**Name：**Introduction to Plancherel Formula for Real Reductive Groups

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**Speaker Information：**

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Prof. Tomasz Przebinda’s research interests focus on representation theory of reductive Groups, classical harmonic analysis with applications to signal processing, harmonic analysis on symmetric spaces, probability and so on.

**Introduction：**Classical Harmonic Analysis concerns a decomposition of a function (signal) into a

superposition of components corresponding to simple harmonics. The analysis of the signal aims at finding these components and the synthesis is the reconstruction of the signal out of them. There is often some “signal processing", dictated by applications, between the analysis and synthesis. The simple harmonics behave well under various symmetries and this is the reason for the decomposition. The fundamental results are Parseval's Theorem (1806) for the Fourier series, [Par06], and Plancherel's Theorem (1910) for the Fourier transform, [Pla10]. Among the best known applications is the Magnetic Resonance Imaging, for which Peter Manseld and Paul Lauterbur were awarded a Nobel

prize in 2003.

If the function is defined on a commutative group, such as the additive group of the real numbers or the multiplicative group of the complex numbers of absolute value one, then the simple harmonics are the eigenvectors under the translations. This is the ultimate symmetry one could expect problems arising in Physics and Number Theory motivated a rapid growth of Harmonic Analysis on non-commutative groups. The earliest examples were the Heisenberg group, necessary for a formulation of the principles of Quantum Mechanics (J. Von Neumann 1926, [vN26]), and the compact Lie groups (Peter-Weyl 1927, [PW27]). Here the simple harmonics are replaced by irreducible unitary representations. All of them may be found by analyzing the square integrable functions on these groups, so that an analog of Plancherel's Theorem may be viewed as the top achievement of the theory.

However, there are plenty of other groups of interest which have irreducible unitary representations occurring outside the space of the square integrable functions on the group. The main class are the non-compact semisimple Lie groups, such as SL(2, R). Though the irreducible unitary representations of most of them are not understood yet, the representations that can be found in the space of the square integrable functions on the group are known and the decomposition of an arbitrary such function in terms of these representations is known as the Plancherel formula. For the group SL(n, C) this formula was first found by Gelfand and Naimark in 1950, [GfN50], and for SL(n, R) by Gelfand and Graev in 1953, [GfG53]. The Plancherel formula on an arbitrary Real Reductive Group was published by Harish-Chandra in 1976, [Har76], and is considered as one of the greatest achievements of Mathematics of the 20th century.

A goal of these five lectures is to explain the ingredients of Harish-Chandra's Plancherel formula, explain how they fit together, study particular cases and go through all the details for the group of the real unimodular matrices of size two. All the necessary information in its original nearly perfect form is contained in Harish-Chandra’s articles [HC14a], [HC14b], [HC14d], [HC14c], [HC18]. The example SL(2, R) is explained in classical books such as [Lan75]. For all of that, a good understanding of the Fourier Transform and Distribution theory on an Euclidean space is needed. Here Hörmander “The Analysis of Linear Partial Differential Operators I" is one of the best references, [Hor83].

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