K-stability of explicit Fano varieties

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Goal

Verify K-(semi,poly)stability of explicit Fano varieties.

- Del Pezzo surfaces.
- Fano threefolds.
- Hypersurfaces and complete intersections.

Conjecture

Every smooth Fano complete intersection $X \subseteq \mathbb{P}^N$ is K-polystable (i.e. admits Kähler-Einstein metric).

Original definition (Tian, Donaldson, Li-Xu)

X is K-semistable (resp. K-stable) iff $\mathrm{Fut}(\mathcal{X}) \geq 0$ (resp. > 0) for all special test configurations $\mathcal{X}$ of X.

Valuative criterion (Fujita, Li, Blum-Xu)

X is K-semistable (resp. K-stable) iff $\beta_X(E) \geq 0$ (resp. > 0) for all divisors E over X.

Tian's criterion

Notation

$X =$ klt Fano variety over $\mathbb{C}$ of dimension n.

Definition (α-invariant)

$\alpha(X) = \sup\{\lambda \geq 0 \,|\, (X, \lambda D) \text{ is lc for every } 0 \leq D \sim_{\mathbb{Q}} -K_X\}$.
It measures singularities of the "worst" anticanonical divisors.

Example: $\alpha(\mathbb{P}^n) = \frac{1}{n+1}$

- $-K_{\mathbb{P}^n} \sim (n+1)H \Rightarrow \alpha(\mathbb{P}^n) \leq \frac{1}{n+1}$.
- $\forall 0 \leq D \sim_{\mathbb{Q}} -K_X$, $\mathrm{mult}_x D \leq n+1 \Rightarrow (X, \frac{1}{n+1} D)$ is lc.

Tian's criterion

Definition (α-invariant)

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Theorem (Tian, '87)

For smooth X, $\alpha(X) > \frac{n}{n+1} \Rightarrow X$ has KE.

Theorem (Odaka-Sano, Fujita)

If $\alpha(X) \geq \frac{n}{n+1}$, then X is K-semistable. If in addition X is smooth and $n \geq 2$, or $\alpha(X) > \frac{n}{n+1}$, then X is K-stable.

Sketch of proof

- Fix a divisor E over X. Want: $\beta(E) = A_X(E) - S(E) \geq 0$.
- Recall $S(E) = \int_0^\infty f(t)\mathrm{d}t$ where $f(t) = \frac{\mathrm{vol}(-K_X - tE)}{(-K_X)^n}$.
- $T = T(E) = \sup\{t \geq 0 \,|\, \mathrm{vol}(-K_X - tE) > 0\}$.
- $S(E) =$ barycenter of $[0, T)$ with density $\rho(t) = -f'(t)$ (integration by part).
- $\rho(t)^{\frac{1}{n-1}}$ is concave (Lazarsfeld-Mustaţă).
- $\frac{n}{n+1} =$ coordinate of barycenter of an n-dim simplex.
- If $\alpha(X) \geq \frac{n}{n+1}$, then $A_X(E) \geq \frac{n}{n+1} T(E) \geq S(E)$.
- If $\alpha(X) = \frac{n}{n+1}$ $(n \geq 2)$ and X smooth, analyze equality condition to show it's never achieved (Fujita).

Application of Tian's criterion

Theorem (Fujita)

If $\alpha(X) \geq \frac{n}{n+1}$ $(n \geq 2)$ and X is smooth, then X is K-stable.

Corollary (Tian, Cheltsov, Fujita)

Every smooth del Pezzo surface of degree ≤ 3 is K-stable.

Corollary (Cheltsov-Park, Fujita)

Every smooth hypersurface $X = X_{n+1} \subseteq \mathbb{P}^{n+1}$ $(n \geq 2)$ is K-stable.

LCT estimate

Proposition (Z.)

X smooth Fano of dimension n, $0 \leq D \sim_{\mathbb{Q}} -K_X$. Assume (X, D) is lc away from isolated points. Then $\mathrm{lct}(X; D) \geq \frac{n}{n+1}$.

Proof

Multiplier ideals + Nadel vanishing.

Application of Tian's criterion

Proposition

X smooth Fano of dimension n, $0 \leq D \sim_{\mathbb{Q}} -K_X$. Assume (X, D) is lc away from isolated points. Then $\mathrm{lct}(X; D) \geq \frac{n}{n+1}$.

Example: $X =$ del Pezzo surface of degree 2

- If $\mathrm{mult}_C D > 1$ for some $D \sim_{\mathbb{Q}} -K_X$ and some curve C, then $(-K_X \cdot C) < 2 \Rightarrow C$ is a (-1)-curve.
- $\pi \colon X \xrightarrow{2:1} \mathbb{P}^2$.
- $-K_X = \pi^*(\mathrm{line}) = \pi^*\pi(C) = C + C'$, C' also a (-1)-curve.
- $-K_X - tC = C' - (t-1)C$ not pseudo-effective for any $t > 1$.
- Proposition + Tian's criterion implies X is K-stable.
- Similar argument for del Pezzo of degree 1 or 3.

Application of Tian's criterion

Proposition

X smooth Fano of dimension n, $0 \leq D \sim_{\mathbb{Q}} -K_X$. Assume (X, D) is lc away from isolated points. Then $\mathrm{lct}(X; D) \geq \frac{n}{n+1}$.

Example: $X = X_{n+1} \subseteq \mathbb{P}^{n+1}$ $(n \geq 2)$

- Pukhlikov showed that $\mathrm{mult}_C D \leq 1$ for all $0 \leq D \sim_{\mathbb{Q}} -K_X$.
- Proposition + Tian's criterion implies X is K-stable.

Equivariant K-stability

Theorem (Datar-Székelyhidi, Liu-Zhu; Li-Wang-Xu, Z.)

- X is K-semistable $\Leftrightarrow$ X is equivariantly K-semistable.
- X is K-(poly)stable $\Leftrightarrow$ X is equivariantly K-(poly)stable $+ \operatorname{Aut}(X) < \infty$ (resp. reductive).

"Proof" when $\operatorname{Aut}(X) < \infty$

- $\pi \colon X \to Y := X/\operatorname{Aut}(X)$.
- $\exists \Delta_Y \geq 0$ such that $K_X = \pi^*(K_Y + \Delta_Y)$.
- X equivariantly K-stable $\Rightarrow$ (Y, Δ_Y) is K-stable $\Rightarrow$ (Y, Δ_Y) admits (conical) KE $\Rightarrow$ X admits KE $\Rightarrow$ X is K-stable.

Equivariant K-stability

Corollary

If $\mathrm{Aut}(X)$ is reductive and $\beta(E) > 0$ for every $\mathrm{Aut}(X)$-invariant divisor E over X, then X is K-polystable.

Corollary (Tian, Odaka-Sano, Fujita + Thm)

If X is smooth, $G = \mathrm{Aut}(X)$ is reductive and $\alpha_G(X) \geq \frac{n}{n+1}$, then X is K-polystable.

Example

- $\mathbb{P}^n$, quadric $Q \subseteq \mathbb{P}^{n+1}$.
- Del Pezzo surfaces of degree $4 \leq d \leq 6$.
- $V_5 = G(2,5) \cap \mathbb{P}^6$, del Pezzo 3-fold of degree 5 (Cheltsov-Shramov).
- Most V_{22} with $\mathrm{Aut} = \infty$ (Donaldson, Cheltsov-Shramov).
- Fermat hypersurfaces $(x_0^d + \cdots + x_{n+1}^d = 0) \subseteq \mathbb{P}^{n+1}$ (Tian, Nadel, Arezzo-Ghigi-Pirola).

Equivariant K-stability

Corollary

If $\mathrm{Aut}(X)$ is reductive and $\beta(E) > 0$ for every $\mathrm{Aut}(X)$-invariant divisor E over X, then X is K-polystable.

Example: del Pezzo surface of degree 4

- X is intersection of two quadrics.
- $X = (\sum x_i^2 = \sum a_i x_i^2 = 0) \subseteq \mathbb{P}^4 \Rightarrow G = (\mathbb{Z}/2\mathbb{Z})^4$ acts on X.
- No G-fixed points $\Rightarrow$ only G-invariant divisors over X are G-invariant curves $C \subseteq X$.
- C is proportional to $-K_X$ and $\beta(C) > 0$.

Birational superrigidity

Definition

A Fano variety X is said to be birationally superrigid (bsr) if

1. Every birational map $X \dashrightarrow Y$ to a terminal Fano variety extends to an isomorphism;
2. $\forall$ dominant $X \dashrightarrow Y$ with $\dim Y, \dim F \geq 1$, the general fiber F is not birational to any Fano variety.

In particular, such varieties are not birational to $\mathbb{P}^n$ ($n \geq 2$).

Theorem (Iskovskih-Manin, Corti, Pukhlikov, Cheltsov, de Fernex-Ein-Mustaţă, de Fernex)

Every smooth hypersurface $X = X_{n+1} \subseteq \mathbb{P}^{n+1}$ ($n \geq 3$) is bsr.

Theorem (Z.)

Every smooth Fano complete intersection $X \subseteq \mathbb{P}^{n+r}$ of Fano index 1, codimension r and dimension $n \geq 10r$ is bsr.

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Remark

Most (mildly singular) Fano varieties of index one are expected to be birationally superrigid.

Birational superrigidity

Theorem (Stibitz-Z)

If X is birationally superrigid and $\alpha(X) \geq \frac{1}{2}$, then X is K-stable.

Sketch of proof

- $T = T(E) = \sup\{t \geq 0 \,|\, \mathrm{vol}(-K_X - tE) > 0\}$.
- $S(E) =$ barycenter of $[0, T)$ with density $\rho(t) = -f'(t)$ where $f(t) = \frac{\mathrm{vol}(-K_X - tE)}{(-K_X)^n}$.
- $\rho(t)^{\frac{1}{n-1}}$ is concave.
- Bsr $\Rightarrow \rho(t)^{\frac{1}{n-1}}$ linearly decrease to 0 when $t \geq A_X(E) - 1$ (Noether-Fano inequality).

Birational superrigidity

Theorem (Stibitz-Z)

If X is birationally superrigid and $\alpha(X) \geq \frac{1}{2}$, then X is K-stable.

Theorem (Z.)

Every smooth Fano complete intersection $X \subseteq \mathbb{P}^{n+r}$ of Fano index 1, codimension r and dimension $n \geq 10r$ is K-stable.

Theorem (Liu-Z, Z.)

Every (klt) hypersurface $X = X_{n+1} \subseteq \mathbb{P}^{n+1}$ of dimension $n \geq 250$ with at worst ordinary singularities (i.e. projective tangent cone is smooth) is K-stable.

Birational superrigidity

Proposition (Z.)

X smooth, L ample line bundle, $0 \leq D \sim_{\mathbb{Q}} L$ such that (X, D) lc outside isolated points. If $h^0(X, \mathcal{O}_X(K_X + L)) \leq n^n/n!$, then $\mathrm{lct}(X; D) \geq \frac{1}{2}$.

Theorem (Z.)

Every smooth Fano complete intersection $X \subseteq \mathbb{P}^{n+r}$ of Fano index 1, codimension r and dimension $n \geq 10r$ is K-stable.

Sketch of proof

- $\forall D \sim_{\mathbb{Q}} -K_X = \mathcal{O}_X(1)$, $\mathrm{mult}_x D \leq 1$ outside a set of dimension $\leq r - 1$ (Suzuki).
- $Y = X \cap \mathbb{P}^{n+1}$, (Y, D_Y) lc outside isolated points.
- Proposition + $n \gg 0 \Rightarrow \mathrm{lct}(Y, D_Y) \geq \frac{1}{2}$.
- Inversion of adjunction $\Rightarrow \mathrm{lct}(X, D) \geq \frac{1}{2}$.