then we have

$$R^{-2} \left| \mathrm{Ric} - \tfrac{1}{3} R g \right|^2 (x_i, t_i) \to 0.$$

Hamilton's 3-manifolds with positive Ricci curvature, VI

In the next lecture we will further discuss singularity analysis and how to prove Hamilton's 1982 result that for any closed Riemannian 3-manifold (M^3, g_0) with $\mathrm{Ric} > 0$, a solution to the **normalized Ricci flow** exists for all time $t \in [0, \infty)$ and as $t \to \infty$, the metric $g(t)$ **converges** to a smooth metric g_∞ on M^3 which satisfies $\mathrm{Ric} = \frac{1}{3} R g$. This, in turn, implies that g_∞ has constant positive sectional curvature.

THANK YOU!